Real Quantum Mechanics in a Kähler Space

Igor Volovich

Steklov Mathemathical Institute, RAS

30 May, 2025

Nonlinearity, Nonlocality and Ultrametricity

Dragovich 80

Outlook

- i) Number Theory & Physics

Outlook

- ii) Quantum Mechanics & Elasticity Theory

Outlook

- iii) Quantum Mechanics as Symplectic Dynamical System
  - Ergodicity of Quantum Mechanics

Outlook

- iv) Quantum Mechanics in

  Hilbert Space & Kähler Space

Number Theory & Physics

I. V. Volovich, "Number theory as the ultimate physical theory". p-adic Numbers, Ultr.Anal.&Appl.(2010) 2, 77; CERN-TH.4781/87, 1987

It was argued that

- application of real numbers in physics is an idealization since only rational numbers could be observed.
- foundamental physicsl laws should be invariant under the change of the number fields

Number Theory & Physics

- at the Planck scale non-Archimedean geometry emigres and one has to use p-adic numbers

I.Arefeva, B.Dragovich, P.H.Frampton, I.V., "Wave function of the universe and p-adic gravity," Int. J. Mod. Phys. A 6 (1991), 4341

Number Theory & Physics

- **Quantum mechanics on p-adic numbers has been constructed**
  V.S. Vladimirov, I. V., E.I. Zelenov, p-adic analysis and mathematical physics, 1994
- **Adelic quantum mechanics**
  B. Dragovich, "Adelic harmonic oscillator," Int. J. Mod. Phys. A (1995), 10, 2349

B. Dragovich, A. Y. Khrennikov, S. V. Kozyrev, I. V., E. I. Zelenov, p-adic mathematical physics: The first 30 years. p-adic Numbers, Ultrametric Analysis and Applications 9, 87–121 (2017)
(about 170 refs.)

Number Theory & Physics

- Riemann zeta function &
  Black Holes/Bose Gas duality

I.Aref'eva and I.V., "Violation of the third law of thermodynamics by black holes, Riemann zeta function and Bose gas in negative dimensions," Eur. Phys. J. Plus (2024) 139, 300

Quantum Mechanics & Elasticity Theory

- **On the equivalence between Schrodinger & Euler-Bernoulli eqs.**

I.V. "On the Equivalence Between the Schrodinger Equation in Quantum Mechanics and the Euler-Bernoulli Equation in Elasticity Theory", 2411.03261

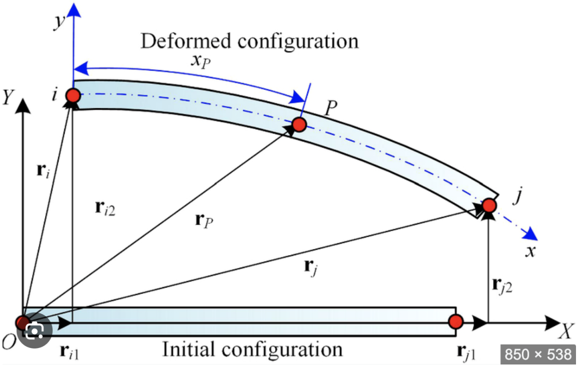
Euler–Bernoulli

In 1750, Euler and Bernoulli proposed an equation to describe the vibrations of beams and plates in elasticity theory.

Euler–Bernoulli eq.: $\ddot{u} + \Delta^2 u = 0,$

Δ - Laplace operator.

