Cosmological solutions of a nonlocal de Sitter $\sqrt{dS}$ gravity

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Motivation

Large cosmological observational findings:

- High orbital speeds of galaxies in clusters. (F.Zwicky, 1933)
- High orbital speeds of stars in spiral galaxies. (Vera Rubin, at the end of 1960es)
- Accelerated expansion of the Universe. (1998)

Problem solving approaches

There are two problem solving approaches:

- Dark matter and energy
- Modification of Einstein theory of gravity

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu},\ c = 1$$

where $T_{\mu\nu}$ is stress-energy tensor, $g_{\mu\nu}$ are the elements of the metric tensor, $R_{\mu\nu}$ is Ricci tensor and R is scalar curvature of metric.

Dark matter and energy

- If Einstein theory of gravity can be applied to the whole Universe then the Universe contains about 4.9% of ordinary matter, 26.8% of dark matter and 68.3% of dark energy.
- It means that 95.1% of total matter, or energy, represents dark side of the Universe, which nature is unknown.
- Dark matter is responsible for orbital speeds in galaxies, and dark energy is responsible for accelerated expansion of the Universe.

Modification of Einstein theory of gravity

Motivation for modification of Einstein theory of gravity

- The validity of General Relativity on cosmological scale is not confirmed.
- Dark matter and dark energy are not yet detected in the laboratory experiments.
- Another cosmological problem is related to the Big Bang singularity. Namely, under rather general conditions, general relativity yields cosmological solutions with zero size of the universe at its beginning, what means an infinite matter density.
- Note that when physical theory contains singularity, it is not valid in the vicinity of singularity and must be appropriately modified.

Approaches to modification of Einstein theory of gravity

There are different approaches to modification of Einstein theory of gravity.

- Einstein General Theory of Relativity

From action $S = \int (\frac{R}{16\pi G} - L_m - 2\Lambda)\sqrt{-g}d^4x$ using variational methods we get field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu},\ c = 1$$

where $T_{\mu\nu}$ is stress-energy tensor, $g_{\mu\nu}$ are the elements of the metric tensor, $R_{\mu\nu}$ is Ricci tensor and R is scalar curvature of metric.

Some of the approaches to modify gravity are:

- f(R) Modified Gravity
- Nonlocal Gravity

Nonlocal Modification of GR

Nonlocal gravity is a modification of Einstein general relativity in such way that Einstein-Hilbert action contains a function f(□, R). Our action is given by

$$S = \frac{1}{16\pi G} \int_M \Big(R - 2\Lambda + P(R)\mathcal{F}(\Box)Q(R) \Big) \sqrt{-g}\; d^4x,$$

where $\Box = \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu$, $\mathcal{F}(\Box) = \sum_{n=1}^{+\infty} f_n \Box^n + \sum_{n=1}^{+\infty} f_{-n}\, \Box^{-n}$.

We use Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)\big(\tfrac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\big),\ k \in \{-1, 0, 1\}.$$

Nonlocal Modification of GR: equations of motion

Equations of motion:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} P\, \mathcal{F}(\Box) Q + R_{\mu\nu} W - K_{\mu\nu} W + \frac{1}{2}\Omega_{\mu\nu} = 0,$$

where

$$W = P'(R)\, \mathcal{F}(\Box)\, Q(R) + Q'(R)\, \mathcal{F}(\Box) P(R), \quad K_{\mu\nu} = \nabla_\mu \nabla_\nu - g_{\mu\nu}\Box,$$

$$\Omega_{\mu\nu} = \sum_{n=1}^{+\infty} f_n \sum_{\ell=0}^{n-1} S_{\mu\nu}(\Box^\ell \mathcal{H}, \Box^{n-1-\ell}\mathcal{G})$$

$$- \sum_{n=1}^{+\infty} f_{-n} \sum_{\ell=0}^{n-1} S_{\mu\nu}(\Box^{-(\ell+1)}\mathcal{H}, \Box^{-(n-\ell)}\mathcal{G}),$$

$$S_{\mu\nu}(A, B) = g_{\mu\nu}\big(\nabla^\alpha A\, \nabla_\alpha B + A\Box B\big) - 2\nabla_\mu A\, \nabla_\nu B.$$

and P' (Q') means derivative of P (Q) with respect to scalar curvature R.

