p-Adic Quantum Mechanics, Continuous-Time Quantum Walks, and Space Discreteness

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The discreteness of space

- In the 1930s, Bronstein showed that general relativity and quantum mechanics imply that the uncertainty Δx of any length measurement satisfies

$$\Delta x \geq L_{\text{Planck}} := \sqrt{\frac{\hbar G}{c^3}}, \tag{1}$$

  where L_{Planck} is the Planck length ($L_{\text{Planck}} \approx 10^{-33}\ cm$).

- A well-accepted interpretation of Bronstein's inequality is that below the Planck length there are no intervals just points (the space has a discrete nature).

The discreteness of space

- This interpretation has a precise mathematical translation: below the Planck length, the space is a totally (or completely) disconnected topological space, which means that the non-trivial connected subsets are points.
- Examples of totally disconnected spaces: Cantor set, the field of p-adic numbers $\mathbb{Q}_p$, etc.
- The choice of $\mathbb{R}$ as a model of the unidimensional space is not compatible with the inequality (1) because $\mathbb{R}$ contains intervals of arbitrarily small lengths.

The discreteness of space

- On the other hand, there are no intervals in $\mathbb{Q}_p$, i.e., the non-trivial connected subsets are points. So $\mathbb{Q}_p$ is the prototype of a 'discrete space' with a very rich mathematical structure.
- In the 80s, Volovich conjectured the p-adic nature of the space at the Planck scale.
- We use the term " discrete space" to mean a totally disconnected topological space.

The naive approach

- Many authors model the discreteness of the space by using a subset of $\mathbb{R}^3$ whose points are located to a finite distance each other. For instance, $\mathbb{Z}^3 \subset \mathbb{R}^3$, i.e., the space is a lattice of the standard Euclidean space.
- This approach is not convenient: this choice does not change the Poincaré group of $\mathbb{R}^3$, and the discrete space $(\mathbb{Z}^3)$ is not invariant under all these symmetries.
- Immediately, special and general relativity are compatible with this notion of discreteness.

p-Adic numbers

The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^{\gamma}\dfrac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma = ord_p(x) := ord(x)$, with $ord(0) := +\infty$, is called the p-adic order of x. Any p-adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j,$$

where $x_j \in \{0, 1, 2, \ldots, p-1\}$ and $x_0 \neq 0$.

p-Adic Analysis

Since $(\mathbb{Q}_p, +)$ is a locally compact topological group, there exists a Haar measure dx, which is invariant under translations, i.e., $d(x+a) = dx$. If we normalize this measure by the condition $\int_{\mathbb{Z}_p} dx = 1$, then dx is unique. We will use $\Omega\left(p^{-r}|x-a|_p\right)$ to denote the characteristic function of the ball $B_r(a) = a + p^{-r}\mathbb{Z}_p$, where

$$\mathbb{Z}_p = \left\{x \in \mathbb{Q}_p ; |x|_p \leq 1\right\}$$

is the unit ball.

p-Adic numbers

We denote by $L^2(\mathbb{Z}_p) := L^2(\mathbb{Z}_p, dx)$, the $\mathbb{C}$-vector space of all the complex valued functions g satisfying

$$\|g\|_2 = \left(\int_{\mathbb{Z}_p} |g(x)|^2 \, dx \right)^{\frac{1}{2}} < \infty,$$

where dx is the normalized Haar measure on $(\mathbb{Q}_p, +)$.

The Dirac-Von Neumann formulation of QM

- The states of a quantum system are described by non-zero vectors from $\mathcal{H}$, a separable complex Hilbert space called the space of states.
- Each observable corresponds to a unique linear self-adjoint operator in $\mathcal{H}$.
- The most important observable of a quantum system is its energy $\boldsymbol{H}$.
- Let $\Psi_0 \in \mathcal{H}$ be the state at time $t = 0$ of a certain quantum system. Then at time t the system is represented by the vector $\Psi(t) = \boldsymbol{U}_t \Psi_0$, where

$$\boldsymbol{U}_t = e^{-it\boldsymbol{H}},\ t \geq 0,$$

is a unitary operator called the evolution operator. The vector function $\Psi(t)$ is the solution Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(t) = \boldsymbol{H}\Psi(t).$$