Euler–Bernoulli eq. for beams and plates



Schrodinger equation

In 1926, Schrodinger discovered the wave equation that governs quantum mechanical particles. For a free particle, this equation is

$$\mathbf{i}\dot{\psi} = -\boldsymbol{\Delta}\psi. \tag{1}$$

Relation of Euler–Bernoulli & Schrodinger equations

From Schrodinger eq. one has

$$\ddot{\psi} + \mathbf{\Delta}^2 \psi = \mathbf{0}. \tag{2}$$

Indeed,

$$\dot{\psi} = \mathbf{i}\mathbf{\Delta}\psi, \qquad \ddot{\psi} = \mathbf{i}\mathbf{\Delta}\dot{\psi}$$

Therefore,

$$\ddot{\psi} = -\mathbf{\Delta}^2 \psi \tag{3}$$

The Cauchy Problem for the Schrodinger Equation

$$i\dot{\psi} = -\Delta\psi, \quad \Delta = \partial_{x_1}^2 + \ldots\partial_{x_d}^2$$
$$\psi(0,x) = \psi_0(x), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}$$

$\psi = \psi(t,x)$ - complex function

Solutions $\psi \in C^\infty(\mathbb{R}^{d+1})$
with initial data $\psi_0 = \psi_0(x)$
belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

Solution to Cauchy problem

is given by the integral

$$\psi(t,x) = \int_{\mathbb{R}^d} \tilde{\psi}_0(k)\, e^{-ik^2 t - ik\cdot x}\, dk,$$

$\tilde{\psi}_0(k)$ - Fourier transform of $\psi_0(x)$,

$$k^2 = \sum_{i=1}^{d} k_i^2 \quad k\cdot x = \sum_{i=1}^{d} k_i x_i$$

Additionally, we have $\dot{\psi}(0,x) = i\Delta\psi_0(x)$.

Symplectic form

Schrodinger eq. as a system of 2 eqs:

$$\psi(t,x) = u(t,x) + iv(t,x),$$

which leads to the system

$$\dot{u} = -\Delta v, \quad \dot{v} = \Delta u \quad (*)$$

(*) represents Hamiltonian dynamics with the Hamiltonian

$$H_{\text{sym}} = \frac{1}{2}\int_{\mathbb{R}^d} [u(-\Delta)u + v(-\Delta)v]\, dx.$$

Theorem

$$i\,\dot{\psi} = -\Delta\psi, \quad \psi(0,x) = \psi_0(x)$$

$$\psi(t,x) = u(t,x) + iv(t,x),$$

$$\psi_0(x) = u_0(x) + iv_0(x)$$

$$\begin{aligned} \ddot{u} + \Delta^2 u = 0, \quad u(0,x) &= u_0(x) \\ \dot{u}(0,x) &= -\Delta v_0(x) \\ \ddot{v} + \Delta^2 v = 0, \quad v(0,x) &= v_0(x), \\ \dot{v}(0,x) &= \Delta u_0(x). \end{aligned}$$

Generalized Euler–Bernoulli Equation with Potential

We start from the Schrodinger eq with a potential:

$$\mathbf{i}\dot{\psi} = \mathbf{H}\psi \, (*) \qquad \mathbf{H} = -\boldsymbol{\Delta} + \mathbf{V} \qquad \psi(\mathbf{0}, \mathbf{x}) = \psi_0$$

Differentiating (*) with respect to time, we obtain a generalization of Euler–Bernoulli eq.

$$\ddot{\psi} + \boldsymbol{H}^2\psi = \mathbf{0}.$$

Ergodicity of Finite-Dimensional Quantum Systems

- **Boltzmann. Ergodicity hypothesis**
- **G.Weyl, Birkhoff**
- **von Neumann, Kolmogorov,...**
- **We will prove that almost any finite dimensional quantum system is ergodic**

