New slow-roll approximations for inflationary models with the Gauss-Bonnet term and nonminimal coupling

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based on

E.O. Pozdeeva, M.A. Skugoreva, A.V. Toporensky, S.Yu. Vernov, JCAP 09 (2024) 050, *arXiv:2403.06147*; JCAP 05 (2025) 081, *arXiv:2502.13008*

NONLINEARITY, NONLOCALITY AND ULTRAMETRICITY
Int. Conf. on the Occasion of Branko Dragovich 80-th Birthday
Belgrade, Serbia, 30.05.2025

MODELS WITH NONMINIMAL COUPLING

The action of a generic model with a nonminimally coupled scalar field ϕ,

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[F(\phi)R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi)\right],$$

includes the coupling function $F(\phi) > 0$ and the potential $V(\phi)$.
In the spatially flat Friedmann-Lemaître-Robertson-Walker metric with

$$ds^2 = -\,dt^2 + a^2(t)\,d\mathbf{x}^2\,,$$

the evolution equations are

$$3FH^2 = \frac{1}{2}\dot\phi^2 + V - 3F_{,\phi}\dot\phi H, \qquad (1)$$

$$2F\dot H = -\,\dot\phi^2 + F_{,\phi}\dot\phi H - F_{,\phi\phi}\dot\phi^2 - F_{,\phi}\ddot\phi, \qquad (2)$$

$$\ddot\phi + 3H\dot\phi + V_{,\phi} - 3F_{,\phi}\left(\dot H + 2H^2\right) = 0, \qquad (3)$$

where dots denote the time derivatives and $A_{,\phi} = \frac{dA(\phi)}{d\phi}$ for any A.

There are 2 ways to transform Eqs. (2) and (3) into a dynamical system. These equations are equivalent to

$$
\begin{aligned}
\dot{\phi} =& \psi, \\
\dot{\psi} =& \frac{1}{E}\left\{ -3F_{,\phi}\left[F_{,\phi\phi}+1\right]\psi^2 + 3\left[F_{,\phi}^2 - 2F\right]H\psi - 2F\left[V_{,\phi} - 6F_{\phi}H^2\right]\right\}, \\
\dot{H} =& \frac{1}{E}\left\{ -\left[F_{,\phi\phi}+1\right]\psi^2 + 4F_{,\phi}H\psi + F_{,\phi}\left[V_{,\phi} - 6F_{\phi}H^2\right]\right\},
\end{aligned}
$$

where

$$E = 3F_{,\phi}^2 + 2F.$$

We consider the e-folding number $N = \ln(a/a_e)$, where a_e is a constant, as a measure of time during inflation. Using the function

$$\chi(N) = \frac{d\phi}{dN} = \frac{\psi}{H},$$

we get Eq. (1) in the following form

$$H^2 = \frac{2V}{6F + 6F_{,\phi}\chi - \chi^2} \tag{4}$$

From system (3) and Eq. (4), we obtain the second order system:

$$\begin{aligned}
\frac{d\phi}{dN} =& \chi, \\
\frac{d\chi}{dN} =& \frac{1}{2EV}\left\{\left[2V\left(F_{,\phi\phi}+1\right)+F_{,\phi}V_{,\phi}\right]\chi^3\right. \\
& + 2\left[FV_{,\phi} - F_{,\phi}V\left(3F_{,\phi\phi}+7\right) - 3F_{,\phi}^2 V_{,\phi}\right]\chi^2 \\
& \left. +\ 6\left[3F_{,\phi}^2 V - 3FF_{,\phi}V_{,\phi} - 2FV\right]\chi + 12F\left(2F_{,\phi}V - FV_{,\phi}\right)\right\}.
\end{aligned} \tag{5}$$

The potential V appears as the first derivative of its logarithm only.

Another way to get a dynamical system is to introduce the function

$$Y=\frac{M_{\rm Pl}}{\sqrt{F}}\left(H+\frac{F_{,\phi}\dot\phi}{2F}\right). \tag{6}$$

and rewrite Eqs. (1)–(3) in the following form:

$$3M_{\rm Pl}^2Y^2=\frac{A}{2}\dot\phi^2+V_{\rm eff}, \tag{7}$$

$$\dot Y=-\frac{A\sqrt{F}}{2M_{\rm Pl}^3}\,\dot\phi^2, \tag{8}$$

$$\ddot\phi=-3\sqrt{\frac{F}{M_{\rm Pl}^2}}Y\dot\phi-\frac{A_{,\phi}}{2A}\dot\phi^2-\frac{V_{{\rm eff},\phi}}{A}, \tag{9}$$

where

$$V_{\rm eff}(\phi)=\frac{M_{\rm Pl}^4V}{F^2},\quad A(\phi)=\frac{M_{\rm Pl}^4}{F^2}\left(1+\frac{3F_{,\phi}^2}{2F}\right), \tag{10}$$

M.A. Skugoreva, A.V. Toporensky and S.Yu. Vernov, Phys. Rev. D **90** (2014) 064044 [arXiv:1404.6226],
A.Yu. Kamenshchik, E.O. Pozdeeva, A. Tribolet, A. Tronconi, G. Venturi and S.Yu. Vernov, Phys. Rev. D **110** (2024) 104011 [arXiv:2406.19762].