The case $Q(R) = P(R)$ and the corresponding EoM

$$G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{g_{\mu\nu}}{2} P(R)\mathcal{F}(\Box)P(R) + R_{\mu\nu}W - K_{\mu\nu}W + \frac{1}{2}\Omega_{\mu\nu} = 0,$$

$$W = 2P'(R)\,\mathcal{F}(\Box)\,P(R), \quad K_{\mu\nu} = \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box,$$

$$\Omega_{\mu\nu} = \sum_{n=1}^{+\infty} f_n \sum_{\ell=0}^{n-1} S_{\mu\nu}(\Box^\ell P, \Box^{n-1-\ell}P) - \sum_{n=1}^{+\infty} f_{-n} \sum_{\ell=0}^{n-1} S_{\mu\nu}(\Box^{-(\ell+1)}P, \Box^{-(n-\ell)}P).$$

Let

$$\Box P(R) = qP(R),$$

then

$$\Box^{-1}P(R) = q^{-1}P(R), \quad \mathcal{F}(\Box)P(R) = \mathcal{F}(q)P(R), \quad q \neq 0,$$

$$W = 2\mathcal{F}(q)P'P, \qquad \Omega_{\mu\nu} = \mathcal{F}'(q)S_{\mu\nu}(P, P),$$

$$S_{\mu\nu}(P, P) = g_{\mu\nu}\left(\nabla^\alpha P\,\nabla_\alpha P + P\Box P\right) - 2\nabla_\mu P\,\nabla_\nu P.$$

$\Box P(R) = qP(R)$, equations of motion

$$G_{\mu\nu}+\Lambda g_{\mu\nu}-\frac{g_{\mu\nu}}{2}\mathcal{F}(q)P^2+2\mathcal{F}(q)R_{\mu\nu}PP'-2\mathcal{F}(q)K_{\mu\nu}PP'+\frac{\mathcal{F}'(q)}{2}S_{\mu\nu}(P,P)=0.$$

The last equation transforms to

$$\left(G_{\mu\nu}+\Lambda g_{\mu\nu}\right)\left(1+2\mathcal{F}(q)PP'\right)+\mathcal{F}(q)g_{\mu\nu}\left(-\frac{1}{2}P^2+PP'(R-2\Lambda)\right)$$
$$-2\mathcal{F}(q)K_{\mu\nu}PP'+\frac{1}{2}\mathcal{F}'(q)S_{\mu\nu}(P,P)=0.$$

Model $P = \sqrt{R - 2\Lambda}$: equations of motion

Let now $P = \sqrt{R-2\Lambda}$, then $PP' = \frac{1}{2}$ and

$$\Box\sqrt{R-2\Lambda} = q\sqrt{R-2\Lambda} = \zeta\Lambda\sqrt{R-2\Lambda}, \quad \zeta\Lambda \neq 0,$$

where $q = \zeta\Lambda$ and $q^{-1} = \zeta^{-1}\Lambda^{-1}$.
Since $P = \sqrt{R-2\Lambda}$, EoM simplify to

$$(G_{\mu\nu} + \Lambda g_{\mu\nu})(1 + \mathcal{F}(q)) + \frac{1}{2}\mathcal{F}'(q)S_{\mu\nu}(\sqrt{R-2\Lambda}, \sqrt{R-2\Lambda}) = 0.$$

It is evident that EoM are satisfied if

$$\mathcal{F}(q) = -1 \quad \text{and} \quad \mathcal{F}'(q) = 0.$$

Model $P = \sqrt{R - 2\Lambda}$: two explicit forms for $\mathcal{F}(\Box)$

It is easy to prove that $\mathcal{F}(\Box)$ presented in the following simple symmetric form:

$$\mathcal{F}(\Box) = \sum_{n=1}^{+\infty} \tilde{f}_n \Big[\Big(\frac{\Box}{q}\Big)^n + \Big(\frac{q}{\Box}\Big)^n \Big] = -\frac{1}{2e}\Big(\frac{\Box}{q} e^{\frac{\Box}{q}} + \frac{q}{\Box} e^{\frac{q}{\Box}}\Big), \quad q \neq 0,$$