Quantum =Classical

- N-dimensional Hilbert space $\mathbb{C}^N$
- $H = (H_{ab})$ Hermitian $N \times N$ matrix, $a, b = 1, ..., N$
- Schr. eq. $i\,\dot{\psi}_a = H_{ab}\,\psi_b$, $\psi_a : \mathbb{R} \to \mathbb{C}$
- $H_{ab} = K_{ab} + iL_{ab} \qquad K_{ab} = K_{ba}$, $L_{ab} = -L_{ba}$ are real
- Real form of the Schr. eq.: $\psi_a = q_a + ip_a$

$$\dot{q}_a = K_{ab}p_b + L_{ab}q_b, \quad \dot{p}_a = -K_{ab}q_b + L_{ab}p_b.$$

Quantum =Classical

- $\dot{q}_a = K_{ab}p_b + L_{ab}q_b, \quad \dot{p}_a = -K_{ab}q_b + L_{ab}p_b.$ are classical Hamiltonian equations with the Hamiltonian

$$H_{sym} = \frac{1}{2}K_{ab}(p_a p_b + q_a q_b) + L_{ab}p_a q_b.$$

- Diagonalization of H by unitary transformation is equivalent to the diagonalization of H_{sym} by symplectic orthogonal transformation. We get

$$H_{sym} = \frac{1}{2}\sum_{a=1}^{N} \lambda_a(\xi_a^2 + \eta_a^2)$$

Ergodicity

- **Definition. A finite-dimensional quantum dynamical system is called ergodic if the associated classical dynamical system $\{M, \varphi_t, \mu\}$ on the surface of the integrals of motion is ergodic.**

- **The time average for an integrable function f coincides with the spatial average almost everywhere**

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\varphi_t(x))dt = \int_M f d\mu.$$

Ergodicity

- Constants of motion $F_a = \xi_a^2 + \eta_a^2, a = 1, \ldots N.$
- In the action-angle variables a linear flow on a torus:
  $\dot{J}_a = 0, \quad \dot{\theta}_a = \lambda_a \ (\text{mod } 1), \quad a = 1, \ldots N$
- Theorem. A finite-dimensional quantum dynamical system with rationally independent eigenvalues of the Hamiltonian is ergodic.
- Thus, in a finite-dim. Hilbert space, almost any quantum dynamical system is ergodic.

Conclusion for points ii) & iii)

- Euler-Bernoulli is equivalent to Schroedinger equation.
- In a finite-dimensional Hilbert space, almost all quantum dynamical systems are ergodic.

QM over real numbers

- QM and real numbers: Birkhoff, von Neumann, Stueckelberg,...

Recent: **M. O. Renou et al, "Quantum theory based on real numbers can be experimentally falsified," Nature (2021)**

Trushechkin et al, "Quantum mechanics based on real numbers: A consistent description," 2503.17307

T. Hoffreumon and M. P. Woods, "Quantum theory does not need complex numbers," 2504.02808 .

T. Feng, C. Ren and V. Vedral, "Locality Implies Complex Numbers in Quantum Mechanics," 2504.07808.

Quantum Mechanics in Real Kähler Space

- In this talk: formulation of real QM in real Kähler space
- Proof of the equivalence of Real Kähler QM to QM in Hilbert space
- Based on: I.V. "Real Quantum Mechanics in a Kähler Space," arXiv:2504.16838

Hilbert Space & Kähler Space

A Kähler space is a real Hilbert space equipped with a symplectic form and automorphism called complex structure.
A Kähler space is a quadruplet

$$(\mathcal{K}, g, \omega, \mathcal{J})$$

$(\mathcal{K}, g, \omega, \mathcal{J})$

- $\mathcal{K}$ is a real vector space,
- $g : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ is a positive defined the linear form,
- $\omega : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ is a a symplectic,
- $J : \mathcal{K} \to \mathcal{K} \quad J^2 = -\mathrm{id}$ is a complex structure,
- $g(\cdot, \cdot) = \omega(\,\cdot\,, J\,\cdot\,)$ and $\omega(J\,\cdot\,, J\,\cdot\,) = \omega(\,\cdot\,, \cdot\,)$,

$(\mathcal{K}, g, \omega, \mathcal{J})$

One can assume that $(\mathcal{K}, g)$ is a real Hilbert space and

$$\mathcal{K} = \mathrm{K} \oplus \mathrm{K}, \quad \psi \Leftrightarrow \begin{pmatrix} q \\ p \end{pmatrix}$$

$$\text{then } \mathcal{J} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -p \\ q \end{pmatrix}, \quad p, q \in \mathrm{K},$$