The Dirac-Von Neumann formulation of QM

- The choice $\Psi(t) \in \mathcal{H} = L^2(\mathbb{R})$ implies that the space is continuous, i.e., given two different points x_0, $x_1 \in \mathbb{R}$ there exists a continuous curve $X(t) : [0,1] \to \mathbb{R}$ such that $X(0) = x_0$, $X(1) = x_1$.
- p-Adic QM is quantum mechanics in the Dirac-von Neumann, with $\mathcal{H} = L^2(\mathbb{Z}_p)$, $\mathcal{H} = L^2(\mathbb{Q}_p)$.
- The Dirac-von Neumann formulation of QM does not rule out the possibility of choosing a 'discrete space,' i.e., we can take $\mathcal{H} = L^2(\mathbb{Z}_p)$; in this case the space $\mathbb{Z}_p$ is a completely disconnected topological space.
- Any continuous function (curve) from $\mathbb{R}$ into $\mathbb{Z}_p$ is constant. Any continuous function maps connected subsets into connected subsets. Then given two points $a, b \in \mathbb{Z}_p$, with $a \neq b$, there is no a continuous curve $X(t) : [0,1] \to \mathbb{Z}_p$ such that $X(0) = a$ and $X(1) = b$.

p-adic QM

- This implies that the word line notion, which is a fundamental pillar in the formulation of special and general relativity, does not exist if we assume, as a model of physical space, a totally disconnected space.

- Consequently, the *p*-adic QM is incompatible with the special and general relativity.

- *p*-Adic QM is a model of the standard QM assuming the discreteness of the space.
- The testability theories like *p*-adic QM, string theory, and quantum gravity, that work at the Planck scale require accessing incredibly high energy levels. So, the physical content of such theories is in question.

p-adic QM

- Recently, we show that *p*-adic Schrödinger equations are related to continuous-time quantum walks (CTQWs) on graphs. This directly connects *p*-adic QM and quantum computing; such a connection does not require accessing incredibly high energy levels.
- This talk aims to show that certain 2-adic Schrödinger equations describe continuous versions of Farhi-Gutmann CTQWs on arbitrary graphs

Quantum nonlocality

We select an evolution operator $e^{\tau \boldsymbol{H}_0}$, $\tau \geq 0$, so that it is a Feller semigroup. Then $u(x,t) = e^{\tau \boldsymbol{H}_0} u_0(x)$ is the solution of the evolution equation (a p-adic heat equation) of the form

$$\frac{\partial}{\partial \tau} u\left(x,\tau\right) = \boldsymbol{H}_0 u\left(x,\tau\right),\ x \in \mathbb{Q}_p,\ \tau \geq 0, \tag{2}$$

with initial datum $u\left(x,0\right) = u_0\left(x\right)$. The Feller condition implies the existence of Markov process in $\mathbb{Q}_p$, with discontinuous paths, attached to equation (2).

Quantum nonlocality

We now apply the Wick rotation $\tau = it$, $t \geq 0$, with $i = \sqrt{-1}$, and $\Psi(x, t) = u(x, it)$, to (2) to obtain the free, p-adic Schrödinger equation:

$$i \frac{\partial}{\partial t} \Psi(x, t) = -\boldsymbol{H}_0 \Psi(x, t), \; x \in \mathbb{Q}_p, \; t \geq 0.$$

It is relevant to mention that all known operators $\boldsymbol{H}_0$ appearing in the p-adic heat equations are **non-local**.