satisfies conditions

$$\mathcal{F}(q) = -1 \quad \text{and} \quad \mathcal{F}'(q) = 0,$$

and has $\sqrt{R - 2\Lambda}$ as its eigenfunction with eigenvalue -1, that is

$$-\frac{1}{2e}\Big(\frac{\Box}{q} e^{\frac{\Box}{q}} + \frac{q}{\Box} e^{\frac{q}{\Box}}\Big) \sqrt{R - 2\Lambda} = -\sqrt{R - 2\Lambda}$$
$$\text{whenever} \quad \Box \sqrt{R - 2\Lambda} = q \sqrt{R - 2\Lambda}.$$

In the sequel we will see that eigenvalue q is proportional to Λ, i.e. $q = \zeta \Lambda$, where $\zeta \neq 0$ is a definite dimensionless constant.

Model $P = \sqrt{R - 2\Lambda}$: two explicit forms for $\mathcal{F}(\Box)$

Since $q = \zeta\Lambda$ we have $\Box\sqrt{R-2\Lambda} = \zeta\Lambda\sqrt{R-2\Lambda}$.
Now, nonlocal operator $\mathcal{F}(\Box)$ can be rewritten as

$$\mathcal{F}(\Box) = \sum_{n=1}^{+\infty} \tilde{f}_n \Big[\Big(\frac{\Box}{\zeta\Lambda}\Big)^n + \Big(\frac{\zeta\Lambda}{\Box}\Big)^n \Big] = -\frac{1}{2e}\Big(\frac{\Box}{\zeta\Lambda} e^{\frac{\Box}{\zeta\Lambda}} + \frac{\zeta\Lambda}{\Box} e^{\frac{\zeta\Lambda}{\Box}}\Big), \quad \zeta\Lambda \neq 0.$$

This representation of $\mathcal{F}(\Box)$ by exponential function is not unique and can be written in the following more general form

$$\mathcal{F}(\Box) = -\frac{1}{2} e^{(\mp 1)} \left(\frac{\Box}{\zeta\Lambda}\, e^{\left(\pm\frac{\Box}{\zeta\Lambda}\right)} + \frac{\zeta\Lambda}{\Box}\, e^{\left(\pm\frac{\zeta\Lambda}{\Box}\right)} \right).$$

Nonlocal square root gravity model

$$S = \frac{1}{16\pi G} \int_M \sqrt{R - 2\Lambda}\, F(\Box)\, \sqrt{R - 2\Lambda}\, \sqrt{-g}\, d^4x,$$

where

$$F(\Box) = 1 + \mathcal{F}(\Box) = 1 + \mathcal{F}_+(\Box) + \mathcal{F}_-(\Box),$$

$$\mathcal{F}_+(\Box) = \sum_{n=1}^{+\infty} f_n\, \Box^n, \ \mathcal{F}_-(\Box) = \sum_{n=1}^{+\infty} f_{-n}\, \Box^{-n}.$$

- Construction

$$R - 2\Lambda = \sqrt{R - 2\Lambda}\,\sqrt{R - 2\Lambda} \to \sqrt{R - 2\Lambda}\, F(\Box)\, \sqrt{R - 2\Lambda}$$
$$= R - 2\Lambda + \sqrt{R - 2\Lambda}\, \mathcal{F}(\Box)\, \sqrt{R - 2\Lambda}$$

Cosmological solutions

Our primary interest is to find cosmological scale factor $a(t)$ that satisfies equation

$$\Box\sqrt{R-2\Lambda} = q\sqrt{R-2\Lambda}.$$

Since we use *FLRW* metric we have

$$\Box = -\frac{\partial^2}{\partial t^2} - 3H(t)\frac{\partial}{\partial t},$$

$$H(t) = \frac{\dot{a}}{a}, \quad R(t) = 6\left(\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2}\right).$$

If $a(t)$ is a solution of equation $\Box\sqrt{R-2\Lambda} = q\sqrt{R-2\Lambda}$, then it is also solution of equations of motion with the corresponding two conditions on nonlocal operator: $\mathcal{F}(q) = -1$ and $\mathcal{F}'(q) = 0$.

