Unification of Conformal and Fuzzy Gravities with Internal Interactions based on the $SO(10)$ GUT

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NONLINEARITY, NONLOCALITY AND ULTRAMETRICITY

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Preliminaries

Second order formulation (Einstein gravity):

- metric tensor $g_{\mu\nu}$
- curvature parametrized by Riemann tensor: $R^{\rho}_{\mu\nu\sigma} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$
- torsion: $T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 0$
- Christoffel Symbols: $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$
- action: $S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \rightarrow$ Einstein Field Equations

First order formulation:

- vierbein and spin connection ${e_\mu}^a, {\omega_\mu}^{ab}$
- curvature parametrized by the curvature 2-form: $R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} - \omega_{\mu ac}{\omega_\nu}^c{}_b - \omega_{\nu ac}{\omega_\mu}^c{}_b$
- torsion: ${T_{\mu\nu}}^a = \partial_\mu {e_\nu}^a - \partial_\nu {e_\mu}^a + {\omega_\mu}^a{}_b {e_\nu}^c - {\omega_\nu}^a{}_b {e_\mu}^c$
- action: $S = \frac{1}{16\pi G}\int \frac{1}{2}\epsilon_{abcd} e^a \wedge e^b \wedge R^{cd}$ (Palatini action)
- $\rightarrow$ Einstein Field Equations + Torsionless condition

Einstein 4d Gravity as a Gauge Theory

The algebra

- Employ the first order formulation of GR
- Gauge theory of Poincaré group ISO(1,3)
- Ten generators (Translations P_a & LT M_{ab})

*see for details:
Utiyama '56, Kibble '61,
Kaku-Townsend-
Nieu/zen '77,
McDowell-Mansuri '77,
Chamseddine-West '77,
Stelle-West, '80,
Ivanov-Niederle '82,
Kibble-Stelle '85,
Witten '88,
Wilczek '98, Ortin '04,
Roumelioti-Stefas-Z '24*

Generators satisfy the commutation relations:

$$[M_{ab}, M_{cd}] = \eta_{ac}M_{db} - \eta_{bc}M_{da} - \eta_{ad}M_{cb} + \eta_{bd}M_{ca}$$
$$[P_a, M_{bc}] = \eta_{ab}P_c - \eta_{ac}P_b\,, \qquad [P_a, P_b] = 0$$

where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$ and $a, b, c, d = 1, \ldots, 4$.

The gauging procedure

- Introduction of a gauge vector field for each generator. For the Poincaré group, $ISO(1,3)$:
  $\rightarrow$ 4 fields ${e_\mu}^a$ for the translation operators P_a
  $\rightarrow$ 6 fields ${\omega_\mu}^{ab}$ for the local $SO(1,3)$ (LT)
- The gauge connection is:

$$A_\mu(x) = {e_\mu}^a(x) P_a + \frac{1}{2}{\omega_\mu}^{ab}(x) M_{ab}$$

- Transforms in the adjoint rep, according to the rule:

$$\delta A_\mu = \partial_\mu \epsilon + [A_\mu, \epsilon]$$

- The gauge transformation parameter, $\epsilon(x)$ is expanded as:

$$\epsilon(x) = \xi^a(x) P_a + \frac{1}{2}\lambda^{ab}(x) M_{ab}$$

- *Combining* the above $\rightarrow$ transformations of the fields:

$$\delta {e_\mu}^a = \partial_\mu \xi^a - {e_\mu}^b {\lambda^a}_b + {\omega_\mu}^{ab}\xi_b$$
$$\delta {\omega_\mu}^{ab} = \partial_\mu \lambda^{ab} - {\lambda^a}_c {\omega_\mu}^{cb} + {\lambda^b}_c {\omega_\mu}^{ca}$$

- Gauge transf $\leftrightarrow$ diffeo transf (imposing cond. torsionless, on shell)

Curvature and Torsion

- Curvatures of the fields are given by:

$$R_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

- Tensor $R_{\mu\nu}$ is also valued in Poincaré algebra:

$$R_{\mu\nu}(A) = T_{\mu\nu}{}^a P_a + \tfrac{1}{2} R_{\mu\nu}{}^{ab} M_{ab}$$

- *Combining* the above $\rightarrow$ component tensor curvatures:

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + e_\mu{}^b \omega_{\nu b}{}^a - e_\nu{}^b \omega_{\mu b}{}^a$$
$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - \omega_\mu{}^{cb} \omega_\nu{}^a{}_c + \omega_\mu{}^{ac} \omega_{\nu c}{}^b$$

- Palatini action is considered
- Torsionless condition + Field equations

Gauge theory of SO(2,3)

- Instead of the Poincaré group - Anti-de Sitter group: SO(2,3)
- Same amount of generators BUT they can be written on equal footing (semisimple group):

$$[\hat{M}_{AB}, \hat{M}_{CD}] = \eta_{AC}\hat{M}_{DB} - \eta_{BC}\hat{M}_{DA} - \eta_{AD}\hat{M}_{CB} + \eta_{BD}\hat{M}_{CA}$$

- η_{AB} is the 5-dim Minkowski metric with two timelike coefficients (1st and 5th) and $A, \dots, D = 1 \dots 5$
- Perform a splitting of the indices $A = (a, 5)$
- Define $\hat{M}_{ab} = M_{ab}$ and $\hat{M}_{a5} = \frac{1}{m}P_a$, $[m] = L^{-1}$
- Gauge connection: $A_\mu = \frac{1}{2}\hat{\omega}_\mu{}^{AB}\hat{M}_{AB} = \frac{1}{2}\omega_\mu{}^{ab}M_{ab} + e_\mu{}^a P_a$
- where $\hat{\omega}_\mu{}^{ab} = \omega_\mu{}^{ab}$ and $\hat{\omega}_\mu{}^{a5} = m e_\mu{}^a$
- The same for the field strength tensor $\hat{R}_{\mu\nu}{}^{AB}$:

$$\hat{R}_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} + 2m^2 e_\mu{}^{[a} e_\mu{}^{b]}\,, \quad \hat{R}_{\mu\nu}{}^{a5} = m T_{\mu\nu}{}^a$$

- Consider the following $SO(2,3)$ invariant quadratic action:

$$S = a_{AdS} \int d^4x \Big(m y^E \epsilon_{ABCDE} \frac{1}{4} \hat{R}_{\mu\nu}{}^{AB} \hat{R}_{\rho\sigma}{}^{CD} \epsilon^{\mu\nu\rho\sigma} + \\ + \lambda \left(y^E y_E + m^{-2} \right) \Big)$$

- y^E an auxiliary scalar field in the vector rep
- vector taken to be gauge fixed towards the 5-th direction:

$$y = y^0 = \left(0, 0, 0, 0, m^{-1}\right).$$

- the non-vanishing value $y^5(x)$ is responsible for the symmetry breaking of $SO(2,3)$ to the $SO(1,3)$

$$S = \frac{a_{AdS}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \hat{R}_{\mu\nu}{}^{ab} \hat{R}_{\rho\sigma}{}^{cd} \epsilon_{abcd}$$
$$= \frac{a_{AdS}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} \left(\mathcal{L}_{RR} + m^2 \mathcal{L}_{eeR} + m^4 \mathcal{L}_{eeee} \right)$$

- $\mathcal{L}_{RR}$: Gauss-Bonnet - no contribution to the e.o.m.
- $\mathcal{L}_{eeR}$: Palatini action (torsionless + Einstein Field Equations)
- $\mathcal{L}_{eeee}$: Plays the role of cosmological constant
- Solution of Einstein Field Equations is the Anti-de Sitter space
- If $m \to 0$: Minkowski spacetime (flat solution).

