# Planck length and metric geometry

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The **Planck length** is the following combination of fundamental constants, having the dimension of length:

$$\ell_{Pl} = \sqrt{\frac{\hbar G}{c^3}}.$$

Numerical value of the Planck length is $\ell_{Pl} \approx 1.61 \cdot 10^{-33} cm$.

The physical meaning of the Planck length is as follows. This is a scale on which it is fundamentally impossible to consider the theory of gravity without taking into account the quantum effects [1], since it is on the Planck scale that the values with the dimension of length inherent for gravity theory (the Schwarzschild radius of a spherically symmetric black hole) coincide with those for quantum theory (the Compton wavelength).

[1] D. Oriti, ed. Approaches to quantum gravity. Toward a new understanding of space, time and matter. Cambridge University Press, 2009.

Really, the Compton wavelength is given by the expression

$$\lambda_C = \frac{\hbar}{mc},$$

and the Schwarzschild radius is

$$r_g = \frac{2Gm}{c^2}.$$

It is easy to see that the equality takes place:

$$\ell_{Pl}^2 = \frac{\lambda_C r_g}{2}.$$

The appearance of a black hole on Planck scales does not allow us to obtain information about the structure of space on scales smaller than the Planck length.

In 1987 I. V. Volovich [2] conjectured that this kind of effect is associated with a fundamental change in the geometry of space on the Planck scale.

Namely, the existence of unmeasurable regions of space is the result of a violation of Archimedes' axiom (the axiom of measurability) in Euclidean geometry. A conjecture about the non-Archimedean nature of space on Planck scales was formulated.

[2] I. V. Volovich, Number Theory as the Ultimate Physical Theory, preprint CERN-TH. 4781/87, CERN, Geneva, 1987 , 11 pp., reproduced in "p-Adic Numbers Ultrametric Anal. Appl. 2 (1), 77–87 (2010).

However, the question of the mechanism of changing the metric from Archimedean to non-Archimedean remains open.

In this paper, an attempt is made to construct a model of metric change using the apparatus of metric geometry. Namely, a geodesic in the **Gromov-Hausdorff space** connecting ultrametric and ordinary metric spaces will be explicitly constructed.

As a model example of an ultrametric space, we will consider the set $\mathbb{Z}_p$ of $p$-adic integers with a metric generated by the standard $p$-adic norm; as a model example of an ordinary metric space, we will choose the unit segment $[0, 1] \subset \mathbb{R}$ with a standard metric generated by the absolute value.

A metric space is a pair $X = (X, d_X)$, where $X$ is a set, $d_X$ is a metric on $X$, that is, a mapping $d_X \colon X \times X \to [0, \infty)$ satisfying the conditions:

- $d_X(x, x') = 0 \iff x = x'$;
- $d_X(x, x') = d_X(x', x)$;
- $d_X(x, x'') \leq d_X(x, x') + d_X(x', x'')$.

If $d_X$ satisfies the condition $d_X(x, x") \leq \max\{d_X(x, x'), d_X(x', x")\}$ then this is ultrametric, the space $(X, d_X)$ is ultrametric (or non-Archimedean).
Important examples for the future are the following.

- $\mathbb{I} = [0, 1], d_{\mathbb{I}}(x, x') = |x - x'|$ – Archimedean space;
- $\mathbb{Z}_p, d_{\mathbb{Z}_p}(x, x') = |x - x'|_p$ – non-Archimedean space;
- $\Delta_m = \{x_1, \dots x_m\}, d_{\Delta_m}(x_i, x_j) = 1, i \neq j, i, j = 1, 2, \dots, m$ – simplex.

We define two operations on metric spaces: direct product and dilation.

Direct product $(X \times Y, d_{X\times Y})$ of the metric spaces $X$ and $Y$ is the Cartesian product of $X \times Y$ with the metric given by the expression

$$d_{X\times Y}\left((x,y),(x',y')\right) = \max\left\{d_X(x,x'), d_Y(y,y')\right\}.$$

Let $\lambda \in \mathbb{R}_+$ be a positive real number. The space $\lambda X$ obtained from the space $(X, d_X)$ by dilation the metric has the form:

$$\lambda X = (X, \lambda d_X).$$

Let $(X, d)$ be a metric space and $H = H(X)$ be a set of compact subsets of $X$. We define the metric (Hausdorff metric) $d_H$ on $H$. Let $A$, $B \in H(X)$,

$$d_H(A,B) = \inf\{\epsilon > 0 :\ B \subset U_\epsilon(A) \text{ and } A \subset U_\epsilon(B)\},$$

where $U_\epsilon(A) = \{x \in X :\ d(x,A) \leq \epsilon\}$.