Cosmological solutions

Nonlocal gravity model

$$S = \frac{1}{16\pi G} \int_M (R - 2\Lambda + \sqrt{R - 2\Lambda}\, \mathcal{F}(\Box) \sqrt{R - 2\Lambda}) \sqrt{-g}\, d^4x,$$

has the following types of exact cosmological solutions:

1) Cosmological solutions in the flat universe $(k = 0)$:

- Solutions of the form $a(t) = A\, t^n\, e^{\gamma t^2}, \ (k = 0)$

$$a_1(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14} t^2}, \quad \mathcal{F}(-\frac{3}{7}\Lambda) = -1, \ \ \mathcal{F}'(-\frac{3}{7}\Lambda) = 0,$$

$$a_2(t) = A\, e^{\frac{\Lambda}{6} t^2}, \quad \mathcal{F}(-\Lambda) = -1, \ \ \mathcal{F}'(-\Lambda) = 0.$$

- Solutions of the form $a(t) = (\alpha\, e^{\lambda t} + \beta\, e^{-\lambda t})^{\gamma}, \ (k = 0)$

$$a_3(t) = A\, \cosh^{\frac{2}{3}}(\sqrt{\frac{3}{8}\Lambda}\, t), \quad \mathcal{F}(\frac{3}{8}\Lambda) = -1, \ \mathcal{F}'(\frac{3}{8}\Lambda) = 0,$$

$$a_4(t) = A\, \sinh^{\frac{2}{3}}(\sqrt{\frac{3}{8}\Lambda}\, t), \quad \mathcal{F}(\frac{3}{8}\Lambda) = -1, \ \mathcal{F}'(\frac{3}{8}\Lambda) = 0.$$

Cosmological solutions

- Solutions of the form $a(t) = (\alpha \ \sin \lambda t + \beta \ \cos \lambda t)^\gamma, \ (k = 0)$

$$a_5(t) = A\left(1 + \sin\left(\sqrt{-\frac{3}{2}\Lambda}\ t\right)\right)^{\frac{1}{3}}, \quad \mathcal{F}(\frac{3}{8}\Lambda) = -1, \ \mathcal{F}'(\frac{3}{8}\Lambda) = 0,$$

$$a_6(t) = A\left(1 - \sin\left(\sqrt{-\frac{3}{2}\Lambda}\ t\right)\right)^{\frac{1}{3}}, \quad \mathcal{F}(\frac{3}{8}\Lambda) = -1, \ \mathcal{F}'(\frac{3}{8}\Lambda) = 0,$$

$$a_7(t) = A\ \sin^{\frac{2}{3}}\left(\sqrt{-\frac{3}{8}\Lambda}\ t\right), \quad \mathcal{F}(\frac{3}{8}\Lambda) = -1, \ \mathcal{F}'(\frac{3}{8}\Lambda) = 0,$$

$$a_8(t) = A\ \cos^{\frac{2}{3}}\left(\sqrt{-\frac{3}{8}\Lambda}\ t\right), \quad \mathcal{F}(\frac{3}{8}\Lambda) = -1, \ \mathcal{F}'(\frac{3}{8}\Lambda) = 0.$$

Cosmological solutions

2) Cosmological solutions in the closed and open universe $(k = \pm 1)$:

- Cosmological solution of the form $a_9(t) = A\, e^{\pm\sqrt{\frac{\Lambda}{6}}t}$, $(k = \pm 1)$
- Solutions of the form $a(t) = (\alpha\, e^{\lambda t} + \beta\, e^{-\lambda t})^{\gamma}$, $(k = \pm 1)$

$$a_{10}(t) = A\, \cosh^{\frac{1}{2}}\, (\sqrt{\frac{2}{3}\Lambda}\, t), \quad \mathcal{F}(\frac{1}{3}\Lambda) = -1,\ \mathcal{F}'(\frac{1}{3}\Lambda) = 0,$$
$$a_{11}(t) = A\, \sinh^{\frac{1}{2}}\, (\sqrt{\frac{2}{3}\Lambda}\, t), \quad \mathcal{F}(\frac{1}{3}\Lambda) = -1,\ \mathcal{F}'(\frac{1}{3}\Lambda) = 0.$$

Cosmological solutions: Case $a_1(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14}t^2}$

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$$S = \frac{1}{16\pi G}\int_M \Big(R - 2\Lambda + \sqrt{R-2\Lambda}\,\mathcal{F}(\Box)\,\sqrt{R-2\Lambda}\Big)\sqrt{-g}\; d^4x,$$

where $\mathcal{F}(\Box) = \sum_{n=1}^{+\infty} f_n \Box^n + \sum_{n=1}^{+\infty} f_{-n}\, \Box^{-n}$.