Conformal 4d Gravity as a Gauge Theory

- Group parametrizing the symmetry: $SO(2,4)$
- 15 generators: 6 LT M_{ab}, 4 translations, P_a, 4 conformal boosts K_a and the dilatation D
- Group generators satisfy the following algebra:

$$
\begin{aligned}
[M_{ab}, M_{cd}] &= \eta_{bc} M_{ad} + \eta_{ad} M_{bc} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}, \\
[M_{ab}, P_c] &= \eta_{bc} P_a - \eta_{ac} P_b, \\
[M_{ab}, K_c] &= \eta_{bc} K_a - \eta_{ac} K_b, \\
[P_a, D] &= P_a, \\
[K_a, D] &= -K_a, \\
[K_a, P_b] &= -2(\eta_{ab} D + M_{ab}),
\end{aligned}
$$

- Following the same procedure one calculates transf of the gauge fields and tensors after defining the gauge connection
- Action is taken of $SO(2,4)$ invariant quadratic form
- Initial symmetry breaks under certain constraints resulting to the *Weyl action* *Kaku,Townsend,Nieu/zen '77,*

  Fradkin, Tseytlin '85
- Initial symmetry breaks spontaneously by introducing a scalar in the adjoint rep fixed in the dilatation direction, or by two scalars in vector reps.

  Roumelioti, Stefas, Z '24

SSB by using a scalar in the adjoint representation

Gauge connection:

$$A_\mu = \frac{1}{2}\omega_\mu{}^{ab}M_{ab} + e_\mu{}^a P_a + b_\mu{}^a K_a + \tilde{a}_\mu D,$$

Field strength tensor:

$$F_{\mu\nu} = \frac{1}{2}R_{\mu\nu}{}^{ab}M_{ab} + \tilde{R}_{\mu\nu}{}^a P_a + R_{\mu\nu}{}^a K_a + R_{\mu\nu}D,$$

where

$$\begin{aligned}
R_{\mu\nu}{}^{ab} &= \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\omega_{\mu c}{}^b - 8e_{[\mu}{}^{[a}b_{\nu]}{}^{b]} \\
&= R^{(0)ab}_{\mu\nu} - 8e_{[\mu}{}^a b_{\nu]}{}^{b]}, \\
\tilde{R}_{\mu\nu}{}^a &= \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^{ab}e_{\nu b} - \omega_\nu{}^{ab}e_{\mu b} - 2\tilde{a}_{[\mu}e_{\nu]}{}^a \\
&= T^{(0)a}_{\mu\nu} - 2\tilde{a}_{[\mu}e_{\nu]}{}^a, \\
R_{\mu\nu}{}^a &= \partial_\mu b_\nu{}^a - \partial_\nu b_\mu{}^a + \omega_\mu{}^{ab}b_{\nu b} - \omega_\nu{}^{ab}b_{\mu b} + 2\tilde{a}_{[\mu}b_{\nu]}{}^a \\
&= T^{(0)a}_{\mu\nu}(b) + 2\tilde{a}_{[\mu}b_{\nu]}{}^a, \\
R_{\mu\nu} &= \partial_\mu\tilde{a}_\nu - \partial_\nu\tilde{a}_\mu + 4e_{[\mu}{}^a b_{\nu]a},
\end{aligned}$$

We start with the parity conserving action, which is quadratic in terms of the field strength tensor and introduce a scalar in the rep 15

$$S_{SO(2,4)} = a_{CG} \int d^4x \left[\mathrm{tr}\, \epsilon^{\mu\nu\rho\sigma} m\phi F_{\mu\nu} F_{\rho\sigma} + \left(\phi^2 - m^{-2} \mathbb{1}_4 \right) \right],$$

The scalar expanded on the generators is:

$$\phi = \phi^{ab} M_{ab} + \tilde{\phi}^a P_a + \phi^a K_a + \tilde{\phi} D,$$

We pick the specific gauge in which ϕ is diagonal of the form $diag(1, 1, -1, -1)$. Specifically we choose ϕ to be only in the direction of the dilatation generator D:

$$\phi = \phi^0 = \tilde{\phi} D \xrightarrow{\phi^2 = m^{-2} \mathbb{1}_4} \phi = -2m^{-1} D.$$

The resulting broken action is (after employing anticommutator relations and the traces over the generators):

$$S_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd}$$

The $\tilde{a}_\mu$ is not present in the action, so we can set it equal to zero.

$R_{\mu\nu}$ is also absent so we can also set it equal to zero

$$R_{\mu\nu} = \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu + 4 e_{[\mu}{}^a b_{\nu]a} = 0 \xrightarrow{\tilde{a}_\mu = 0}$$

$$e_\mu{}^a b_{\nu a} - e_\nu{}^a b_{\mu a} = 0$$

We examine two possible solutions of the above equation:

- $b_\mu{}^a = a e_\mu{}^a$, *Chamseddine '03*
- $b_\mu{}^a = -\frac{1}{4}\left(R_\mu{}^a + \frac{1}{6} R e_\mu{}^a\right)$ *Kaku, Townsend, Nieu/zen, '78*

Freedman, Van Proyen 'Supergravity' '12

The first choice leads to the Einstein-Hilbert action, while the second leads to Weyl action.

→ Similar results are obtained using two scalars in the vector rep.

Einstein-Hilbert action

- When $b_\mu{}^a = a e_\mu{}^a$, the broken action becomes:

$$S_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4}\int d^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd} \implies$$

$$S_{\mathrm{SO}(1,3)} = \frac{a_{CG}}{4}\int d^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\Big[R^{(0)ab}_{\mu\nu}R^{(0)cd}_{\rho\sigma} - 16m^2 a R^{(0)ab}_{\mu\nu}e_\rho{}^c e_\sigma{}^d + \\ + 64m^4a^2 e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d\Big]$$

This action consists of three terms: one G-B topological term, the E-H action, and a cosmological constant. For $a < 0$ describes GR in AdS space.