Quantum nonlocality

The simplest choice for $\boldsymbol{H}_0$ is $\boldsymbol{D}^{\alpha}$, $\alpha > 0$, the Taibleson-Vladimirov fractional,

$$\boldsymbol{D}^{\alpha}\varphi(x) = \frac{1-p^{\alpha}}{1-p^{-\alpha-1}} \int_{\mathbb{Q}_p} \frac{\varphi(z)-\varphi(x)}{|z-x|_p^{\alpha+1}}\, dz,$$

for φ a locally constant function with compact support. To see the non-local nature of this operator, we take $\varphi(x) = 1$ if $|x|_p \leq 1$, otherwise $\varphi(x) = 0$, then

$$\boldsymbol{D}^{\alpha}\varphi(x) = \begin{cases} -\frac{1-p^{\alpha}}{1-p^{-\alpha-1}} \left(\int\limits_{|z|_p>1} \frac{dz}{|z|_p^{\alpha+1}} \right) & \text{if} \quad |x|_p \leq 1 \\ \frac{1-p^{\alpha}}{1-p^{-\alpha-1}} \frac{1}{|x|_p^{\alpha+1}} & \text{if} \quad |x|_p > 1. \end{cases}$$

Quantum nonlocality

- By definition, p-adic QM is a nonlocal theory. Hence, the violation of Bell's inequality (i.e., the paradigm: the universe is not locally real) does not cause any trouble in p-adic QM.
- The mentioned paradigm causes serious trouble for standard QM since standard QM is supposed to be a local theory, and abandoning the idea that objects have definite properties independent of observation seems to have profound epistemological consequences.
- **Do p-adic Schrodinger equations describe physical systems?**

The double-slit experiment

In *Zúñiga-Galindo W. A., The p-Adic Schrödinger equation and the two-slit experiment in quantum mechanics. Ann. Physics 469 (2024), Paper No. 169747*,

a p-adic model of the double-slit experiment was studied; in this model, each particle goes through one slit only. A similar description of the two-slit experiment was given in

Aharonov Y., Cohen E., Colombo F., Landsberger T., Sabadini I., Struppa D., and Tollaksen J., Finally making sense of the double-slit experiment. Proc. Natl. Acad. Sci. U. S. A. 114, 6480 (2017):

"Instead of a quantum wave passing through both slits, we have a localized particle with nonlocal interactions with the other slit. " in our paper, the same conclusion was obtained, but in the p-adic framework, the nonlocal interactions are a consequence of the discreteness of the space $\mathbb{Q}_p^3$.

Breaking of the Lorentz symmetry and the violation of Einstein causality

- Taking $\mathbb{R} \times \mathbb{Q}_p^3$ as a space-time model, in p-adic QM, the Lorentz symmetry is broken, since the time and position are not interchangeable.

- In the last thirty-five years, the experimental and theoretical studies of the Lorentz breaking symmetry have been an area of intense research.

Breaking of the Lorentz symmetry and the violation of Einstein causality

In

Zúñiga-Galindo W. A., p-adic quantum mechanics, the Dirac equation, and the violation of Einstein causality. J. Phys. A 57 (2024), no. 30, Paper No. 305301, 29 pp.,

we introduced a p-adic Dirac equation that shares many properties with the standard one. In particular, the new equation also predicts the existence of pairs of particles and antiparticles and a charge conjugation symmetry.

Breaking of the Lorentz symmetry and the violation of Einstein causality

- The p-adic Dirac equation admits space-localized planes waves $\Psi_{\boldsymbol{rnj}}(t, \boldsymbol{x})$ for any time $t \geq 0$, which is, $\operatorname{supp} \Psi_{\boldsymbol{rnj}}(t, \cdot)$ is contained in a compact subset of $\mathbb{Q}_p^3$. This phenomenon does not occur in the standard case.

- We compute the transition probability from a localized state at time $t = 0$ to another localized state at $t > 0$, assuming that the space supports of the states are arbitrarily far away.
- It turns out that this transition probability is greater than zero for any time $t \in (0, \epsilon)$, for arbitrarily small ϵ.
- Since this probability is nonzero for some arbitrarily small t, the system has a nonzero probability of getting between the mentioned localized states arbitrarily shortly, thereby propagating with superluminal speed in $\mathbb{R} \times \mathbb{Q}_p^3$.