THE EINSTEIN FRAME

The conformal transformation

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu} \tag{11}$$

with $\Omega^2 = \frac{M_{\rm Pl}^2}{F(\phi)}$ gives the GR action with a minimally coupled scalar field. Let us express the Einstein frame Hubble parameter $\tilde{H}$ through Jordan frame variables,

$$\tilde{H} = \frac{d\ln(\tilde{a})}{d\tilde{t}} = \Omega^{-1}\left(H + \frac{d\ln\Omega}{dt}\right) = \frac{M_{\rm Pl}}{\sqrt{F}}\left(H + \frac{F_{,\phi}\dot{\phi}}{2F}\right) \equiv Y. \tag{12}$$

The last expression is nothing else, but the variable Y introduced in Eq. (6) without any connections to the Einstein frame.

SLOW-ROLL PARAMETERS

The slow-roll parameter

$$\tilde{\varepsilon}_1 = -\frac{1}{\tilde{H}^2}\frac{d\tilde{H}}{d\tilde{t}} = \varepsilon_1 + \frac{\zeta_1(1-\varepsilon_1)}{2+\zeta_1} + \frac{2\zeta_1\zeta_2}{(2+\zeta_1)^2} \approx \varepsilon_1 + \frac{1}{2}\zeta_1 \,, \qquad (13)$$

where

$$\varepsilon_1 = -\frac{\dot{H}}{H^2} = \frac{d\ln(H^{-1})}{dN} = -\frac{1}{2}\frac{d\ln(H^2)}{dN}\,, \qquad \varepsilon_n = \frac{\dot{\varepsilon}_{n-1}}{H\varepsilon_{n-1}} = \frac{d\ln(\varepsilon_{n-1})}{dN}\,,$$

$$\zeta_1 = \frac{\dot{F}}{HF} = \frac{d\ln(F)}{dN}, \qquad \zeta_n = \frac{\dot{\zeta}_{n-1}}{H\zeta_{n-1}} = \frac{d\ln(\zeta_{n-1})}{dN}\,.$$

So, there are two sets of the slow-roll parameters.
We get Eq. (7) in the following form

$$3M_{\rm Pl}^4H^2\left(1+\frac{1}{2}\zeta_1\right)^2 = \frac{AF}{2}\dot{\phi}^2 + FV_{\rm eff}. \qquad (14)$$

If $V(\phi) > 0$ and $F(\phi) > 0$ for all ϕ in some interval $\phi_1 < \phi \leqslant \phi_0$ and $\zeta_1(\phi_0) > -2$, then $\zeta_1(\phi) > -2$ for all ϕ in this interval.

INFLATIONARY PARAMETERS

The inflationary parameters are

$$r \approx 8\left|2\varepsilon_1 + \zeta_1\right| = 8\left|\frac{d(\ln(F/H^2))}{d\phi}\chi\right|, \tag{15}$$

$$n_s \approx 1 - 2\varepsilon_1 - \zeta_1 - \frac{2\varepsilon_1\varepsilon_2 + \zeta_1\zeta_2}{2\varepsilon_1 + \zeta_1} = 1 + \frac{r}{8} - \frac{d\ln(r)}{dN}, \tag{16}$$

$$A_s \approx \frac{2H^2}{\pi^2 F\, r}. \tag{17}$$

The inflationary parameters are constrained by the combined analysis of Planck, BICEP/Keck and other observations as follows[1]:

$$A_s = (2.10 \pm 0.03) \times 10^{-9}, \qquad n_s = 0.9654 \pm 0.0040\,, \qquad r < 0.028\,.$$

[1] G. Galloni, N. Bartolo, S. Matarrese, M. Migliaccio, A. Ricciardone and N. Vittorio, JCAP **04** (2023) 062 [arXiv:2208.00188].

In the simplest approximation, one assumes that all slow-roll parameters are negligibly small and get the following system of approximate equations,

$$H^2 \approx \frac{V}{3F}, \tag{18}$$

$$3H\dot{\phi} + V_{,\phi} - 6F_{,\phi}H^2 \approx 0. \tag{19}$$

The expressions of $\chi(\phi)$ and $\zeta_1(\phi)$ are as follows:

$$\chi(\phi) = \frac{\dot{\phi}H}{H^2} \approx 2F_{,\phi} - \frac{FV_{,\phi}}{V}, \tag{20}$$

$$\zeta_1(\phi) = \frac{F_{,\phi}}{F}\chi(\phi) \approx \frac{2F_{,\phi}^2}{F} - \frac{F_{,\phi}V_{,\phi}}{V}. \tag{21}$$

We come to the expression

$$\varepsilon_1(\phi) = -\frac{1}{2}\frac{dH^2}{d\phi}\chi(\phi) \approx \frac{FV_{,\phi} - 2F_{,\phi}V}{2V}\left(\frac{V_{,\phi}}{V} - \frac{F_{,\phi}}{F}\right). \tag{22}$$

A MORE ACCURATE APPROXIMATION

This approximation is derived using the transition between Jordan and Einstein frames. One obtains the following slow-roll equations[2]:

$$H^2(\phi) \approx \frac{V}{3F} \tag{23}$$

and

$$\chi(\phi) = \frac{\dot{\phi} H}{H^2} \approx \frac{2F\left(2F_{,\phi}V - FV_{,\phi}\right)}{V\left(2F + 3F_{,\phi}^2\right)}. \tag{24}$$

The slow-roll parameters $\varepsilon_1(\phi)$ and $\zeta_1(\phi)$ are

$$\varepsilon_1(\phi) \approx -\frac{F\left(2F_{,\phi}V - FV_{,\phi}\right)}{V(2F + 3F_{,\phi}^2)}\left(\frac{V_{,\phi}}{V} - \frac{F_{,\phi}}{F}\right), \tag{25}$$

$$\zeta_1(\phi) = \frac{F_{,\phi}}{F}\chi(\phi) \approx \frac{2F_{,\phi}\left(2F_{,\phi}V - FV_{,\phi}\right)}{V\left(2F + 3F_{,\phi}^2\right)}. \tag{26}$$

[2] D.I. Kaiser, Phys. Rev. D **52** (1995) 4295 [arXiv:astro-ph/9408044].