Weyl action

- When $b_\mu{}^a = -\frac{1}{4}(R_\mu{}^a + \frac{1}{6}Re_\mu{}^a)$, the broken action becomes

$$S = \frac{a_{CG}}{4}\int d^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\Big[R^{(0)ab}_{\mu\nu} - \frac{1}{2}\left(\tilde{e}_\mu{}^{[a}R_\nu{}^{b]} - \tilde{e}_\nu{}^{[a}R_\mu{}^{b]}\right) + + \frac{1}{3}R\tilde{e}_\mu{}^{[a}\tilde{e}_\nu{}^{b]}\Big] \Big[R^{(0)cd}_{\rho\sigma} - \frac{1}{2}\left(\tilde{e}_\rho{}^{[c}R_\sigma{}^{d]} - \tilde{e}_\sigma{}^{[c}R_\rho{}^{d]}\right) + + \frac{1}{3}R\tilde{e}_\rho{}^{[c}\tilde{e}_\sigma{}^{d]}\Big],$$

where $\tilde{e}_\mu{}^a = me_\mu{}^a$ is the rescaled vierbein. The above action is equal to

$$S = \frac{a_{CG}}{4}\int d^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}C_{\mu\nu}{}^{ab}C_{\rho\sigma}{}^{cd}$$

$$\Rightarrow \quad 2a_{CG}\int \mathrm{d}^4x\left(R_{\mu\nu}R^{\nu\mu} - \frac{1}{3}R^2\right),$$

where $C_{\mu\nu}{}^{ab}$ is the Weyl conformal tensor.

The NC framework & gauge theories

- Quantization of phase space of $x^i, p_j \rightarrow$ replaced by Herm operators: $\hat{x}^i, \hat{p}_j$ satisfying:$[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j$
- Noncommutative space $\rightarrow$ quantization of space: x^i $\rightarrow$ replace with operators $\hat{x}^i$ $(\in \mathcal{A})$ satisfying: $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(\hat{x})$

Connes '94, Madore '99

- Antisymmetric tensor $\theta^{ij}(\hat{x})$ - defines the NC of the space
  - Canonical case: $\theta^{ij}(\hat{x}) = \theta^{ij}, i, j = 1, \ldots, N$

    For $N = 2 \rightarrow$ *Moyal plane*
  - Lie-type case: $\theta^{ij}(\hat{x}) = C^{ij}{}_k \hat{x}^k, i, j = 1, \ldots, N$

    For $N = 3 \rightarrow$ Noncommutative *(fuzzy) sphere* (SU(2))
- NC framework admits a matrix representation (operators)
  - Derivation: $e_i(A) = [d_i, A]$, $d_i \in \mathcal{A}$
  - Integration $\rightarrow$ Trace

For Reviews:
Szabo '01, Douglas-Nekrasov '01

The NC Gauge Fields & transformations

Madore-Wess et al. '00

Consider a field $\phi(X_a)$ on a fuzzy space described by NC coordinates X_a. An infinitesimal gauge transformation

$$\delta\phi(X_a) = \lambda(X_a)\phi(X_a)\,,$$

where $\lambda(X_a)$ is a gauge transf parameter:

- $U(1)$ if $\lambda(X_a)$ is antihermitian function of X_a
- $U(P)$ if $\lambda(X_a)$ is valued in Lie algebra of $P \times P$ matrices

Coordinates $\phi(X_a)$ are invariant under gauge transformation, i.e. $\delta(X_a) = 0$. Therefore:

- $\delta(X_a\phi) = X_a\lambda(X_a)\phi \neq \lambda(X_a)X_a\phi$
- $\delta(\mathcal{X}_a\phi) = \lambda(X_a)\delta\phi_a\phi$,
  which holds if: $\delta(\mathcal{X}_a) = [\lambda(X_a), \phi_a]$
- where $\mathcal{X}_a = X_a + A_a$ the covariant coordinate $\rightarrow$ NC analogue of cov. der. and A_a are interpreted as gauge fields

The NC Gauge Fields & transformations (2)

Note that the transformation of A_a is:

$$\delta A_a = -[X_a, \lambda] + [\lambda, A_a],$$

supporting the interpretation of A_a as gauge field.

Correspondingly, define:

$$\begin{aligned} F_{ab} =& [X_a, A_b] - [X_b, A_a] + [A_a, A_b] = -C^c{}_{ab} A_c \\ =& [\phi_a, \phi_b] - C^c{}_{ab} \phi_c, \end{aligned}$$

an analogue of the field strength tensor whose transformation is given by:

$$\delta F_{ab} = [\lambda, F_{ab}]$$

Non-Abelian case

▷ *In nonabelian case, where are the gauge fields valued?*

- Let us consider the CR of two elements of an algebra:

$$[\epsilon, A] = [\epsilon^A T^A, A^B T^B] = \frac{1}{2}\{\epsilon^A, A^B\}[T^A, T^B] + \frac{1}{2}[\epsilon^A, A^B]\{T^A, T^B\}$$

- *Not* possible to restrict to a matrix algebra:
  last term neither *vanishes* in NC nor is an *algebra element*

- There are two options to overpass the difficulty:

Ćirić-Gočanin-Konjik-Radovanović '18

  - Consider the universal enveloping algebra
  - Extend the generators and/or fix the rep so that the anticommutators close

▷ *We employ the second option*

The 4d covariant noncommutative space

Motivation for a 4d covariant NC space

- Constructing field theories on NC spaces is non-trivial: NC deformations break Lorentz invariance
- such an example is the fuzzy sphere (2d space) - coords are identified as rescaled SU(2) generators *Madore '92*
  Hammou-Lagraa-Sheikh Jabbari '02
  Vitale-Wallet '13, Vitale '14
  Jurman-Steinacker '14
  Chatzistavrakidis-Jonke-Jurman-Manolakos-Manousselis-GZ '18
- Previous work on 3d NC gravity on the covariant spaces $R^3_\lambda (R^{1,2}_\lambda)$
- Need of 4d covariant NC space to construct a gravity gauge theory

Construction of the 4d covariant NC space

- dS_4: homogeneous spacetime with constant curvature (positive)
- Described by the embedding $\eta^{AB} X_A X_B = R^2$ into M_5
- Aim for a NC version of dS_4

- Introduce a natural minimal length
- Assign the spacetime coordinates to elements of the 4-d dS group, $SO(1,4)$

- The SO(1,4) generators, $J_{mn}, m, n = 0, \ldots, 4,$ satisfy the commutation relation:

$$[J_{mn}, J_{rs}] = i(\eta_{mr}J_{ns} + \eta_{ns}J_{mr} - \eta_{nr}J_{ms} - \eta_{ms}J_{nr})$$