$(H(X), d_H)$ is a metric space, and it is true that $H(X)$ is compact if and only if $X$ is compact.
By means of $GH$, we denote the set of **isometry classes of compact metric spaces**. We introduce the metric on the set $GH$ as follows [3].

The **realization** of the pair $X$, $Y$ of compact metric spaces is called the triple $(Z, X', Y')$, where $Z$ is a metric space, $X' \subset Z$, $Y' \subset Z$, $X$, $Y$ are isometric to $X'$, $Y'$, respectively, and $d_Z|_{X'} = d_X, d_Z|_{Y'} = d_Y$.

$$d_{GH}(X, Y) = \inf_{\text{realizations of X,Y}} d_H(X', Y').$$

$(GH, d_{GH})$ is a complete separable metric space.

[3] M. Gromov. Metric structures for Riemannian and non-Riemannian spaces. Birkhäuser Boston, MA, 2007.
D. Burago, Yu. Burago, S. Ivanov. A course in metric geometry. AMS, 2001.

The following Theorems are valid.

**Theorem 1**

$$d_{GH}(\mathbb{I}, \mathbb{Z}_p) = \frac{1}{2}.$$

**Theorem 2**

*Let $X$ be a connected compact metric space,* $\operatorname{diam} X = 1$. *Then we have:*

$$d_{GH}(X, \mathbb{Z}_p) = \frac{1}{2}.$$

**Theorem 3**

*Let $k$ be a positive integer such that the inequalities $p^k < q < p^{k+1}$ are satisfied. Then equality is valid:*

$$2d_{GH}(\mathbb{Z}_p, \mathbb{Z}_q) = 1 - \frac{1}{p^k}.$$

A subset of $R(X, Y) \subset X \times Y$ of the direct product of the sets $X$ and $Y$ is called a **correspondence** if the projections of this subset onto the components of the product are surjective: $\mathrm{pr}_X R(X, Y) = X$, $\mathrm{pr}_Y R(X, Y) = Y$.

The **distortion** $\mathrm{dist} R(X, Y)$ of a corresponence $R(X, Y)$ is the following number:

$$\mathrm{dist} R(X, Y) = \sup_{(x,y),(x',y') \in R(X,Y)} |d_X(x, x') - d_Y(y, y')|$$

.
The following statement [4] is true:

$$d_{GH}(X, Y) = \frac{1}{2} \inf_{\text{correspondences } R(X,Y)} \mathrm{dist} R(X, Y).$$

This statement provides a convenient way to calculate distances in the Gromov-Hausdorff space.

[4]D. Burago, Yu. Burago, S. Ivanov. A course in metric geometry. AMS, 2001.

Here are some simple examples.

## Example 1

*Let* $R(X, Y) = X \times Y$*, then*

$$\operatorname{dist} R(X, Y) = \max\{\operatorname{diam} X, \operatorname{diam} Y\}.$$

*Therefore,*

$$2d_{GH}(X, Y) \leq \max\{\operatorname{diam} X, \operatorname{diam} Y\}.$$

## Example 2

$2d_{GH}(X, \Delta_1) = \operatorname{diam} X$. *Using the triangle inequality*

$$d_{GH}(X, \Delta_1) \leq d_{GH}(X, Y) + d_{GH}(Y, \Delta_1),$$

*we get:*

$$2d_{GH}(X, Y) \geq |\operatorname{diam} X - \operatorname{diam} Y|.$$

Example 3

*Let $f: X \to Y$ be surjective. Then the graph $\{(x, f(x)), x \in X\}$ is a correspondence.*

There are two important points:

- there is (not unique) optimal correspondence

$$R_{opt}(X, Y)\colon 2d_{GH} = \mathrm{dist} R_{opt}(X, Y);$$

- to calculate distances in the Gromov-Hausdorff space, it is enough to consider only closed correspondences.

Proof of Theorem 1.

Let $R(\mathbb{Z}_p, \mathbb{I})$ be an arbitrary closed correspondence.