NEW APPROXIMATION I

Neglecting terms proportional to $\dot\phi^2$ and $\ddot\phi$, we reduce Eqs. (1) and (3) as follows

$$3H^2F \approx V - 3F_{,\phi}\dot\phi H, \tag{27}$$

$$3H\dot\phi + V_{,\phi} - 3F_{,\phi}(\dot H + 2H^2) \approx 0. \tag{28}$$

Substituting (27) and (2) into Eq. (28) and again neglecting terms proportional to $\dot\phi^2$ and $\ddot\phi$, we find

$$3H\dot\phi \approx -2\left(\frac{FV_{,\phi} - 2F_{,\phi}V}{2F + 3F_{,\phi}^2}\right), \tag{29}$$

Therefore,

$$H^2(\phi) \approx \frac{2FV - F_{,\phi}^2V + 2FF_{,\phi}V_{,\phi}}{3F(2F + 3F_{,\phi}^2)}. \tag{30}$$

$$\chi(\phi) = \frac{\dot\phi H}{H^2} \approx -\frac{2F(FV_{,\phi} - 2F_{,\phi}V)}{2FV - {F_{,\phi}}^2V + 2FF_{,\phi}V_{,\phi}}. \tag{31}$$

NEW APPROXIMATIONS II and III

These slow-roll approximations are based on system (7)–(9). Neglecting the proportional to $\dot{\phi}^2$ term in Eq. (7), we get

$$Y^2 \approx \frac{V_{\rm eff}}{3M_{\rm Pl}^2} = \frac{M_{\rm Pl}^2 V}{3F^2}\,. \tag{32}$$

Differentiating this equation over time and using Eq. (8), we obtain

$$\psi \approx -\frac{M_{\rm Pl} V_{{\rm eff},\phi}}{3YA\sqrt{F}} = -\frac{2M_{\rm Pl}\left(V_{,\phi}F - 2VF_{,\phi}\right)}{3Y\sqrt{F}\left(2F+3F_{,\phi}^2\right)} = \frac{-\,2\left(V_{,\phi}F - 2VF_{,\phi}\right)}{3H\left(1+\frac{\zeta_1}{2}\right)\left(2F+3F_{,\phi}^2\right)}\,.$$

and a linear algebraic equation for $\chi(\phi)$:

$$\chi \approx -\frac{2\left(V_{,\phi}F - 2VF_{,\phi}\right)}{3H^2\left(1+\frac{\zeta_1}{2}\right)\left(2F+3F_{,\phi}^2\right)} = -\frac{2F\left(V_{,\phi}F - 2VF_{,\phi}\right)}{V\left(2F+3F_{,\phi}^2\right)}\left(1+\frac{F_{,\phi}}{2F}\chi\right)\,.$$

We obtain the first-order differential equation that defines slow-roll dynamic of ϕ:

$$\chi = \frac{d\phi}{dN} = \frac{2F\left(2VF_{,\phi} - V_{,\phi}F\right)}{2VF + V_{,\phi}F_{,\phi}F + VF_{,\phi}^2}\,. \tag{33}$$

So, we find

$$\zeta_1(\phi) = \frac{2F_{,\phi}(2VF_{,\phi} - V_{,\phi}F)}{2VF + V_{,\phi}FF_{,\phi} + VF_{,\phi}^2}. \tag{34}$$

If we neglect the term proportional to ζ_1^2 in

$$H^2 = \frac{FY^2}{M_{\mathrm{Pl}}^2\left(1+\frac{1}{2}\zeta_1\right)^2} \approx \frac{V}{3F\left(1+\zeta_1\right)}, \tag{35}$$

then we come to **the Approximation II.**

If we do not neglect any terms in Eq. (35) and use Eq. (33), then we obtain **the Approximation III**:

$$H^2 \approx \frac{V}{3F}\left(1+\frac{1}{2}\zeta_1\right)^{-2} = \frac{\left(2FV + FF_{,\phi}V_{,\phi} + F_{,\phi}^2V\right)^2}{3FV\left(2F + 3F_{,\phi}^2\right)^2}\,, \tag{36}$$

- **The Approximation III** corresponds to the Einstein frame approximation, because all terms in Eq. (7) represent terms from the corresponding Friedmann equation in the Einstein frame.
- The function Y is the Hubble parameter in the Einstein frame. Equation (7) differs from the standard form of the Friedmann equation, because there is no transition to the Einstein frame scalar field, so that the kinetic term in Eq. (7) is a non-standard one.
- The time t is the cosmic time in the Jordan frame and a parametric one in the Einstein frame.
- The moments of the end of inflation in the Jordan and Einstein frames do not coincide, because

$$\tilde{\varepsilon}_1 = -\frac{1}{\tilde{H}^2}\frac{d\tilde{H}}{d\tilde{t}} \approx \varepsilon_1 + \frac{1}{2}\zeta_1 \,. \tag{37}$$

This means that despite the above-mentioned correspondence, calculations in the Approximation III give results different from ones obtained directly in the Einstein frame.