Quantum nonlocality and faster-than-light communication

- In 1988, Eberhard and Ross, **using** $\mathbb{R} \times \mathbb{R}^3$ as a **space-time model**, showed that the relativistic quantum field theory inherently forbids faster-than-light communication.
- This result is known as the no-communication theorem. It preserves the principle of causality in quantum mechanics and ensures that information transfer does not violate special relativity by exceeding the speed of light.
- **So, if the space is not discrete at the Planck length, then faster-than-light communication is impossible.**

Quantum nonlocality and faster-than-light communication

- The no-communication theorem **does not rule out the possible superluminal speed in** $\mathbb{R} \times \mathbb{Q}_p^3$.
- We have a theory on the space-time $\mathbb{R} \times \mathbb{R}^3$, and want a copy of it on the space-time $\mathbb{R} \times \mathbb{Q}_p^3$. This is possible if there exists

$$\mathbb{Q}_p \hookrightarrow \mathbb{R} \text{ preserving the algebraic, topological and analytic properties of } \mathbb{Q}_p.$$

Such an arrow does not exist!

- **The no-communication theorem under the hypothesis that space is completely disconnected is an open problem**.

2-Adic Schrödinger equations and quantum networks

- In p-adic QM, the Schrödinger equations are obtained from p-adic heat equations by performing a Wick rotation.
- These equations are associated with Markov processes, which are generalizations of the random motion of a particle in a fractal, such as $\mathbb{Z}_p$ or $\mathbb{Q}_p$.
- In *Zúñiga-Galindo, W. A., Ultrametric diffusion, rugged energy landscapes and transition networks. Phys. A 597 (2022), Paper No. 127221, 19 pp.*, we introduce a new type of stochastic networks, which are p-adic continuous analogs of the standard Markov state models constructed using master equations.

2-Adic Heat equations and ultrametric networks

The evolution equation

$$\frac{du\left(x,\tau\right)}{d\tau}=\int\limits_{\mathcal{K}}\left\{j(x\mid y)u(y,\tau)-j(y\mid x)u(x,\tau)\right\}dy,\ \tau\geq 0,x\in\mathcal{K}, \quad (3)$$

is a 2-adic heat equation: there exists a probability measure $p_{\tau}\left(x,\cdot\right)$, $t\in[0,T]$, with $T=T(u_0)$, $x\in\mathcal{K}$, on the Borel σ-algebra of $\mathcal{K}$, such that the IVP:

$$\begin{cases} u\left(\cdot,\tau\right)\in\mathcal{C}^{1}\left(\left[0,T\right],\mathcal{C}\left(\mathcal{K},\mathbb{R}\right)\right); \\ \frac{du(x,\tau)}{d\tau}=\int\limits_{\mathcal{K}}\left\{j(x\mid y)u(y,\tau)-j(y\mid x)u(x,\tau)\right\}dy, \quad \tau\in\left[0,T\right],x\in\mathcal{K}; \\ u\left(x,0\right)=u_0\left(x\right)\in\mathcal{C}\left(\mathcal{K},\mathbb{R}_{+}\right). \end{cases}$$

2-Adic Heat equations and ultrametric networks

has a unique solution of the form

$$u(x,\tau)=\int\limits_{\mathcal{K}} u_0(y)p_\tau\left(x,dy\right).$$

In addition, $p_\tau\left(x,\cdot\right)$ is the transition function of a Markov process $\mathfrak{X}$ whose paths are right continuous and have no discontinuities other than jumps.

2-Adic Schrödinger equations coming from master equations

We now perform a Wick rotation ($\tau = it$, $t \geq 0$, with $i = \sqrt{-1}$, and $\Psi(x, t) = u(x, it)$) in (3) to obtain a Schrödinger equation.

It is more convenient to change the notation. We set $A(x, y) = j(x \mid y)$, $B(x, y) = j(y \mid x)$, where $A(x, y)$, $B(x, y)$ are non-negative, continuous, symmetric functions ($A(x, y) = A(y, x)$, $B(x, y) = B(y, x)$).

With this notation, Schrödinger equation takes the form

$$i \frac{\partial}{\partial t} \Psi(x, t) = -\int_{\mathcal{K}} \{A(x, y)\Psi(y, t) - B(x, y)\Psi(x, t)\} \, dy$$

for $t \geq 0, x \in \mathcal{K}$.