Cosmological solution: $a_1(t) = A\, t^{\frac{2}{3}} e^{\frac{\Lambda}{14}t^2}, \quad k = 0, \;\; \Lambda \neq 0$

- $R = \frac{4}{3}t^{-2} + \frac{22}{7}\Lambda + \frac{12}{49}\Lambda^2 t^2$
- $\Box\sqrt{R-2\Lambda} = -\frac{3}{7}\Lambda\sqrt{R-2\Lambda}$
- $\mathcal{F}(-\frac{3}{7}\Lambda) = -1, \quad \mathcal{F}'(-\frac{3}{7}\Lambda) = 0$
- mimics dark matter $t^{\frac{2}{3}}$ and dark energy $e^{\frac{\Lambda}{14}t^2}$
- $H(t) = \frac{\dot{a}}{a} = \frac{2}{3}t^{-1} + \frac{1}{7}\Lambda t$ – Hubble parameter

Cosmological solutions: Case $a_1(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14} t^2}$

The EOM can be rewritten in the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu}.$$

Effective Friedmann equations are:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p}) + \frac{\Lambda}{3}, \quad \frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G}{3}\bar{\rho} + \frac{\Lambda}{3},$$

where $\bar{\rho}$ and $\bar{p}$ are counterparts of the energy density and pressure in the standard model of cosmology, respectively.

From the effective Friedmann equations we have

$$\bar\rho(t) = \frac{3}{8\pi G}\Big(\frac{\dot a^2 + k}{a^2} - \frac{\Lambda}{3}\Big), \quad \bar p(t) = \frac{1}{8\pi G}\Big(\Lambda - 2\frac{\ddot a}{a} - \frac{\dot a^2 + k}{a^2}\Big).$$

Then the equation of state is

$$\bar p(t) = \bar w(t)\, \bar\rho(t),$$

where $\bar w(t)$ is the corresponding effective state parameter.
Corresponding effective density and pressure for solutin $a_1(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14} t^2}$ are:

- $\bar\rho = \frac{2t^{-2} + \frac{9}{98}\Lambda^2 t^2 - \frac{9}{14}\Lambda}{12\pi G}$,
- $\bar p = -\frac{\Lambda}{56\pi G}\big(\frac{3}{7}\Lambda t^2 - 1\big)$.

We have $\bar w = \frac{\bar p}{\bar\rho} \to -1 \quad$ when $\quad t \to \infty$.

Planck data

The Planck 2018 data for the ΛCDM universe are:

- $H_0 = (67.40 \pm 0.50)$ km/s/Mpc – Hubble parameter;
- $\Omega_m = 0.315 \pm 0.007$ – matter density parameter;
- $\Omega_\Lambda = 0.685$ – Λ density parameter;
- $t_0 = (13.801 \pm 0.024) \cdot 10^9$ yr – age of the universe;
- $w_0 = -1.03 \pm 0.03$ – ratio of pressure to energy density.

Cosmological solutions: Case $a_1(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14} t^2}$

- Recall that $H_0 = \frac{2}{3} t_0^{-1} + \frac{1}{7} \Lambda_1 t_0$.
- Taking the above Planck results for t_0 and H_0 one obtains $\Lambda_1 = 1.05 \cdot 10^{-35}\, s^{-2}$ (in $c = 1$ units), that differs from Λ in ΛCDM model, where $\Lambda = 3 H_0^2 \Omega_\Lambda = 0.98 \cdot 10^{-35} s^{-2}$.
- One can also calculate time (t_m) for which the Hubble parameter has minimum value H_m, i.e. $t_m = 21.1 \cdot 10^9$ yr and $H_m = 61.72$ km/s/Mpc.