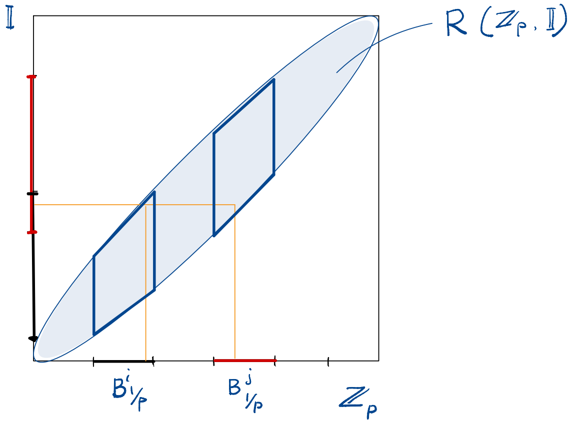
Let's consider $\mathbb{Z}_p$ as a disjoint union of $p$ balls of radius $1/p$, $\mathbb{Z}_p = \sqcup_{i=1,\dots,p} B^i_{1/p}$. The family of subsets $\mathbb{I}$ of the form $\{\mathrm{pr}_{\mathbb{I}} R(B^i_{1/p}, \mathbb{I}), i = 1, \dots, p\}$ forms a covering of the segment $\mathbb{I}$ by closed subsets.

Since $\mathbb{I}$ is connected, at least two sets of our coverage have a common point. The projections on $\mathbb{Z}_p$ of the preimages of this common point lie in different balls $B_i$ and $B_j$.

Therefore, the distance in $\mathbb{Z}_p$ between the projections of the preimages is equal to one. Thus, $\mathrm{dist} R(\mathbb{Z}_p, \mathbb{I}) \geq 1$. Since this is true for any correspondence, choosing the optimal one yields $2d_{GH}(\mathbb{Z}_p, \mathbb{I}) \geq 1$.

On the other hand, $2d_{GH}(\mathbb{Z}_p, \mathbb{I}) \leq \max\{\mathrm{diam}\, \mathbb{Z}_p, \mathrm{diam} \mathbb{I}\} = 1$.

Since we used only the connectivity of the space $\mathbb{I}$, the same proof works in the case of Theorem 2.



𝕀
R(ℤp, 𝕀)
B^i_{1/p}
B^j_{1/p}
ℤp

Proof of Theorem 3.

First, let's prove the estimate from below: $1 - \frac{1}{p^k} \leq 2d_{GH}(\mathbb{Z}_p, \mathbb{Z}_q)$.

Let $N$ be a positive integer. Let's consider $\mathbb{Z}_p$ as a disjoint union of $p^N$ balls of radius $\epsilon = p^{-N}$. We will choose one point in each ball of the constructed partition. The set $X_N^{(p)}$ obtained in this way, consisting of $p^N$ points, is provided with the metric $d_N$, induced by the metric on $\mathbb{Z}_p$. As a result, we get the metric space $(X_N^{(p)}, d_N)$. It is useful to note that the space $X_1^{(p)}$ is nothing but a simplex $\Delta_p$.

Note that the Hausdorff distance between $\mathbb{Z}_p$ and $X_N^{(p)}$ is equal to $\epsilon$. This immediately implies the validity of the evaluation of $d_{GH}(\mathbb{Z}_p, X_N^{(p)}) \leq \epsilon$ (it suffices to consider the realization of the pair $(\mathbb{Z}_p, X_N^{(p)})$ of the form $Z = \mathbb{Z}_p = Y', X' = X_N^{(p)}$).

The following simple Lemma follows from the triangle inequality.

### Lemma 1

*For any metric compact $X$, the inequality holds:*

$$\left| d_{GH}(X, \mathbb{Z}_p) - d_{GH}(X, X_N^{(p)}) \right| \le p^{-N}.$$

Let $MST(X_N^{(p)})$ be the **minimum spanning tree** of a finite metric space $X_N^{(p)}$.

By means of $\sigma(X_N^{(p)})$, we denote the **mst-spectrum** of the space $X_N^{(p)}$, that is, the sequence of edge lengths of the minimum spanning tree in decreasing order.

The following Lemma is valid.

## Lemma 2

$$\sigma(X_N^{(p)}) = \left\{ \underbrace{1, \ldots, 1}_{p-1} \underbrace{\frac{1}{p}, \ldots, \frac{1}{p}}_{p(p-1)}, \underbrace{\frac{1}{p^2}, \ldots, \frac{1}{p^2}}_{p^2(p-1)} \cdots, \underbrace{\frac{1}{p^{N-1}}, \ldots, \frac{1}{p^{N-1}}}_{p^{N-1}(p-1)} \right\}.$$

Let's decompose $\mathbb{Z}_p$ into a disjoint union of $p$ balls of radius $1/p$: $\mathbb{Z}_p = \sqcup_i^p B_{1/p}^i$. In each of the partition balls, we will choose one element from the set $X_N^{(p)}$. The pairwise distances between the various elements of this set are equal to one, that is, it is a simplex $\Delta_p$. It follows directly from this that $MST(X_N^{(p)})$ has exactly $p-1$ edge of length 1.