- Consider decomposition of $SO(1,4)$ to max subgroup, $SO(1,3)$
- Convert the generators to physical quantities by setting $\Theta_{ij} = \hbar J_{ij}\,, X_i = \lambda J_{i4}\,;\, \lambda$ a length parameter
- Thus, the commutation relations regarding the operators $\Theta_{\mu\nu}$ and X_μ are:

$$[\Theta_{ij}, \Theta_{kl}] = i\hbar\left(\eta_{ik}\Theta_{jl} + \eta_{jl}\Theta_{ik} - \eta_{jk}\Theta_{il} - \eta_{il}\Theta_{jk}\right),$$
$$[\Theta_{ij}, X_k] = i\hbar\left(\eta_{ik}X_j - \eta_{jk}X_i\right),$$
$$[X_i, X_j] = \frac{i\lambda^2}{\hbar}\Theta_{ij}$$

- The noncommutativity of coordinates becomes manifest

Yang's Model '47

- Extending covariance to include also momenta generators $\rightarrow$ use a group with larger symmetry $\rightarrow$ min extension: $SO(1,5)$

Yang '47
Kimura '02, Heckman-Verlinde '15
Steinacker '16
Sperling-Steinacker '17,'19
Burić-Madore '14,'15
Manolakos-Manousselis-GZ '19,'21

- The SO(1,5) generators, $J_{MN}, M, N = 0, \ldots, 5$, satisfy the commutation relation:

$$[J_{MN}, J_{P\Sigma}] = i(\eta_{MP}J_{N\Sigma} + \eta_{N\Sigma}J_{MP} - \eta_{NP}J_{M\Sigma} - \eta_{M\Sigma}J_{NP})$$

- Employ a *2-step* decomposition $SO(1,5) \supset SO(1,4) \supset SO(1,3)$

Yang's Model '47 (Continued)

- Convert the generators to physical quantities by identifying $\Theta_{ij} = \hbar J_{ij}, \;\; X_i = \lambda J_{i5}, \;\; P_i = \frac{\hbar}{\lambda} J_{i4}, \;\; h = J_{45}$
- Thus, the commutation relations regarding all the operators $\Theta_{\mu\nu}, X_\mu, P_\mu, h$ are:

$$
\begin{aligned}
[\Theta_{\mu\nu}, \Theta_{\rho\sigma}] &= i\hbar(\eta_{\mu\rho}\Theta_{\nu\sigma} + \eta_{\nu\sigma}\Theta_{\mu\rho} - \eta_{\nu\rho}\Theta_{\mu\sigma} - \eta_{\mu\sigma}\Theta_{\nu\rho})\,, \\
[\Theta_{\mu\nu}, X_\rho] &= i\hbar(\eta_{\mu\rho}X_\nu - \eta_{\nu\rho}X_\mu) \\
[\Theta_{\mu\nu}, P_\rho] &= i\hbar(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
[P_\mu, P_\nu] &= i\frac{\hbar}{\lambda^2}\Theta_{\mu\nu}\,, \qquad [X_\mu, X_\nu] = i\frac{\lambda^2}{\hbar}\Theta_{\mu\nu}\,, \\
[P_\mu, h] &= -i\frac{\hbar}{\lambda^2}X_\mu\,, \qquad [X_\mu, h] = i\frac{\lambda^2}{\hbar}P_\mu\,, \\
[P_\mu, X_\nu] &= i\hbar\eta_{\mu\nu}h\,, \qquad [\Theta_{\mu\nu}, h] = 0
\end{aligned}
$$

- The above relations describe the noncommutative space

Noncommutative gauge theory of 4d gravity

- Formulation of gravity on the above space
- Noncommutative gauge theory construction + the procedure described in the Einstein gravity case

Kimura '02, Heckman-Verlinde '15

- Gauge the isometry group of the space, $\mathrm{SO}(1,4)$ as seen as a subgroup of the $\mathrm{SO}(1,5)$ we ended up
- Anticommutators do not close $\rightarrow$ enlargement of the algebra + fix the representation

Aschieri-Castellani '09

Chatzistavrakidis-Jonke-Jurman-Manolakos-Manousselis-Z '18

- Noncommutative gauge theory of $\mathrm{SO}(2,4) \times \mathrm{U}(1)$

Manolakos-Manousselis-Z '19, '21

Roumelioti-Stefas-Z '24

- The generators of the group are represented by combinations of the 4×4 gamma matrices
- Specifically, the generators are expressed by:
  - six Lorentz rotation generators: $M_{ab} = -\frac{i}{4}\left[\gamma_a, \gamma_b\right]$
  - four generators for conformal boosts: $K_a = \frac{1}{2}\gamma_a(1 + \gamma_5)$
  - four generators for translations: $P_a = -\frac{1}{2}\gamma_a(1 - \gamma_5)$
  - one generator for special conformal transformations: $\mathrm{D} = -\frac{1}{2}\gamma_5$
  - one U(1) generator: $\mathbb{1}$
- The above expressions of the generators allow the calculation of the algebra they satisfy:

$$
\begin{aligned}
&[M_{ab}, M_{cd}] = \eta_{bc}M_{ad} + \eta_{ad}M_{bc} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}, \\
&[K_a, P_b] = -2\left(\eta_{ab}D + M_{ab}\right),\ [P_a, D] = P_a,\ [K_a, D] = -K_a, \\
&[M_{ab}, K_c] = \eta_{bc}K_a - \eta_{ac}K_b,\ [M_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b
\end{aligned}
$$

- Generators satisfy the following anticommutation relations:

Smolin '03

$$
\begin{aligned}
\{M_{ab}, M_{cd}\} &= \frac{1}{2}\left(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad}\right) - i\epsilon_{abcd}D, \\
\{M_{ab}, P_c\} &= +i\epsilon_{abcd}P^d, \\
\{M_{ab}, K_c\} &= -i\epsilon_{abcd}K^d, \\
\{M_{ab}, D\} &= 2M_{ab}D, \\
\{P_a, K_b\} &= 4M_{ab}D + \eta_{ab}, \\
\{K_a, K_b\} &= \{P_a, P_b\} = -\eta_{ab}, \\
\{P_a, D\} &= \{K_a, D\} = 0.
\end{aligned}
$$

- We will introduce gauge fields in a motivated way
- Use the general treatment of NC gauge theories

NC gauge theory

Manolakos-Manousselis-Z '21

- Since the gauge group is determined to be $SO(2,4) \times U(1)$, we can move on with the gauging procedure.
- Consider the *covariant coordinate* $\mathcal{X}_\mu = X_\mu + A_\mu$
- Determine appropriate *covariant field strength tensor* $\mathcal{R}_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - i\frac{\lambda^2}{\hbar}\hat{\Theta}_{\mu\nu}$,
  where $\hat{\Theta}_{\mu\nu} = \Theta_{\mu\nu} + \mathcal{B}_{\mu\nu}$, the *covariant noncommutative tensor*
- For the SSB to take place we:
  - Introduce scalar field $\Phi(X)$ belonging in the 2nd rank antisym. of $SO(4)$, *charged* under $U(1) \rightarrow U(1)$ breaks and doesn't appear in final action
  - Gauge fix $\Phi(X)$ in the direction that leads to Lorentz group