Let us consider the well-known inflationary model[3] with the induced gravity term and the fourth degree monomial potential,

$$F(\phi) = M_{\mathrm{Pl}}^2 + \xi\phi^2, \qquad V(\phi) = \frac{\lambda}{4}\phi^4, \tag{38}$$

where ξ and λ are dimensionless positive constants.
Note that the function $\phi(N)$, a solution of Eq. (5), as well as functions $\chi(\phi)$, $\varepsilon_i(\phi)$ and $\zeta_1(\phi)$ calculated in any slow-roll approximation do not depend on λ.
Hence, the values of inflationary parameters n_s and r do not depend on λ and this parameter is important to define the inflationary parameter A_s.
With an additional assumption that ϕ is the Standard model Higgs boson, one chooses $\xi = 17367$ and $\lambda = 0.05$.

[3] B.L. Spokoiny, *Phys. Lett. B* **147** (1984) 39;
A.O. Barvinsky and A.Yu. Kamenshchik, *Phys. Lett. B* **332** (1994) 270;
F.L. Bezrukov and M. Shaposhnikov, *Phys. Lett. B* **659** (2008) 703;
A.O. Barvinsky, A.Yu. Kamenshchik, and A.A. Starobinsky, *JCAP* **11** (2008) 021;
F. Bezrukov, A. Magnin, M. Shaposhnikov, and S. Sibiryakov, *JHEP* **01** (2011) 016;
F. Bezrukov, *Class. Quant. Grav.* **30** (2013) 214001

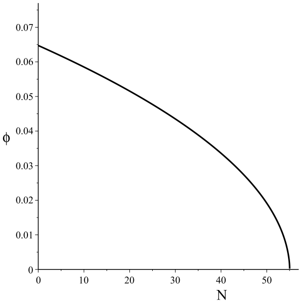

Figure: The evolution of $\phi(N)$ and $H^2(N)$ in units of M_{Pl} during inflation, obtained by the numerical integration of the system (5).

In Fig. 2, one can see the behavior of the slow-roll parameters ε_1 and ζ_1 as functions of ϕ.

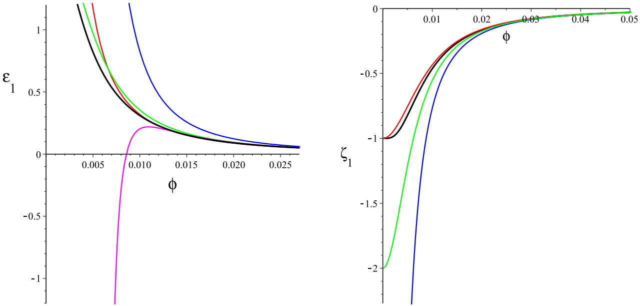

Figure: The functions $\varepsilon_1(\phi)$ (left) and $\zeta_1(\phi)$ (right). The black lines are the result of the numerical integration of the system (5), the blue curves are obtained in the known approximation, the red curves are obtained in the approximation I, the magenta curve is obtained in the approximation II and the green curves are obtained in the approximation III.

A more interesting and unexpected result is that new approximations I and III give essentially more precise values of the inflationary parameters r and A_s.

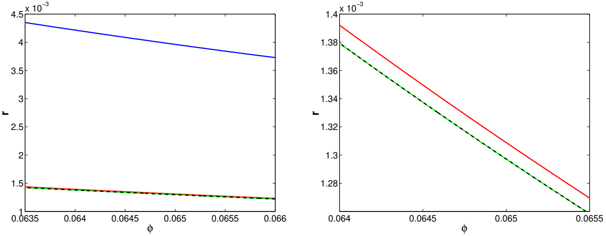

Figure: The dependence r of the scalar field ϕ for the model with the potential $V(\phi) = 0.0125\phi^4$ and the coupling function $F(\phi) = 1 + 17367\phi^2$. The black lines are the result of the numerical integration of the system (5), the blue curves are obtained in the known approximation, the red curves are obtained in the approximation I and the green curves are obtained in the approximation III.

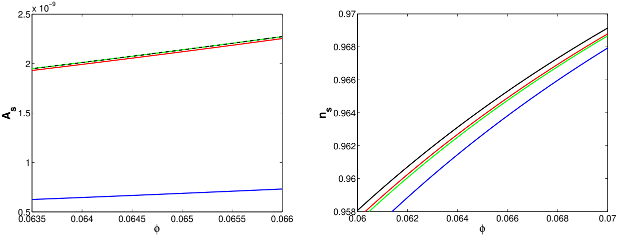

Figure: The functions $A_s(\phi)$ (left) and $n_s(\phi)$ (right). The black lines are the result of the numerical integration of the system (5), the blue curves are obtained in the known approximation, the red curves are obtained in the approximation I, and the green curves are obtained in the approximation III.

For other values of the model parameters:
$\xi = 1,\ \lambda = 2 \cdot 10^{-10}$
and
$\xi = 2.4 \cdot 10^{9},\ \lambda = 8 \cdot 10^{8}$
we get similar results.

MODELS WITH THE GAUSS–BONNET TERM

We consider models with the Gauss–Bonnet term, described by the following action:

$$S=\int d^4x\sqrt{-g}\left[U_0R-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)-\frac{1}{2}\xi(\phi)\mathcal{G}\right], \qquad (39)$$

where $U_0=\frac{M_{\mathrm{Pl}}^2}{2}=\frac{1}{16\pi G}$,
the functions $V(\phi)$ and $\xi(\phi)$ are differentiable ones,
R is the Ricci scalar and

$$\mathcal{G}=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2$$

is the Gauss–Bonnet term.