K is a real vector space

From Hilbert to Kähler

- **If** $\langle .,. \rangle$ **is inner product in the complex Hilbert space then**

$$\langle .,. \rangle = \mathbf{g}(.,.) + \mathbf{i}\omega(.,.)$$

- $g(.,.)$ – **positive defined**

  $\omega(.,.)$ – **skew-symmetric**

γ & γ^{-1}

- $\psi \xrightarrow{\gamma^{-1}} \begin{pmatrix} q \\ p \end{pmatrix}, \qquad \gamma^{-1}\psi = \begin{pmatrix} q \\ p \end{pmatrix} =$

  $= q|+\rangle + p|-\rangle, \; |+\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |-\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- $\gamma \begin{pmatrix} q \\ p \end{pmatrix} = q + ip = \psi$

From Kähler to Hilbert

Let $\mathcal{K} = \boldsymbol{K} \oplus \boldsymbol{K}$

Define $\psi \in \boldsymbol{\mathcal{H}}$ as $\psi = \boldsymbol{q} + i\boldsymbol{p}, \quad \boldsymbol{q}, \boldsymbol{p} \in \boldsymbol{K}$.

The inner product on $\boldsymbol{\mathcal{H}}$

$$\begin{aligned} \langle \psi_1, \psi_2 \rangle &= \boldsymbol{g}\big((\boldsymbol{q}_1, \boldsymbol{p}_1), (\boldsymbol{q}_2, \boldsymbol{p}_2)\big) \\ &+ i\,\boldsymbol{\omega}\big((\boldsymbol{q}_1, \boldsymbol{p}_1), (\boldsymbol{q}_2, \boldsymbol{p}_2)\big) \end{aligned}$$

$\boldsymbol{g}$ and $\boldsymbol{\omega}$ are the metric and symplectic form on $\mathcal{K}$

$(g, \omega, \mathcal{J})$ on $\mathbb{R}^{2N} = \mathbb{R}^N \oplus \mathbb{R}^N$

- the scalar product on $\mathbb{R}^{2N}$

$$g\left(\begin{pmatrix} q_1 \\ p_1 \end{pmatrix}, \begin{pmatrix} q_2 \\ p_2 \end{pmatrix}\right) = \sum_{a=1}^{N} (q_{1,a} q_{2,a} + p_{1,a} p_{2,a})$$

- the symplectic form on $\mathbb{R}^{2N}$

$$\omega\left(\begin{pmatrix} q_1 \\ p_1 \end{pmatrix}, \begin{pmatrix} q_2 \\ p_2 \end{pmatrix}\right) = \sum_{a=1}^{N} (q_{1,a} p_{2,a} - q_{2,a} p_{1,a})$$

- the complex structure $\mathcal{J}$

$$\mathcal{J}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -p \\ q \end{pmatrix}$$

Consistency condition on (g, ω, J)

- One can check explicitly

$$\langle \psi_1, \psi_2 \rangle = g(\gamma^{-1}\psi_1, \gamma^{-1}\psi_2) + i\omega(\gamma^{-1}\psi_1, \gamma^{-1}\psi_2)$$

Tensor product & Paradox

- $$\mathbb{C} = \mathbb{R}^2,$$
- $$\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$$
- $$\mathbb{R}^2 \otimes \mathbb{R}^2 = \mathbb{R}^4$$
- $$\mathbb{R}^2 =?\mathbb{R}^4$$

Operators

Let $\boldsymbol{L} : \mathbb{C}^{\boldsymbol{N}} \to \mathbb{C}^{\boldsymbol{N}}$ be a linear operator.