Now each of the balls $B_{1/p}^i$, $i = 1, 2, \ldots, p$ of our partition let's decompose into disjoint union of $p$ balls of radius $1/p^2$ (in total, we get $p^2$ balls of radius $1/p^2$) and let's do a similar reasoning for each of these balls. Continuing these arguments $N$ times, we obtain the statement of the Lemma.

From the triangle inequality for the spaces $\mathbb{Z}_p, \mathbb{Z}_q, \Delta_{p^k}$ we obtain:

$$d_{GH}(\mathbb{Z}_p \mathbb{Z}_q) \geq d_{GH}(\Delta_{p^k}, \mathbb{Z}_q) - d_{GH}(\Delta_{p^k}, \mathbb{Z}_p).$$

Next, we will use the results of paper[5] (Theorem 3.3). The above theorem states, in particular, the following:

$$2d_{GH}(\Delta_m, X) = \max\{\sigma_1(X) - 1, \sigma_m(X), 1 - \sigma_{m-1}(X)\},$$

where $X$ is a finite ultrametric space consisting of $n$ points, and $1 < m < n$.

[5] A. O. Ivanov, A. A. Tuzhilin. The Gromov-Hausdorff Distances between Simplexes and Ultrametric Spaces. arXiv:1907.03828v1, 2019.

Let's choose a positive integer $N$, $q < p^N$. Then the equality

$$2d_{GH}(\Delta_{p^k}, X_N^{(q)}) = 1$$

is valid, because

$$\sigma_1(X_N^{(q)}) = \sigma_{p-1}(X_N^{(q)}) = \sigma_{p^k}(X_N^{(q)}) = 1.$$

In addition, the following equality is true

$$2d_{GH}(\Delta_{p^k}, X_N^{(p)}) = 1/p^k,$$

because in this case

$$\sigma_1(X_N^{(p)}) = \sigma_{p-1}(X_N^{(p)}) = 1, \ \sigma_{p^k}(X_N^{(p)}) = 1/p^k.$$

Taking into account the last equalities for sufficiently large $N$, we obtain the required estimate from below.

To obtain an estimate from above, we construct the correspondence $R(\mathbb{Z}_p, \mathbb{Z}_q)$ explicitly and calculate its distortion.

Let's represent the number $q$ as the sum of positive integers of the following form:

$$q = q_1 + q_2 + \cdots + q_{p^k},\ 1 \leq q_i \leq p,\ \ i = 1, 2, \ldots, p^k.$$

Note that in this representation, at least one of the terms is not equal to 1 (since $p^k < q$).

Let's decompose $\mathbb{Z}_p$ into a disjoint union of $p^k$ balls of radius $p^{-k}$:

$$\mathbb{Z}_p = \sqcup_{i=1}^{p^k} B^i_{p^{-k}}.$$

Let's represent $\mathbb{Z}_q$ as a disjoint union of balls of radius $q^{-1}$ in accordance with the above decomposition of the number $q$:

$$\mathbb{Z}_q = \sqcup_{i_1=1}^{q_1} B^{i_1}_{q^{-1}} \sqcup_{i_2=1}^{q_2} B^{i_2}_{q^{-1}} \cdots \sqcup_{i_1=1}^{q_{p^k}} B^{i_{p^k}}_{q^{-1}}.$$

Since any compact totally disconnected spaces are homeomorphic, there exists a homeomorphism $\phi\colon \mathbb{Z}_q \to \mathbb{Z}_p$ such that for all $j = 1, 2, \ldots p^k$ the conditions

$$\phi\left(\sqcup_{i_j=1}^{q_j} B_{q^{-1}}^{i_j}\right) = B_{p^{-k}}^{j}$$

are fulfilled.

As the desired correspondence, $R(\mathbb{Z}_q, \mathbb{Z}_p)$ let's take the graph of the map $\phi$.
We'll show that the distortion of this correspondence is $1 - \frac{1}{p^k}$.