The perturbation theory for such types of models has been developed in
C. Cartier, J.c. Hwang, E.J. Copeland, Phys. Rev. D **64** (2001) 103504
J. c. Hwang and H. Noh, Phys. Rev. D **71** (2005) 063536

INFLATIONARY MODELS

Inflationary models with the Gauss–Bonnet term have been studied in many papers:

Z.K. Guo and D.J. Schwarz, Phys. Rev. D **81**, 123520 (2010)
A. De Felice, S. Tsujikawa, J. Elliston, R. Tavakol, JCAP **08** (2011) 021
M. De Laurentis, M. Paolella and S. Capozziello, Phys. Rev. D **91** (2015) 083531,
G. Hikmawan, J. Soda, A. Suroso, and F.P. Zen, Phys. Rev. D **93**, 068301 (2016)
C. van de Bruck and C. Longden, Phys. Rev. D **93** (2016) 063519
S. Koh, B.H. Lee and G. Tumurtushaa, Phys. Rev. D **95** (2017) 123509,
K. Nozari and N. Rashidi, Phys. Rev. D **95** (2017) 123518
S.D. Odintsov and V.K. Oikonomou, Phys. Rev. D **98** (2018) 044039
Z. Yi and Y. Gong, Universe **5** (2019) 200
E.O. Pozdeeva, *Eur. Phys. J.* C **80** (2020) 612
E.O. Pozdeeva, S.Yu. Vernov, Eur. Phys. J. C **81** (2021) 633
R. Kawaguchi and S. Tsujikawa, Phys. Rev. D **107** (2023) 063508
S.D. Odintsov, V.K. Oikonomou, F.P. Fronimos, Phys. Rev. D **107** (2023) 08
Yogesh, I.A. Bhat and M.R. Gangopadhyay, [arXiv:2408.01670].

EQUATIONS IN THE FLRW METRIC

In the spatially flat FLRW metric, one gets the evolution equations

$$12H^2\left(U_0-2\xi_{,\phi}\psi H\right) = \psi^2+2V, \quad (40)$$

$$4\dot{H}\left(U_0-2\xi_{,\phi}\psi H\right) = 4H^2\left(\xi_{,\phi\phi}\psi^2+\xi_{,\phi}\dot{\psi}-H\xi_{,\phi}\psi\right)-\psi^2, \quad (41)$$

$$\dot{\psi}+3H\psi = -V_{,\phi}-12H^2\xi_{,\phi}\left(\dot{H}+H^2\right). \quad (42)$$

Using the relation $\frac{d}{dt}=H\frac{d}{dN}$ and introducing $\chi=\frac{\psi}{H}$, we get

$$\frac{d\phi}{dN}=\chi,$$

$$\frac{d\chi}{dN}=\frac{1}{H^2\left(B-2\xi_{,\phi}H^2\chi\right)}\left\{3\left[3-4\xi_{,\phi\phi}H^2\right]\xi_{,\phi}H^4\chi^2\right.$$

$$\left.+\left[3B+2\xi_{,\phi}V_{,\phi}-6U_0\right]H^2\chi-\frac{V^2}{U_0}X\right\}-\frac{\chi}{2H^2}\frac{dH^2}{dN}, \quad (43)$$

$$\frac{dH^2}{dN}=\frac{H^2}{2\left(B-2\xi_{,\phi}H^2\chi\right)}\left[\left(4\xi_{,\phi\phi}H^2-1\right)\chi^2-16\xi_{,\phi}H^2\chi-4\frac{V^2}{U_0^2}\xi_{,\phi}X\right],$$

where $B=12\xi_{,\phi}^2H^4+U_0$ and $X=\frac{U_0^2}{V^2}\left(12\xi_{,\phi}H^4+V_{,\phi}\right)$.

SLOW-ROLL PARAMETERS

Following

Z. K. Guo and D. J. Schwarz, Phys. Rev. D **81** (2010) 123520,
C. van de Bruck and C. Longden, Phys. Rev. D **93** (2016) 063519,
E.O. Pozdeeva, M.R. Gangopadhyay, M. Sami, A.V. Toporensky, S.Yu. Vernov, Phys. Rev. D **102** (2020) 043525,

we consider the slow-roll parameters:

$$\varepsilon_1 = -\frac{\dot{H}}{H^2} = -\frac{d\ln(H)}{dN}, \qquad \varepsilon_{i+1} = \frac{d\ln|\varepsilon_i|}{dN}, \quad i \geqslant 1, \tag{44}$$

$$\delta_1 = \frac{2}{U_0}\xi_{,\phi}H\psi = \frac{2}{U_0}\xi_{,\phi}H^2\chi, \qquad \delta_{i+1} = \frac{d\ln|\delta_i|}{dN}, \quad i \geqslant 1. \tag{45}$$

It is useful, to rewrite evolution equations in terms of the slow-roll parameters. Equations (40) and (41) are equivalent to

$$12U_0H^2(1-\delta_1) = \psi^2 + 2V = \frac{U_0^2\delta_1^2}{4\xi_{,\phi}^2H^2} + 2V\,, \tag{46}$$

$$4U_0\dot{H}(1-\delta_1) = -\psi^2 + 2U_0H^2\delta_1(\delta_2+\varepsilon_1-1)\,.$$

INFLATIONARY PARAMETERS

The spectral index n_s and the tensor-to-scalar ratio r are connected with the slow-roll parameters as follows[4],

$$n_s = 1 - 2\varepsilon_1 - \frac{2\varepsilon_1\varepsilon_2 - \delta_1\delta_2}{2\varepsilon_1 - \delta_1} = 1 - 2\varepsilon_1 - \frac{d\ln(r)}{dN} = 1 + \frac{d}{dN}\ln\left(\frac{H^2}{U_0 r}\right), \quad (47)$$

$$r = 8|2\varepsilon_1 - \delta_1|. \quad (48)$$

The scalar perturbations amplitude

$$A_s = \frac{H^2}{\pi^2 U_0\, r}\,. \quad (49)$$

The inflationary parameters are constrained by the combined analysis of Planck, BICEP/Keck and other observations as follows[5]:

$$A_s = (2.10 \pm 0.03) \times 10^{-9}, \qquad n_s = 0.9654 \pm 0.0040\,, \qquad r < 0.028\,.$$

[4] Z.K. Guo and D.J. Schwarz, Phys. Rev. D **81** (2010), 123520 [arXiv:1001.1897]

[5] G. Galloni, N. Bartolo, S. Matarrese, M. Migliaccio, A. Ricciardone and N. Vittorio, JCAP **04** (2023) 062 [arXiv:2208.00188].