2-Adic Schrödinger equations coming from master equations

The operator

$$\begin{aligned}\Psi(x,t) &\rightarrow -\int_{\mathcal{K}} \{A(x,y)\Psi(y,t) - B(x,y)\Psi(x,t)\}\,dy \\ &= : \boldsymbol{H}\Psi(x,t),\end{aligned}$$

for $t \geq 0$, is self-adjoint on $L^2(\mathcal{K})$.

2-Adic Schrödinger equations coming from master equations

Now since $\boldsymbol{H}$ is self-adjoint on $L^2(\mathcal{K})$, by Stone's theorem on one-parameter unitary groups, there exists a one-paremeter family of unitary operators $\left\{e^{-it\boldsymbol{H}}\right\}_{t\geq 0}$, such that $\Psi\left(x,t\right)=e^{-it\boldsymbol{H}}\Psi_0\left(x\right)$ is the unique solution of the Cauchy problem

$$\begin{cases} \Psi\left(\cdot,t\right)\in L^2(\mathcal{K}),\ t\geq 0;\ \Psi\left(x,\cdot\right)\in\mathcal{C}^1(\mathbb{R}_+),\ x\in\mathcal{K} \\ \\ i\frac{\partial}{\partial t}\Psi\left(x,t\right)=\boldsymbol{H}\Psi\left(x,t\right),\ x\in\mathcal{K},t\geq 0 \\ \\ \Psi\left(x,0\right)=\Psi_0\left(x\right)\in L^2(\mathcal{K}). \end{cases} \tag{4}$$

Construction of CTQWs

We now take $\mathcal{K} = \bigsqcup_{I \in G_I^0} \left(I + 2^l \mathbb{Z}_2 \right)$, where G_I^0 is a finite subset of $\mathbb{Z}_2$, $\Psi_I(x) := 2^{\frac{l}{2}} \Omega\left(2^l |x - I|_2\right)$, with $\Omega\left(2^l |x - I|_2\right)$ denoting the characteristic function of the ball $I + 2^l \mathbb{Z}_2$, and $\Psi(x,t) = e^{-it\boldsymbol{H}} \Psi_I(x)$ as before. Notice that

$$1 = \|\Psi_I(x)\|_2 = \|\Psi(x,t)\|_2 = \sqrt{\int_{\mathcal{K}} |\Psi(x,t)|^2 \, dx};$$

then, by Born's rule,

$$\int_B |\Psi(x,t)|^2 \, dx$$

gives the probability of finding the system in a state supported in $B \subset \mathcal{K}$ (a Borel subset) given that at time zero the state of the system was given by $\Psi_I(x)$.

Construction of CTQWs I

Therefore,

$$\widetilde{\pi}_{J,I}(t) = \int_{J+2^{l}\mathbb{Z}_2} \left|\Psi(x,t)\right|^2 dx \tag{5}$$

is a transition probability between a state supported in the ball $I + 2^{l}\mathbb{Z}_2$ to a state supported in the ball $J + 2^{l}\mathbb{Z}_2$ at the time t. Notice that

$$\sum_{J\in G_l^0} \widetilde{\pi}_{J,I}(t) = 1. \tag{6}$$

Then, if we identify the ball $I + 2^{l}\mathbb{Z}_2$ with vertex $I \in G_l^0$ of a complete graph, the matrix $\left[\widetilde{\pi}_{J,I}(t)\right]$ defines a quantum Markov chain on the graph (i.e. a CTQW).

Construction of CTQWs

This approach was introduced in
Zúñiga-Galindo W. A., Mayes Nathanniel P., p-Adic quantum mechanics, infinite potential wells, and continuous-time quantum walks. arXiv:2410.13048.

The drawback of this approach is that it requires the solution of Cauchy problem (4), and that the constructed CTQWs are exclusively defined on complete graphs.
In

Zúñiga-Galindo W. A., 2-Adic quantum mechanics, continuous-time quantum walks, and the space discreteness,arXiv:2502.16416,

we provide a different approach to the construction of CTQWs based on the discretization of (4)

Quantum networks (Standard Construction)