Let $x, x' \in \mathbb{Z}_q\colon |x - x'|_q \leq \frac{1}{q}$, then the inequality $|\phi(x) - \phi(x')|_p \leq \frac{1}{p^k}$ is fulfilled by the definition of the map $\phi$. Indeed, the inequality $|x - x'|_q \leq \frac{1}{q}$ means that $x$ and $x'$ lie inside a ball of radius $1/q$, and the image of each such ball lies inside a ball of radius $1/p^k$ in $\mathbb{Z}_p$. Therefore, for all such $x$ and $x'$, the inequality $||x - x'|_q - |\phi(x) - \phi(x')|_p| \leq p^{-k}$ holds.

Now let $x$ and $x'$ lie in different balls of radius $1/q$ in $\mathbb{Z}_q$ (in this case, $|x - x'|_q = 1$). There are two possible cases here. The first is when $x$ and $x'$ lie in different groups of balls, and the second is when they lie in the same group of balls.

In the first case, we have $|\phi(x) - \phi(x')|_p \geq p^{-k+1}$, since $\phi(x)$ and $\phi(x')$ lie in different balls of radius $p^{-k}$ in $\mathbb{Z}_p$. Therefore, the inequality

$$\big||x - x'|_q - |\phi(x) - \phi(x')||_p\big| \leq 1 - p^{-k+1}$$

holds.

In the second case, $|\phi(x) - \phi(x')|_p \leq p^{-k}$, since $\phi(x)$ and $\phi(x')$ lie in the same ball of radius $p^{-k}$ in $\mathbb{Z}_p$.

Now we will impose an additional condition on the map $\phi$.

As noted earlier, in our partition of a set consisting of $q$ balls of radius $1/q$ in $\mathbb{Z}_q$ into $p^k$ groups of balls, there are groups (at least one) consisting of $q_k$ balls such that the inequalities $2 \leq q_k \leq p$ are satisfied.

The image of each such group under the map $\phi$, is a ball of radius $p^{-k}$ in $\mathbb{Z}_p$ (each group has its own).

Let's decompose this ball into a disjoint union of $p$ balls of radius $p^{-k-1}$, and divide this set into $q_k$ groups (recall that $q_k \leq p$).

We will construct the map $\phi$ in such a way that each of the $q_k$ balls is mapped into its own group.

In this case, if $x$ and $x'$ lie in different balls from the group of $q_k$ balls, then their images lie in different balls of radius $p^{-k-1}$ inside a ball of radius $p^{-k}$ and, thus, $|\phi(x) - \phi(x')|_p = p^{-k}$.

Thus, we have obtained the following properties of the map $\phi$:

$$\big||x-x'|_q - |\phi(x)-\phi(x')|_p\big| \leq \frac{1}{p^k}, \text{ if } |x-x'|_q \leq \frac{1}{q},$$

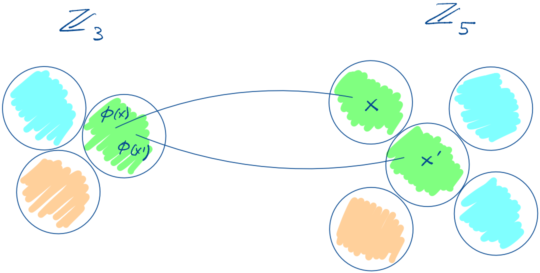
and in the case of $|x-x'|_q = 1$:

$$\big||x-x'|_q - |\phi(x)-\phi(x')|_p\big| \leq 1 - \frac{1}{p^{k-1}}$$

or

$$\big||x-x'|_q - |\phi(x)-\phi(x')|_p\big| = 1 - \frac{1}{p^k}.$$

It follows directly from the last formulas that the graph of the constructed map $\phi$ has a distortion equal to $1-\frac{1}{p^k}$. Therefore, the inequality is valid $2d_{GH}\left(\mathbb{Z}_p, \mathbb{Z}_q\right) \leq 1-\frac{1}{p^k}$. The theorem has been proved.




$$|x - x'|_5 = 1$$

$$|\phi(x) - \phi(x')| = \frac{1}{3}$$

$$5 = 1 + 2 + 2$$

Note that the correspondence constructed during the proof of theorem 3 (the graph of the map $\phi\colon \mathbb{Z}_q \to \mathbb{Z}_p$) is optimal.

It is not difficult to construct an optimal correspondence $R(\mathbb{Z}_p, \mathbb{I})$ between $\mathbb{Z}_p$ and the unit interval $\mathbb{I}$.