THE STANDARD APPROXIMATION

The standard approximate equations have been proposed in
Z.K. Guo, D.J. Schwarz, Phys. Rev. D **81** (2010), 123520
This way assumes that all inflationary parameters are negligibly small and can be removed from equations. In this slow-roll approximation, the leading order equations have the following form:

$$\begin{aligned} H^2 &\simeq \frac{V}{6U_0}, \\ \dot{H} &\simeq -\frac{\dot{\phi}^2}{4U_0} - \frac{\xi_{,\phi}H^3\dot{\phi}}{U_0}, \\ \dot{\phi} &\simeq -\frac{V_{,\phi}+12\xi_{,\phi}H^4}{3H}. \end{aligned} \tag{50}$$

THE STANDARD APPROXIMATION

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Z.K. Guo, D.J. Schwarz, Phys. Rev. D **81** (2010), 123520
This way assumes that all inflationary parameters are negligibly small and can be removed from equations. In this slow-roll approximation, the leading order equations have the following form:

$$\begin{aligned} H^2 &\simeq \frac{V}{6U_0}\,, \\ \dot{H} &\simeq -\frac{\dot{\phi}^2}{4U_0} - \frac{\xi_{,\phi} H^3 \dot{\phi}}{U_0}\,, \\ \dot{\phi} &\simeq -\frac{V_{,\phi} + 12\xi_{,\phi} H^4}{3H}. \end{aligned} \tag{50}$$

It has been shown by numerical calculations in
C. van de Bruck and C. Longden, Phys. Rev. D **93** (2016) 063519
that the model with the potential $V = V_0\phi^4$ and $\xi = \xi_0/V$, where V_0 and ξ_0 are positive constants, has no exit from inflation, whereas the standard slow-roll approximation shows that this exit does exist.
So, it is important to improve this slow-roll approximation.

QUADRATIC EQUATION IN H^2

Multiplying (46) to H^2 and substituting ψ in terms of the slow-roll parameter δ_1, we obtain:

$$12\,U_0\,(1-\delta_1)\,H^4 - 2VH^2 - \frac{\delta_1^2 U_0^2}{4\,\xi_{,\phi}^2} = 0\,.$$

We consider the positive H^2 at $\delta_1 < 1$:

$$H^2 = \frac{V}{12\,U_0\,(1-\delta_1)} + \frac{\sqrt{V^2\xi_{,\phi}^2 + 3\,U_0^3\delta_1^2\,(1-\delta_1)}}{12\,U_0\,(1-\delta_1)\,|\xi_{,\phi}|}\,. \qquad (51)$$

NEW APPROXIMATION I

We expand the obtained expression (51) to series with respect to the slow-roll parameter $\delta_1 \ll 1$:

$$H^2 \approx \frac{V}{6\,U_0} + \frac{V}{6\,U_0}\delta_1 + \mathcal{O}(\delta_1^2)\,. \tag{52}$$

For $\dot{H}$, we get

$$\dot{H} \simeq -\frac{H^2\delta_1}{2} - \frac{U_0\delta_1^2}{16\xi_{,\phi}^2 H^2} - \frac{H^2\delta_1^2}{2}\,. \tag{53}$$

We neglect terms proportional to $\ddot{\phi}$ and $\dot{\phi}^2$ in the field equation and get the following approximate equation:

$$\frac{3U_0\delta_1}{2\xi_{,\phi}} = -V_{,\phi} - 12H^2\xi_{,\phi}\left(\dot{H} + H^2\right). \tag{54}$$

Substituting H^2 and $\dot{H}$ into here and neglecting terms, proportional to δ_1^n, where $n \geqslant 2$, we get

$$\delta_1(\phi) = -\frac{2\,V^2\xi_{,\phi}\,V_{eff\,,\phi}}{V^2\xi_{,\phi}^2 + 3\,U_0^3}\,.$$

The knowledge of $\delta_1(\phi)$ allows us to obtain $H(\phi)$ and $\chi(\phi)$.

$$H^2 \simeq \frac{V}{6U_0}\left[1 - \frac{2V^2\xi_{,\phi}V_{eff\,,\phi}}{V^2\xi_{,\phi}^2 + 3U_0^3}\right] = \frac{V\left(9U_0^3 - 6U_0^2\xi_{,\phi}\,V_{,\phi} + \xi_{,\phi}^2V^2\right)}{18U_0\left(3U_0^3 + \xi_{,\phi}^2V^2\right)}. \tag{55}$$

$$\chi = \frac{d\phi}{dN} = \frac{U_0\delta_1}{2\xi_{,\phi}H^2} \simeq -\frac{6U_0^2VV_{eff\,,\phi}}{V^2\xi_{,\phi}^2 + 3U_0^3 - 2V^2\xi_{,\phi}V_{eff\,,\phi}}\,, \tag{56}$$

where

$$V_{eff}(\phi) = -\frac{U_0^2}{V(\phi)} + \frac{1}{3}\xi(\phi). \tag{57}$$

We get the slow-roll parameters as functions of ϕ:

$$\varepsilon_1(\phi) = -\frac{1}{2}\frac{d\phi}{dN}\frac{d\ln(H^2)}{d\phi}, \quad \varepsilon_2(\phi) = \frac{U_0\delta_1}{2\xi_{,\phi}H^2\varepsilon_1}\varepsilon_{1,\phi}\,, \quad \delta_2 = \frac{U_0}{2H^2\xi_{,\phi}}\delta_{1,\phi}.$$