From now on, $\mathcal{H}$ denotes the Hilbert space $\mathbb{C}^{2^l}$, with norm $\|\cdot\|$, and canonical basis as $\{|e_I\rangle\}_{I\in G_l}$.
We assume that $\boldsymbol{H}^{(l)}$ is a Hermitian matrix so $\exp(-it\boldsymbol{H}^{(l)})$ is unitary matrix. We identify G_l with an graph with vertices $I \in G_l$.
We define the transition probability $\pi_{I,J}(t)$ from J to I as

$$\pi_{I,J}(t) = \left|\langle e_I|\, e^{-it\boldsymbol{H}^{(l)}}\, |e_J\rangle\right|^2, \text{ for } J, I \in G_l.$$

Note that

$$\sum_{I\in G_l} \pi_{I,J}(t) =_{I\in G_l} \left|\langle e_I|\, e^{-it\boldsymbol{H}^{(l)}}\, |e_J\rangle\right|^2 = 1.$$

The continuous-time Markov chain on G_l determined by the transition probabilities $[\pi_{I,J}(t)]_{I,J\in G_l}$, is the quantum network associated with the discrete 2-adic Schrödinger equation. This construction works if we replace G_l with a subset G_l^0 of it.

CTQWs on graphs

The CTQWs on graphs play a central role in quantum computing. We show that this type of CTQWs can be obtained from a suitable 2-adic Schrödinger equation. Let $\mathcal{G}$ be an undirected, finite graph with vertices $I \in G_l^0 \subset G_l$, and adjacency matrix $[A_{JI}]_{J,I\in G_l^0}$, with

$$A_{JI} := \begin{cases} 1 & \text{if the vertices } J \text{ and } I \text{ are connected} \\ \\ 0 & \text{otherwise.} \end{cases}$$

We fix and l such that $\#G_l^0 \leq 2^l$, and set

$$\mathcal{K} = \mathcal{K}_l := \bigsqcup_{I\in G_l^0} \left(I + 2^l \mathbb{Z}_2 \right), \tag{7}$$

which is an open compact subset of $\mathbb{Z}_2$.

CTQWs on graphs

We also define

$$J^{(l)}(x,y)=2^{l}\sum_{J\in G_{l}^{0}}\sum_{K\in G_{l}^{0}}A_{JK}\Omega\left(2^{l}\left|x-J\right|_{p}\right)\Omega\left(2^{l}\left|y-K\right|_{p}\right), \qquad (8)$$

$x,\ y\in\mathbb{Z}_{2}$, where $[A_{JI}]_{J,I\in G_{l}^{0}}$ is the adjacency matrix of graph $\mathcal{G}$. Notice that $J^{(l)}(x,y)$ is a real-valued test function on $\mathcal{K}_{l}\times\mathcal{K}_{l}$. We now introduce the linear operator

$$\boldsymbol{J}_{\mathcal{G}}\varphi\left(x\right):=\int_{\mathcal{K}_{l}}\left\{\varphi\left(y\right)-\varphi\left(x\right)\right\}J^{(l)}(x,y)dy,\text{ for }\varphi\in C\left(\mathcal{K}_{l}\right).$$

This operator extends to linear bounded operator in $L^{2}(\mathcal{K}_{l})$.

CTQWs on graphs

The Shrödinger equation attached to operator

$$\boldsymbol{J}_{\mathcal{G}}\varphi(x)=\int_{\mathcal{K}_l}\{\varphi(y)-\varphi(x)\}\,J^{(l)}(x,y)dy$$

is

$$\begin{cases} i\frac{\partial}{\partial t}\Psi(x,t)=-m\boldsymbol{J}_{\mathcal{G}}\Psi(x,t), & x\in\mathcal{K}_l,\ t\geq 0 \\ \Psi(x,0)=\Psi_0(x)\in L^2(\mathcal{K}_l). \end{cases}$$

The discretization is obtained by computing the matrix of $\boldsymbol{J}_{\mathcal{G}}|_{\mathcal{X}_l}$ assuming that

$$\Psi^{(l)}(x,t)=\sum_{I\in G_l^0}\Psi_I^{(l)}(t)\,2^{\frac{l}{2}}\Omega\left(2^l\,|x-I|_2\right).$$

We identify $\Psi^{(l)}(x,t)$ with the column vector $\left[\Psi_I^{(l)}(t)\right]$.