Cosmological solutions: Case $a_1(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14} t^2}$

Similarly, from

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{2}{9} t^{-2} + \frac{\Lambda}{3} + \frac{\Lambda^2 t^2}{49},$$

we get that accelerated expansion of the Universe started at $t_a = 7.84 \cdot 10^9 \, years$ or in other words $5.96 \cdot 10^9 \, years$ ago.

Cosmological solutions: Case $a_1(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14} t^2}$

We calculated the critical energy density ρ_c and the energy density of the dark matter $\bar{\rho}$:

$$\rho_c = \frac{3}{8\pi G} H_0^2 = 8.51 \cdot 10^{-30} \frac{g}{cm^3}$$

$$\bar{\rho} = \frac{3}{8\pi G}\Big(H_0^2 - \frac{\Lambda_1}{3}\Big) = \frac{3}{8\pi G}\Big(\frac{4}{9} t_0^{-2} - \frac{\Lambda_1}{7} + \frac{\Lambda_1^2}{49} t_0^2\Big) = 2.26 \cdot 10^{-30} \frac{g}{cm^3}$$

Note that energy density ρ in Einstein theory of gravity with Λ-term is

$$\rho = \frac{3}{8\pi G}\Big(H_0^2 - \frac{\Lambda}{3}\Big) = 2.68 \cdot 10^{-30} \frac{g}{cm^3}.$$

Cosmological solutions: Case $a_1(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14} t^2}$

Then we have

$$\rho - \bar{\rho} = \frac{\Lambda_1 - \Lambda}{8\pi G} = \rho_{\Lambda_1} - \rho_\Lambda = 0.42 \cdot 10^{-30} \frac{g}{cm^3},$$

where

$$\rho_{\Lambda_1} = \frac{\Lambda_1}{8\pi G} = 6.25 \cdot 10^{-30} \frac{g}{cm^3}, \quad \rho_\Lambda = \frac{\Lambda}{8\pi G} = 5.83 \cdot 10^{-30} \frac{g}{cm^3}$$

are vacuum energy density of background solution $a_1(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14} t^2}$ and ΛCDM model, respectively.

Cosmological solutions: Case $a_1(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14} t^2}$

We get

$$\Omega_{\Lambda_1} = \frac{\rho_{\Lambda_1}}{\rho_c} = 0.734, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = 0.685, \; \triangle\Omega_\Lambda = \Omega_{\Lambda_1} - \Omega_\Lambda = 0.049,$$
$$\Omega_m = \frac{\rho}{\rho_c} = 0.315, \quad \Omega_{m_1} = \frac{\bar{\rho}}{\rho_c} = 0.266, \; \triangle\Omega_m = \Omega_m - \Omega_{m_1} = 0.049.$$

We obtain that $\Omega_{m_1} = 26.6\%$ corresponds to dark matter and $\triangle\Omega_m = \triangle\Omega_\Lambda = 4.9\%$ is related to visible matter, what is in a very good agreement with the standard model of cosmology.

Concluding Remarks

- We point out nonlocal square root gravity model

$$S = \frac{1}{16\pi G} \int_M \Big(R - 2\Lambda + \sqrt{R - 2\Lambda}\, \mathcal{F}(\Box) \sqrt{R - 2\Lambda} \Big) \sqrt{-g}\; d^4x$$

where $\mathcal{F}(\Box) = -\frac{1}{2} e^{(\mp 1)} \left(\frac{\Box}{\zeta\Lambda}\, e^{\left(\pm \frac{\Box}{\zeta\Lambda}\right)} + \frac{\zeta\Lambda}{\Box}\, e^{\left(\pm \frac{\zeta\Lambda}{\Box}\right)} \right)$.

- We have found exact vacuum cosmological solution

$$a(t) = A\, t^{\frac{2}{3}} e^{\frac{\Lambda}{14} t^2}, \quad \Lambda \neq 0, \;\; k = 0$$

which mimics dark matter and dark energy.

- Computed cosmological parameters are in good agreement with observations.
- Eight new exact vacuum cosmological solutions were found by detail analysis of two classes of functions: $a(t) = (\alpha\, e^{\lambda t} + \beta\, e^{-\lambda t})^{\gamma}$ and $a(t) = (\alpha\, \sin \lambda t + \beta\, \cos \lambda t)^{\gamma}$.

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