Gauge connection and field strength tensor decompose as:

$$A_\mu(X) = e_\mu{}^a \otimes P_a + \omega_\mu{}^{ab} \otimes M_{ab} + b_\mu{}^a \otimes K_a + \tilde{a}_\mu \otimes D + a_\mu \otimes \mathbf{I}_4 \,.$$

$$\mathcal{R}_{\mu\nu}(X) = \tilde{R}_{\mu\nu}{}^a \otimes P_a + R_{\mu\nu}{}^{ab} \otimes M_{ab} + R_{\mu\nu}{}^a \otimes K_a + \tilde{R}_{\mu\nu} \otimes D + R_{\mu\nu} \otimes \mathbf{I}_4 \,.$$

The component curvatures:

$$R_{\mu\nu} = [X_\mu, a_\nu] - [X_\nu, a_\mu] + [a_\mu, a_\nu] + [b_\mu{}^a, b_{\nu a}] + [\tilde{a}_\mu, \tilde{a}_\nu] + \frac{1}{2}[\omega_\mu{}^{ab}, \omega_{\nu ab}]$$
$$+ [e_{\mu a}, e_\nu{}^a] - \frac{i\hbar}{\lambda^2} B_{\mu\nu}$$

$$\tilde{R}_{\mu\nu} = [X_\mu, \tilde{a}_\nu] + [a_\mu, \tilde{a}_\nu] - [X_\nu, \tilde{a}_\mu] - [a_\nu, \tilde{a}_\mu] - i\{b_{\mu a}, e_\nu{}^a\} + i\{b_{\nu a}, e_\mu{}^a\}$$
$$+ \frac{1}{2}\epsilon_{abcd}[\omega_\mu{}^{ab}, \omega_\nu{}^{cd}] - \frac{i\hbar}{\lambda^2}\tilde{B}_{\mu\nu}$$

$$R_{\mu\nu}{}^a = [X_\mu, b_\nu{}^a] + [a_\mu, b_\nu{}^a] - [X_\nu, b_\mu{}^a] - [a_\nu, b_\mu{}^a] + i\{b_{\mu b}, \omega_\mu{}^{ab}\} - i\{b_{\nu b}, \omega_\mu{}^{ab}\}$$
$$+ i\{\tilde{a}_\mu, e_\nu{}^a\} - i\{\tilde{a}_\nu, e_\mu{}^a\} + \epsilon_{abcd}([e_\mu{}^b, \omega_\nu{}^{cd}] - [e_\nu{}^b, \omega_\mu{}^{cd}]) - \frac{i\hbar}{\lambda^2} B_{\mu\nu}{}^a$$

$$\tilde{R}_{\mu\nu}{}^a = [X_\mu, e_\nu{}^a] + [a_\mu, e_\nu{}^a] - [X_\nu, e_\mu{}^a] - [a_\nu, e_\mu{}^a] + i\{b_\mu{}^a, \tilde{a}_\nu\} - i\{b_\nu{}^a, \tilde{a}_\mu\}$$
$$- ([b_\mu{}^b, \omega_\nu{}^{cd}] - [b_\nu{}^b, \omega_\mu{}^{cd}])\epsilon_{abcd} - i\{\omega_\mu{}^{ab}, e_{\nu b}\} + i\{\omega_\nu{}^{ab}, e_{\mu b}\} - \frac{i\hbar}{\lambda^2}\tilde{B}_{\mu\nu}{}^a$$

$$R_{\mu\nu}{}^{ab} = [X_\mu, \omega_n{}^{ab}] + [a_\mu, \omega_\nu{}^{ab}] - [X_\nu, \omega_\mu{}^{ab}] - [a_\nu, \omega_m{}^{ab}] + 2i\{b_\mu{}^a, b_\nu{}^b\} + ([b_\mu{}^c, e_\nu{}^d]$$
$$- [b_\nu{}^c, e_\mu{}^d])\epsilon_{abcd} + \frac{1}{2}([\tilde{a}_\mu, \omega_\nu{}^{cd}] - [\tilde{a}_\nu, \omega_\mu{}^{cd}])\epsilon_{abcd} + 2i\{\omega_\mu{}^{ac}, \omega_\nu{}^b{}_c\}$$
$$+ 2i\{e_\mu{}^a, e_\nu{}^b\} - \frac{i\hbar}{\lambda^2} B_{\mu\nu}{}^{ab}$$

Symmetry breaking

Introduction of auxiliary field $\Phi(X)$ charged under $U(1)$:

$$\Phi = \tilde{\phi}^a \otimes P_a + \phi^{ab} \otimes M_{ab} + \phi^a \otimes K_a + \phi \otimes \mathbf{I}_4 + \tilde{\phi} \otimes D$$

into the action:

$$\mathcal{S} = \mathrm{Trtr}_G \, \lambda \Phi(X) \mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} + \eta(\Phi(X)^2 - \lambda^{-2} \mathbf{I}_N \otimes \mathbf{I}_4) \, ,$$

induces a symmetry breaking:

$$\boxed{\mathcal{S}_{br} = \mathrm{Tr} \left(\frac{\sqrt{2}}{4} \varepsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd} - 4 R_{\mu\nu} \tilde{R}_{\rho\sigma} \right) \varepsilon^{\mu\nu\rho\sigma}}$$

when the auxiliary field is gauge fixed as:

$$\Phi(X) = \tilde{\phi}(X) \otimes D|_{\tilde{\phi} = -2\lambda^{-1}} = -2\lambda^{-1} \mathbf{I}_N \otimes D$$

Residual symmetry: $SO(1,3) \times U(1)$

The constraints that correspond to the above breaking are:

Chamseddine '02

$R_{\mu\nu}{}^a = \frac{i}{2} \tilde{R}_{\mu\nu}{}^a = 0$ leading to $\tilde{a}_\mu = 0$, $b_\mu{}^a = \frac{i}{2} e_\mu{}^a$ and $B_{\mu\nu}{}^a = \frac{i}{2} \tilde{B}_{\mu\nu}{}^a$

The commutative limit

- The 2-form field, $\mathcal{B}_{\mu\nu}$ and a_μ decouple
- The commutators of functions vanish: $[f(x), g(x)] \to 0$
- The anticommutators of functions reduce to product: $\{f(x), g(x)\} \to 2f(x)g(x)$
- The inner derivation becomes: $[X_\mu, f] \to \partial_\mu f$
- Trace reduces to integration: $\frac{\sqrt{2}}{4}\mathrm{Tr} \to \int d^4x$
- We also regard the following reparametrizations:

- $e_\mu{}^a \to ime_\mu{}^a\,, \quad P_a \to -\frac{i}{m}P_a\,, \quad \tilde{R}_{\mu\nu}{}^a \to imT_{\mu\nu}{}^a$
- $\omega_\mu{}^{ab} \to -\frac{i}{2}\omega_\mu{}^{ab}\,, \quad M_{ab} \to 2iM_{ab}\,, \quad R_{\mu\nu}{}^{ab} \to -\frac{i}{2}R_{\mu\nu}{}^{ab}$

- When the commutative limit of the action is considered, it reduces to the Palatini action, which is equivalent to EG, with a cosmological constant term present.
- The tensor components transformations are given in: *Manousselis, Manolakos, Z, '18* (See App. I).