NEW APPROXIMATION II

The second way to get $\delta_1(\phi)$ is the following.
We neglect the term proportional to δ_1^2 and get a nonzero solution:

$$H^2 = \frac{V}{6U_0(1-\delta_1)}. \tag{58}$$

Considering the differential of H^2 and using the definition of the slow-roll parameters, we get

$$\frac{dH^2}{dN} = \frac{V_{,\phi}\,\delta_1}{12\xi_{,\phi}H^2(1-\delta_1)} + \frac{V\delta_1\delta_2}{6U_0(1-\delta_1)^2} = \frac{U_0\,V_{,\phi}\,\delta_1}{2\xi_{,\phi}V} + \frac{V\delta_1\delta_2}{6U_0(1-\delta_1)^2}\,.$$

and

$$\varepsilon_1 = -\frac{3\,U_0^2\,V_{,\phi}}{2V^2\xi_{,\phi}}\delta_1(1-\delta_1) - \frac{\delta_1\delta_2}{2\,(1-\delta_1)}\,. \tag{59}$$

From definition of slow-roll parameters, we get

$$\dot{\psi} \approx \frac{U_0\delta_1}{2\xi_\phi}\left(\delta_2 + \varepsilon_1 - \frac{3U_0^2\xi_{,\phi\phi}\delta_1}{V\,\xi_{,\phi}^2}\right)\,. \tag{60}$$

Substituting H^2, ε_1, $\dot{\psi}$ into the field equation, multiplying it to $(1-\delta_1)^2$, and supposing that any products of the slow-roll parameters are negligible, we get

$$\delta_1(\phi) = -\frac{2\xi_{,\phi}\left(3U_0^2 V_{,\phi} + V^2\xi_{,\phi}\right)}{9U_0^2\left(U_0 - \xi_{,\phi}V_{,\phi}\right)}. \tag{61}$$

Now we can express H^2, χ, $N_{,\phi}$, and ε_1 via ϕ:

$$H^2(\phi) \simeq \frac{3U_0 V(U_0 - \xi_{,\phi}V_{,\phi})}{2(9U_0^3 - 3U_0^2\xi_{,\phi}V_{,\phi} + 2\xi_{,\phi}^2 V^2)}, \tag{62}$$

$$\chi = \frac{U_0\delta_1}{2\xi_{,\phi}H^2} \simeq -\frac{2(3U_0^2V_{,\phi} + \xi_{,\phi}V^2)(9U_0^3 - 3U_0^2\xi_{,\phi}V_{,\phi} + 2\xi_{,\phi}^2V^2)}{27U_0^2V(U_0 - \xi_{,\phi}V_{,\phi})^2}, \tag{63}$$

$$\frac{dN}{d\phi} = \chi^{-1}, \quad \varepsilon_1(\phi) = -\frac{\chi}{2}\frac{d\ln(H^2)}{d\phi}, \tag{64}$$

We propose models with the potential

$$V = V_0\phi^n, \tag{65}$$

where $n = 2$ or $n = 4$, and

$$\xi = \frac{CU_0^2}{V + \Lambda}, \tag{66}$$

where C and Λ are positive constants.
Such a modification eliminate a singular behavior at $\phi = 0$ and allowing the Universe to exit the inflationary epoch when ϕ becomes sufficiently small.
The initial value of the scalar field ϕ_0 is positive and ϕ tends to zero during inflation.

THE EFFECTIVE POTENTIAL

To analyze the stability of de Sitter solutions in model (39) the effective potential has been proposed[6]:

$$V_{eff}(\phi) = -\frac{U_0^2}{V(\phi)} + \frac{1}{3}\xi(\phi). \tag{67}$$

Calculating the derivative of the effective potential (67),

$$V_{eff\,,\phi} = \frac{U_0^2 n\left(V_0^2\,(3-C)\,\phi^{2\,n} + 6\,\Lambda\,V_0\phi^n + 3\,\Lambda^2\right)}{3V_0\phi^{n+1}\,(V_0\,\phi^n + \Lambda)^2}\,, \tag{68}$$

we find that $V_{eff\,,\phi} > 0$ for any $\phi > 0$ at $C < 3$.
It is a sufficient condition that a de Sitter solution does not exist at any $\phi > 0$.
This condition allows us to get an inflationary model without any fine-tuning of the initial data.

[6] E.O. Pozdeeva, M. Sami, A.V. Toporensky and S.Yu. Vernov, Phys. Rev. D **100** (2019) 083527.

FOURTH-ORDER POTENTIAL

We propose the model with the fourth-order potential $V = V_0\phi^4$.
For parameters

$$V_0 = 3.4 \times 10^{-11}, \quad C = 2.856, \quad \Lambda = 5.95 \times 10^{-13} M_{\rm Pl}^4 .$$

numeric calculations show that the inflation scenario does not contradict the current observation data.
We fix the number of e-folding to be equal $N = 60.6$ and get unappropriated results for the standard approximations.
New approximations work essentially better (see Table 1).