An associated operator $\boldsymbol{\mathcal{L}} : \mathbb{R}^{\boldsymbol{2N}} \to \mathbb{R}^{\boldsymbol{2N}}$ which is defined by using

$$\langle\gamma(\cdot), \boldsymbol{L}\gamma(\cdot)\rangle = \boldsymbol{g}(\cdot, \boldsymbol{\mathcal{L}}\cdot) + \boldsymbol{i}\boldsymbol{\omega}(\cdot, \boldsymbol{\mathcal{L}}\cdot).$$

$\mathcal{K}$-Hermitian operators

Hermitian $\boldsymbol{L}$ in $\mathbb{C}^{\boldsymbol{N}} \Leftrightarrow \mathcal{K}$-Hermitian $\boldsymbol{\mathcal{L}}$

Hermiticity $\langle \boldsymbol{\psi}, \boldsymbol{L}\boldsymbol{\phi} \rangle = \langle \boldsymbol{L}\boldsymbol{\psi}, \boldsymbol{\phi} \rangle$

$$\Downarrow$$

$$\begin{aligned} \boldsymbol{g}\left(\boldsymbol{\mathcal{L}}(\boldsymbol{q},\boldsymbol{p}),(\boldsymbol{r},\boldsymbol{s})\right) &= \boldsymbol{g}\left((\boldsymbol{q},\boldsymbol{p}),\boldsymbol{\mathcal{L}}(\boldsymbol{r},\boldsymbol{s})\right) \\ \boldsymbol{\omega}\left(\boldsymbol{\mathcal{L}}(\boldsymbol{q},\boldsymbol{p}),(\boldsymbol{r},\boldsymbol{s})\right) &= \boldsymbol{\omega}\left((\boldsymbol{q},\boldsymbol{p}),\boldsymbol{\mathcal{L}}(\boldsymbol{r},\boldsymbol{s})\right) \end{aligned}$$

$$\Downarrow$$

$$\boldsymbol{\mathcal{L}}^{\mathrm{T}} = \boldsymbol{\mathcal{L}}, \quad \boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{J}} = \boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{L}}$$

Quantum Mechanics in Real Kähler Space

- i) To a physical system one assigns a real Kähler space $\mathcal{K}$;
  its state is represented by vectors $\eta \in \mathcal{K}$, $g(\eta, \eta) = 1$,
  $g(\cdot, \cdot)$ - an inner product in $\mathcal{K}$, (or a density matrix ρ)
- ii) To the observable L corresponds the $\mathcal{K}$- Hermitian operator $\mathcal{L}$, which spectrum is observable.

The spectral decomposition for $\mathcal{L}$:

$$\sum_{i=1}^{n} \lambda_i \mathcal{E}_i = \mathcal{L}, \quad \sum_{i=1}^{n} \mathcal{E}_i = \mathcal{I}, \quad \text{rank}(\mathcal{E}_i) \geq 2.$$

Quantum Mechanics in Real Kähler Space

- iii) Born rule: if we measure L in the normalized state η, the probability of obtaining result λ_i is given by $g(\eta, \mathcal{E}_i\eta)/\mathrm{rank}(\mathcal{E}_i)$.
- iv) the Kähler space $\mathcal{K}$ corresponding to the composition of two systems $\mathfrak{N}$ and $\mathfrak{M}$ is $\mathcal{K}_{\mathfrak{N}} \bigotimes_{\mathcal{K}} \mathcal{K}_{\mathfrak{M}}$.
- v*) A compact Lie group $\mathfrak{G}$ of internal symmetries is realized in the Kähler space $\mathcal{K}$ by symplectic orthogonal representation $\mathcal{U}(\mathfrak{g}), \mathfrak{g} \in \mathfrak{G}$.
- vi*) time and space - symplectic orthogonal representation of the Galilean group

To be continued in the next talk by I.Aref'eva

Conclusion

- **Euler-Bernoulli is equivalent to Schroedinger equation.**
- **In a finite-dim. Hilbert space, almost all quantum dynamical systems are ergodic.**
- **Formulation of real QM in real Kähler space and proof of the equivalence of real Kähler QM to QM in Hilbert space are given**