Unification of gravity theories with Internal Interactions

- So far in the gauge theoretic approach of gravity, general relativity is described by gauging the symmetry of the tangent manifold in four dimensions.
- Usually the dimension of the tangent space is considered to be equal to the dimension of the curved manifold. However, the tangent group of a manifold of dimension d is not necessarily SO_d.

Weinberg '84

- It has been suggested that by gauging an enlarged symmetry of the tangent space in four dimensions one could unify gravity with internal interactions.

Chamseddine, Mukhanov '10

- We aim to unify gravities as a gauge theory with internal interactions under one unification gauge group.
- Further attempts of unification for the case of Einstein gravity: *Percacci, '91; Manolakos et al, '23; Konitopoulos, Roumelioti, Z, '23.*

Unification of Conformal and Fuzzy Gravities with Internal Interactions

Unification Group

- Weyl gravity is based on gauging the $SO(2,4)$, while Fuzzy gravity on $SO(2,4) \times U(1)$.
- Internal Interactions by $SO(10)$ (GUT).
- Spontaneous symmetry breakings are used in all cases.

Roumelioti, Stefas, Z, '24
Roumelioti, Stefas, Z, '24

Usually to have a Chiral theory we need a $SO(4n+2)$ group. The smallest unification group in which both Majorana and Weyl condition can be imposed is $SO(2,16)$ from which:

$$SO(2,16) \xrightarrow{SSB} SO(2,4) \times SO(12)$$

and

$$SO(12) \xrightarrow{SSB} SO(10) \times [U(1)].$$

Breakings and branching rules

We start from $SO(2,16) \sim SO(18)$

- For CG we gauge $SO(2,4) \sim SU(2,2) \sim SO(6) \sim SU(4)$
- For FG we gauge $SO(2,4) \times U(1) \sim SO(6) \times U(1) \sim U(4)$
- For internal interactions we require $SO(10)$ GUT.

$$C_{SO(2,16)}(SO(2,4)) = SO(10)$$

and

$$C_{SO(2,16)}(SO(2,4) \times U(1)) = SO(10) \times U(1).$$

Breakings and branching rules (Continued)

$$SO(18) \supset SU(4) \times SO(12)$$

$$\begin{aligned}
18 &= (6,1) + (1,12) && \text{vector} \\
153 &= (15,1) + (6,12) + (1,66) && \text{adjoint} \\
256 &= (4,\overline{32}) + (\bar{4},32) && \text{spinor} \\
170 &= (1,1) + (6,12) + (20',1) + (1,77) && \text{2nd rank symmetric}
\end{aligned}$$

VEV in the $\langle 1,1 \rangle$ component of a scalar in 170 leads to $SU(4) \times SO(12)$.

Breakings and branching rules (Continued)

We break the $SO(12)$ down to $SO(10) \times U(1)$ or to $SO(10)$ with the 66 rep or the 77 rep.

$$
\begin{aligned}
& SO(12) \supset SO(10) \times U(1) \\
& \qquad 66 = (1)(0) + (10)(2) + (10)(-2) + (45)(0) \\
& \qquad 77 = (1)(4) + (1)(0) + (1)(-4) + (10)(2) + (10)(-2) + (54)(0)
\end{aligned}
$$

by giving VEV to the $\langle(1)(0)\rangle$ of the 66 rep we obtain $SO(10) \times U(1)$.

by giving VEV to the $\langle(1)(4)\rangle$ of the 77 rep we obtain $SO(10)$.

Breakings and branching rules (Continued)

We break $SU(4)$ in 2 steps:

- First step: Breaking $SU(4) \to Sp_4$:

$$SU(4) \supset Sp4$$
$$4 = 4$$
$$6 = 1 + 5$$

giving VEV to a scalar in 6 rep in the $\langle 1 \rangle$ component, the $SU(4)$ breaks down to the Sp_4.

- Second step: Breaking $Sp_4 \to SU(2) \times SU(2)$

$$Sp_4 \supset SU(2) \times SU(2)$$
$$5 = (1,1) + (2,2)$$
$$4 = (2,1) + (1,2).$$

giving VEV in $\langle 1, 1 \rangle$ of a scalar in the 5 rep we obtain eventually the Lorentz group $SU(2) \times SU(2) \sim SO(1,3)$.

Fermions

Weyl condition: $\Gamma^{D+1}\psi_{\pm} = \pm\psi_{\pm}, \quad D = even.$

Note that since $\Gamma^{D+1} = \gamma^5 \otimes \gamma^{d+1}$, the eigenvalues of γ^5 and γ^{d+1} are interrelated. However the choice of the eigenvalue of Γ^{D+1} does not impose the eigenvalue on γ^5!

Majorana condition: $\psi = C\bar{\psi}^T$

Weyl-Majorana spinors can exist when $D = 4n + 2$.
Type of spinors of $SO(p, q)$ depends on signature $(p - q) mod 8$.
For $p + q = even$:

- 0: real rep
- 4: quaternionic rep
- 2 or 6: complex rep

Chapline & Slansky, 1982; Polchinski, 1998; D'Auria et al., 2001; Figueroa-O'Farrill, n.d.

Fermions (Continued)

In the case of $SO(2,16)$ the signature is 6, and imposing the Weyl and Majorana conditions is permitted. Dirac spinors are defined as

direct sum of Weyl spinors and the Weyl condition chooses one of them, say $\sigma_{18} = 256$.