As such a correspondence, consider the graph of the Monna map.

Let $\mathbb{Z}_p \ni x = x_0 + x_1 p + \dots x_k p^k + \dots$. The Monna map $\mu\colon \mathbb{Z}_p \to \mathbb{I}$ is given by the expression

$$\mu(x) = \frac{1}{p}(x_0 + x_1 p^{-1} + \dots + x_k p^{-k} \dots).$$

Let's calculate the distortion of the Monna map's graph.

Let $x, x' \in \mathbb{Z}_p\colon |x - x'|_p = p^{-n}$. This means that $x_0 = x'_0, x_1 = x'_1, \dots, x_{n-1} = x'_{n-1}, x_n \neq x'_n$. Then the inequality is valid

$$|\mu(x) - \mu(x')| \leq p^{-n}.$$

Therefore, for all $x, x'$: $|x - x'| < 1$, the estimate

$$\big||x - x'|_p - |\mu(x) - \mu(x')|\big| \leq 1/p$$

is valid.
Now let $|x - x'|_p = 1$, that is, $x_0 \neq x_0'$. Let $x_0 > x_0'$ be for certainty, then $(x - x')_0 = x_0 - x_0'$ and the inequalities are valid

$$\frac{x_0 - x_0'}{p} \leq |\mu(x) - \mu(x')| \leq \frac{x_0 - x_0' + 1}{p}.$$

It immediately follows that the distortion of the Monna map's graph is $1 - \frac{1}{p}$. Taking into account theorem 1, it can be concluded that the Monna map's graph defines the optimal correspondence between $\mathbb{Z}_p$ and $\mathbb{I}$.

Our task is to construct a geodesic connecting $\mathbb{Z}_p$ and $\mathbb{I}$ in the Gromov-Hausdorff space. To do this, we will use the following result from the paper [6]:

### Proposition 1

*Let $(X, d_X), (Y, d_Y)$ be compact metric spaces, then for any optimal correspondence $R_{opt}(X, Y)$ there is a family of compact metric spaces $R_t$ such that $R_0 = X$, $R_1 = Y$ and for $t \in (0, 1)$ $R_t = (R_{opt}(X, Y), d_t)$, where*

$$d_t\left((x, y), (x', y')\right) = (1 - t)d_X(x, x') + td_Y(y, y')$$

*defines the shortest curve in GH connecting the spaces $X$ and $Y$.*

[6] A. O. Ivanov, S. Iliadis, A. A. Tuzhilin. Realization of Gromov-Hausdorff Distance. arXiv:1603.08850v1, 2016.

Thus, the following statement is true.

## Theorem 4

*The family of spaces* $(\mathbb{Z}_p, d_0 = |\cdot|_p)$, $(\Gamma_\mu, d_t)$, $(\mathbb{I}, d_1 = |\cdot|)$, *where* $\Gamma_\mu \subset \mathbb{Z}_p \times \mathbb{I}$ *denotes the graph of the Monna map,*

$$d_t\left((x,\mu(x)),(y,\mu(y))\right) = (1-t)|x-y|_p + t|\mu(x)-\mu(y)|, \ t \in (0,1),$$

*defines the shortest curve connecting* $\mathbb{Z}_p$ *and* $\mathbb{I}$ *in the Gromov-Hausdorff space.*

A geodesic connecting $\mathbb{Z}_p$ and $\mathbb{Z}_q$ is constructed in a similar way. To do this, instead of the graph of the Monna map, we need to take the graph of the map $\phi$ from the proof of theorem 3.

$(X, d)$ is a metric space. Let $M_r$ be the space of positive symmetric $r \times r$ matrices and let $K_r(X) \subset M_r$ be the subset realizable by the distances among $r$-tuples of points of $X$:

$$(m_{ij}) \in K_r(X) \iff \exists x_1, \dots, x_r : d(x_i, x_j) = m_{ij}, \; i, j = 1, \dots, r.$$

## Proposition 2

*Two compact metric spaces $X$ and $Y$ are isometric if and only if $K_r(X) = K_r(Y) \; \forall \, r = 1, 2 \dots.$*

## Question 1

*What is the $K$-curvature class of $(\Gamma_\mu, d_t)$?*

## Question 2

*What is the degree of ultrametricity [7] of $(\Gamma_\mu, d_t)$?*

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[7] R. Rammal, J. C. Angles d'Auriac, B. Doucot. On the degree of ultrametricity. J. Physique Lett. 46 (1985).

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