Table: **3. Numerical and approximate values of parameters, characterizing the inflationary dynamic in the model with the quartic potential.**

Parameter	Numeric result	Standard Approx	Approx I	Approx II
$\phi_0/M_{\rm Pl}$	1.4019	4.9705	1.4898	1.3974
$10^9 A_s(\phi_0)$	2.096	117.2	2.599	2.017
$n_s(\phi_0)$	0.965	0.953	0.965	0.965
$r(\phi_0)$	0.0044	0.0120	0.0045	0.0045
$\phi_{end}/M_{\rm Pl}$	0.2000	0.8899	0.3048	0.3037
$\delta_1(\phi_{end})$	0.885	1.80	4.23	0.577
$N(\phi_{end})$	60.6	60.6	60.6	60.6

$$\varepsilon_1(\phi_{end}) = 1. \tag{69}$$

Table: **4. Values of the inflationary parameters for the model with the quartic potential in different approximations.**

Parameter	Standard Approx	Approx I	Approx II
$\phi_{in}/M_{\rm Pl}$	2.5555	1.4104	1.4116
$10^9 A_s(\phi_{in})$	2.10	2.10	2.10
$n_s(\phi_{in})$	0.817	0.964	0.965
$r(\phi_{in})$	0.0466	0.0045	0.0045
$N(\phi_{end}) - N(\phi_{in})$	13.5	54.6	61.8

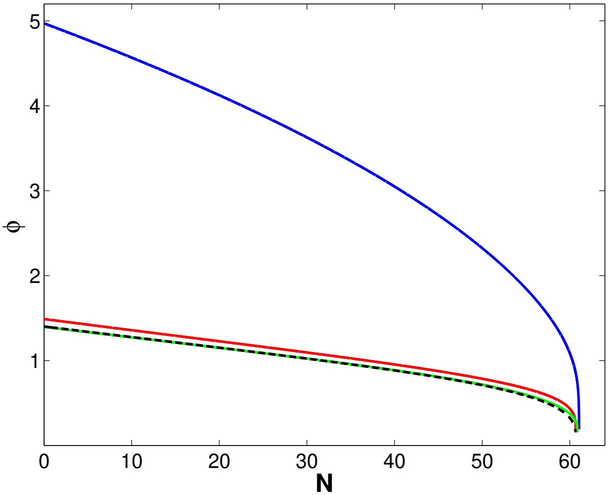

Figure: *4.* The inflationary model with $V(\phi) = V_0\phi^4$. Values of the function $\phi(N)$ in units of $M_{\rm Pl}$. The black line is the result of the numerical integration. The blue curve is obtained in the standard approximation, red — in the approximation I , green — in the approximation II. The initial values $\phi(0) = \phi_0$ are given in Table 1.

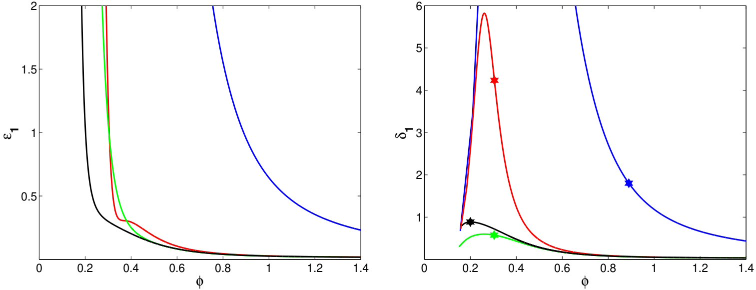

Figure: *5.* The slow-roll parameters $\varepsilon_1(\phi)$ (left panel) and $\delta_1(\phi)$ (right panel) for the model with $V(\phi) = V_0\phi^4$. The black line is the result of the numerical integration, blue curves are obtained in the standard approximation, red curves — in the approximation I, and green curves — in the approximation II. The stars denote the end of the inflation (when $\varepsilon_1 = 1$). Values of ϕ are given in units of M_{Pl}.

Similar results have been obtained for $V = m^2\phi^2$.

CONCLUSIONS

- The standard approach to analyze an inflationary model with a nonminimally coupled scalar field involves a conformal transformation of the metric and the construction of a corresponding model in the Einstein frame. Note that this method cannot be applied to more complicated modified gravity models, for example, to models with the Gauss–Bonnet term.
- We propose new slow-roll approximations for inflationary models with nonminimal coupling. We find more precise expressions for the slow-roll parameters as functions of the scalar field.
- To test the accuracy of the proposed approximations we consider the Higgs-driven inflationary model. The proposed versions of the slow-roll approximation not only give more accurate results at the end of inflation but also provide much more precise estimations for the tensor-to-scalar ratio r and the amplitude of scalar perturbations A_s.

CONCLUSIONS

- We propose new slow-roll approximations for inflationary models with the Gauss–Bonnet term. We find more accurate expressions for the standard slow-roll parameters as functions of the scalar field. The construction of a higher accuracy slow-roll approximation is based on the use of the function $H(\phi, \delta_1)$ rather than the function $H(\phi)$. To get $H(\phi)$ we need to obtain $\delta_1(\phi)$.
- To verify the accuracy of approximations considered we construct inflationary models with quadratic and quartic monomial potentials, $V = V_0\phi^n$, and the function $\xi = \frac{CU_0^2}{V+\Lambda}$. Numerical analysis of these models indicates that the proposed inflationary scenarios are consistent with observation data.
- The obtained numerical solutions have been compared with slow-roll solutions. As for the standard approximation, we show that it is not accurate enough to get correct values of inflationary parameters and correct number of e-folding during inflation. On the contrary, the proposed approximations give solutions that are close enough to the numerical solutions. The observation data calculated using these approximations remained within the allowed ranges.

CONCLUSIONS

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Thank you for your attention