Spinor rep branching rules are:

$$SO(18) \supset SU(4) \times SO(12)$$
$$256 = (4, \overline{32}) + (\bar{4}, 32)$$

Imposing Majorana condition the fermions are in the $(\bar{4}, 32)$. Then

$$SO(12) \supset SO(10) \times [U(1)]$$
$$32 = (\overline{16})(1) + (16)(-1)$$

On the other hand

$$SU(4) \to Sp_4 \to SU(2) \times SU(2)$$
$$4 = 4 = (2,1) + (1,2).$$

Fermions (Continued)

After all the breakings:

$$
\begin{aligned}
&SU(2) \times SU(2) \times SO(10) \times [U(1)] \\
&\{[(2,1) + (1,2)\}\{(16)(-1) + (\bar{16})(1)\} \\
&\qquad = 16_L(-1) + \bar{16}_L(1) + 16_R(-1) + \bar{16}_R(1)
\end{aligned}
$$

and since $\bar{16}_R(1) = 16_L(-1)$ and $\bar{16}_L(1) = 16_R(-1)$,

$$= 2 \times 16_L(-1) + 2 \times 16_R(-1).$$

Finally, keeping only the left-handed part we obtain:

$$2 \times 16_L(-1)$$

Imposing also the Majorana condition in lower dims we obtain

$$16_L(-1) \quad \text{of} \quad SO(10) \times [U(1)]$$

Fermions in Fuzzy Gravity and Unification with Internal Interactions

- Fermions should be chiral in the original theory to have a chance to survive in low energies
- they should appear in a matrix representation since FG is a matrix model

Fortunately the way out was suggested in unification schemes with extra fuzzy dimensions *Chatzistavrakidis, Steinacker, Z '10*

Instead of using fermions in fundamental, spinor or adjoint reps of an $SU(N)$, we can use bi-fundamental reps of cross product $SU(N)$ groups.
Interesting example $N = 1$, $SU(N)^k$ models:

$$SU(N)_1 \times SU(N)_2 \times ... \times SU(N)^k$$

with matter content

$$(N, \bar{N}, 1, ..., 1) + (1, N, \bar{N}, ..., 1) + ... + (\bar{N}, 1, 1, ..., N)$$

Ma, Mondragon, Z, '04

with successful phenomenology, $N = 1$, $SU(3)^3$.

Fermions in Fuzzy Gravity and Unification with Internal Interactions (Continued)

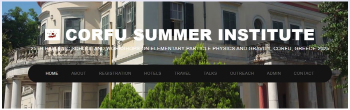
- In FG choosing to start with the $SO(6) \times SO(12)$ as the initial gauge theory with fermions in the $(4, \overline{32})$ we satisfy the criteria to obtain chiral fermions in tensorial representation.
- Weyl and Majorana conditions do not concern the global or local nature of the gauge part of the theory. Therefore all the discussion of unifying conformal conformal gravity with internal interactions can be repeated.
- The gauge $U(1)$ of FG due to the anticommutation relations, is identified with the one appearing in the $SO(12) \supset SO(10) \times U(1)$.

Further studies:

- on various breakings and their scales *Patellis, Trakas, Z, '24*
- including possible gravitational signals from cosmic strings due to the $SO(10)$ breakings *Patellis, Roumelioti, Stefas, Z, '25*

Thank you for your attention!

MY DEAR FRIEND BRANKO, HAPPY BIRTHDAY!!!



Events

Workshop on Future Accelerators,

Apr 27 - May 04, 2025.

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Workshop on the Standard Model and Beyond,

Aug 24 - Sep 03, 2025.

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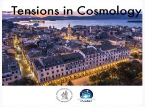

Tensions in Cosmology,

Sep 02 - Sep 08, 2025.

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Workshop on Quantum Gravity and Strings,

Sep 07 - Sep 14, 2025.

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Cost Action CaLISTA General Meeting 2025,

Sep 14 - Sep 22, 2025.

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ISMD 2025, 53rd International Symposium on Multiparticle Dynamics 2025,

Sep 21 - Sep 28, 2025.

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Appendix I: Fields and transformations

- The gauge connection, A_μ, as an element of the $SO(2,4)$ algebra, can be expanded in terms of the generators as

$$A_\mu = \frac{1}{2}\omega_\mu{}^{ab} M_{ab} + e_\mu{}^a P_a + b_\mu{}^a K_a + \tilde{a}_\mu D,$$

- It obeys the following infinitesimal transformation rule,

$$\delta A_\mu = D_\mu \epsilon = \partial_\mu \epsilon + [A_\mu, \epsilon],$$

where $\epsilon = \epsilon(x)$ a gauge algebra parameter which be expanded too as,

$$\epsilon = \xi^a P_a + \frac{1}{2}\lambda^{ab} M_{ab} + \kappa D + \rho^a K_a.$$

- From the above, the transf rule of the fields can be found:

$$\begin{aligned}
\delta e_\mu{}^a &= \partial_\mu \xi^a + \omega_\mu{}^a{}_b \xi^b - b_\mu \xi^a - \lambda^a{}_b e_\mu{}^b + \kappa e_\mu{}^a, \\
\delta \omega_\mu{}^{ab} &= \partial_\mu \lambda^{ab} - 2\omega_\mu{}^{ac} \lambda^b{}_c - 4 f_\mu{}^{[a} \xi^{b]} - 4 e_\mu{}^{[a} \rho^{b]}, \\
\delta \tilde{a}_\mu &= \partial_\mu \kappa - 2\xi^a f_{\mu a} + 2\rho^a e_{\mu a}, \\
\delta b_\mu{}^a &= \partial_\mu \rho^a + \omega_\mu{}^{ab} \rho_b + b_\mu \rho^a - \lambda^{ab} f_{\mu b} - \kappa f_\mu{}^a.
\end{aligned}$$

Appendix I: Fields and transformations (2)

The transformations of the fields:

$$\delta\omega_m^{\ ab} = -i[X_m, \lambda^{ab}] - i[a_m, \lambda^{ab}] + i[\epsilon_0, \omega_m^{\ ab}] - 2\{\xi^a, b_m^{\ b}\} - \frac{1}{2}\{\lambda^a{}_c, \omega_m^{\ bc}\}$$
$$- \frac{1}{2}\{\tilde{\xi}^a, e_m^{\ b}\} + i[\xi^c, e_m^{\ d}]\epsilon_{abcd} + \frac{i}{2}[\tilde{\epsilon}_0, \omega_m^{\ cd}]\epsilon_{abcd} + \frac{i}{2}[\lambda^{cd}, \tilde{a}_m]\epsilon_{abcd} - i[\tilde{\xi}^c, b_m^{\ d}]\epsilon_{abcd}$$
$$\delta e_m^{\ a} = -i[X_m, \tilde{\xi}^a] - i[a_m, \tilde{\xi}^a] + i[\epsilon_0, e_m^{\ a}] - \{\xi^a, \tilde{a}_m\} + \{\tilde{\epsilon}_0, b_m^{\ a}\} + \frac{1}{4}\{\lambda^a{}_b, e_m^{\ b}\}$$
$$- \frac{1}{4}\{\tilde{\xi}_b, \omega_m^{\ ab}\} + i[\xi^c, \omega_m^{\ bd}]\epsilon_{abcd} - i[\lambda^{cd}, b_m^{\ b}]\epsilon_{abcd}$$
$$\delta b_m^{\ a} = -i[X_m, \xi^a] - i[a_m, \xi^a] + i[\epsilon_0, b_m^{\ a}] - \{\xi_b, \omega_m^{\ ab}\} - 2\{\tilde{\epsilon}_0, e_m^{\ a}\} + \frac{1}{2}\{\lambda^a{}_b, b_m^{\ b}\}$$
$$+ \{\tilde{\xi}^a, \tilde{a}_m\} + i[\lambda^{bc}, e_m^{\ d}]\epsilon_{abcd} + i[\tilde{\xi}^b, \omega_m^{\ cd}]\epsilon_{abcd}$$
$$\delta a_m = -i[X_m, \epsilon_0] - i[a_m, \epsilon_0] + i[\xi^a, b_m^{\ a}] + i[\tilde{\epsilon}_0, \tilde{a}_m] + \frac{i}{2}[\lambda_{ab}, \omega_m^{\ ab}] + \frac{i}{2}[\tilde{\xi}_a, e_m^{\ a}]$$
$$\delta\tilde{a}_m = -i[X_m, \tilde{\epsilon}_0] - i[a_m, \tilde{\epsilon}_0] + i[\epsilon_0, \tilde{a}_m] + \{\xi_a, e_m^{\ a}\} - \{\tilde{\xi}_a, b_m^{\ a}\} + \frac{i}{2}[\lambda^{ad}, \omega_m^{\ bc}]\epsilon_{abcd}$$