CTQWs on graphs

We denote by $\mathcal{X}_I(\mathbb{Z}_2) \subset \mathcal{D}_I(\mathbb{Z}_2)$, the $\mathbb{C}$-vector space consisting of all the test functions supported in $\mathcal{K}_I$ having the form

$$\varphi(x) = \sum_{J \in G_I^0} \varphi_J 2^{\frac{l}{2}} \Omega\left(2^l |x - J|_2\right), \tag{9}$$

where $\varphi_J \in \mathbb{C}$.

$\boldsymbol{J}_{\mathcal{G}} : \mathcal{X}_I(\mathbb{Z}_2) \to \mathcal{X}_I(\mathbb{Z}_2)$ is a linear bounded operator satisfying $\|\boldsymbol{J}_{\mathcal{G}}\| \leq 2\gamma_{\mathcal{G}}$, where $\gamma_{\mathcal{G}} := \max_{I \in G_I^0} \gamma_I$, with $\gamma_I := \sum_{J \in G_I^0} A_{IJ}$.

Notice that $\gamma_I = \mathrm{val}(I)$, the valence of I, i.e., it is the number of connections from I to its other vertices.

CTQWs on graphs

The discretization is obtained by computing the matrix of $\boldsymbol{J}_{\mathcal{G}}|_{\mathcal{X}_l}$. We identify $\Psi^{(l)}(x,t)$ with the column vector $\left[\Psi_I^{(l)}(t)\right]$. The computation of the matrix of $\boldsymbol{J}_{\mathcal{G}}|_{\mathcal{X}_l}$:

$$\boldsymbol{J}_{\mathcal{G}}\left(2^{\frac{l}{2}}\Omega\left(2^l|x-I|_2\right)\right)=\sum_{J\in G_l^0}\{A_{JI}-\gamma_I\delta_{JI}\}\,2^{\frac{l}{2}}\Omega\left(2^l|x-J|_2\right),$$

where δ_{JI} is the Konecker delta.

CTQWs on graphs

We set

$$H^{(l)} = -mJ_{\mathcal{G}}^{(l)} = \left[H_{J,I}^{(l)} \right]_{J,I \in G_l^0}, \tag{10}$$

where

$$H_{J,I}^{(l)} = \begin{cases} -m & \text{if} \quad J \neq I \text{ and } A_{JI} = 1 \\ 0 & \text{if} \quad J \neq I \text{ and } A_{JI} = 0 \\ m\mathrm{val}(I) + V_I & \text{if} \quad J = I. \end{cases}$$

The discretization of the 2-adic Schrödinger equation takes the form

$$i \frac{\partial}{\partial t} \left[\Psi_I^{(l)}(t) \right] = H^{(l)} \left[\Psi_I^{(l)}(t) \right], \ t \geq 0. \tag{11}$$

The Farhi-Gutmann CTQWs

Let $\mathcal{G}$ be a finite graph. We take $G_I^0 = V(\mathcal{G})$, the set of vertices, and

$$\Psi(t) := \sum_{I \in V(\mathcal{G})} \Psi_I^{(I)}(t) \left| e_I \right\rangle = \sum_{I \in V(\mathcal{G})} \left\langle e_I \right| \Psi(t) \rangle \left| e_I \right\rangle ,$$

with

$$\left\| \Psi(t) \right\|^2 = \sum_{I \in V(\mathcal{G})} \left| \left\langle e_I \right| \Psi(t) \rangle \right|^2 = 1.$$

Now, we set $\left\langle e_I \right| \widehat{H} \left| e_K \right\rangle := H_{I,K}^{(I)}$. Then, equation (11) can be rewritten as

$$i \frac{\partial}{\partial t} \left\langle e_I \right| \Psi(t) \rangle = \sum_{K \in V(\mathcal{G})} \left\langle e_I \right| \widehat{H} \left| e_K \right\rangle \left\langle e_K \right| \Psi(t) \rangle , \qquad (12)$$

which is the Schrödinger equation for the Farhi-Gutmann CTQWs.
Farhi E., Gutmann S., Quantum computation and decision trees. Phys. Rev. A (3)58(1998), no.2, 915–928.

Questions?

Thank you!