(Transformations of the component of $\mathcal{B}_{mn}$ are calculated as well)

Appendix II: SSB of Weyl Gravity to EG

- The result of the breaking can be seen by considering the decomp. of **15** of $SU(4)$ under $SU(2) \times SU(2) \times U(1)$

$$SU(4) \xrightarrow{<\mathbf{15}>} SU(2) \times SU(2) \times U(1)$$
$$\mathbf{15} = [(\mathbf{3},\mathbf{1})_0 + (\mathbf{1},\mathbf{3})_0] + (\mathbf{1},\mathbf{1})_0 + (\mathbf{2},\mathbf{2})_2 + (\mathbf{2},\mathbf{2})_2\,,$$

$\rightarrow$ $[(\mathbf{3},\mathbf{1})_0 + (\mathbf{1},\mathbf{3})_0]$ describes the generators of the Lorentz gauge group, M_{ab}
$\rightarrow$ $(\mathbf{1},\mathbf{1})_0$ the generator of dilatations, D
$\rightarrow$ $(\mathbf{2},\mathbf{2})_2$ the generators of the translations, P_a
$\rightarrow$ $(\mathbf{2},\mathbf{2})_2$ the generators of conformal transformations, K_a

- The generators P_a and K_a are broken due to the SSB of the scalar **15**-plet

Appendix II: SSB of Weyl Gravity to EG (2)

- Similarly, the decomposition of the 15 generators of $SU(4)$ under the $SO(5)$ to which it breaks after the SSB of the scalar **6**-plet is,

$$SU(4) \xrightarrow{<\mathbf{6}>} SO(5)$$
$$\mathbf{15} = \mathbf{10} + \mathbf{5}\,,$$

$\rightarrow$ **10** the generators of the unbroken gauge group, $SO(5)$, and
$\rightarrow$ **5** the broken generators

- To identify the unbroken and the broken generators above we consider the decomp of reps **10** and **5** of $SO(5)$ under the $SU(2) \times SU(2)$ describing the Lorentz gauge group,

$$SO(5) \supset SU(2) \times SU(2)$$
$$\mathbf{10} = (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})\,,$$
$$\mathbf{5} = (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})\,.$$

$\rightarrow$ Ten unbroken gen. from the SSB of the scalar **6**-plet correspond to M_{ab} and the P_a (which were broken by the $<\mathbf{15}>$)
$\rightarrow$ Five broken gen. are the $(\mathbf{1},\mathbf{1})$ of D and the $(\mathbf{2},\mathbf{2})$ of K_a.

Appendix II: SSB of Weyl Gravity to EG (3)

- In summary, $<\mathbf{15}>$ breaks the generators of P_a and K_a, leaving unbroken the Lorentz rotation generators, M_{ab} and the dilaton generator, D, while $<\mathbf{6}>$ breaks the dilaton generator, D and gives an additional contribution to the breaking of the generators K_a (and to the masses of the corresponding gauge bosons).

Appendix III: Equivalence of Gauge and Diffeo transf

- We calculate the difference between a diffeomorphism and a gauge transformation of the fields:

$$\tilde{\delta} e^a_\mu - \delta e^a_\mu = \left(v^\nu \partial_\nu e^a_\mu + \partial_\mu (v^\nu e^a_\nu) - v^\nu \partial_\mu e^a_\nu \right) \\ - \left(\partial_\mu \xi^a + \omega_\mu{}^a{}_b \xi^b - b_\mu \xi^a - \lambda^{ab} e^b_\mu + \kappa e^a_\mu \right)$$

- Setting $\xi^a = v^\mu e_\mu{}^a$, $\lambda^{ab} = v^\mu \omega_\mu{}^{ab}$, $\kappa = v^\mu b_\mu$, and $\rho^a = v^\mu f_\mu{}^a$:

$$\tilde{\delta} e_\mu{}^a - \delta e_\mu{}^a = v^\nu \left(\partial_\nu e_\mu{}^a - \partial_\mu e_\nu{}^a - \omega_\mu{}^a{}_b e_\nu{}^b + \omega_\nu{}^a{}_b e_\mu{}^b + b_\mu e_\nu{}^a - b_\nu e_\mu{}^a \right) \\ = - v^\nu \tilde{R}_{\mu\nu}{}^a .$$

$\rightarrow$ the constraint needed for getting rid of the translational part, with a coordinate transformation making up for them, is the vanishing of the torsion,

$$\tilde{R}_{\mu\nu}{}^a = 0 \, .$$

Appendix III: Equiv. of Gauge and Diffeo transf (2)

- Similarly, the difference between a diffeomorphism and the gauge transformation $\tilde{\delta} b_\mu{}^a - \delta b_\mu{}^a$ leads to

$$R_{\mu\nu}{}^a = 0\,,$$

while the corresponding difference $\tilde{\delta}\omega_\mu{}^{ab} - \delta\omega_\mu{}^{ab}$ results to

$$R_{\mu\nu}{}^{ab} = 0\,.$$

- As already mentioned the generators P_a and K_a are broken due to the SSB of the scalar 15-plet, i.e. the two torsionless conditions are resulting from the SSB of the scalar 15-plet.
- The two torsionless conditions and the vanishing of the curvature tensor (which is satisfied on-shell) guarantee the equivalence of the diffeomorphisms and gauge transformations.
  $\rightarrow$ The gauge theory based on the $SO(2,4)$ group describes the 4-d conformal gravity.