Lebesgue-like measure and integration theory on the Levi-Civita field

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Outline for Section 2

- Let $\mathscr{R} = \{f : \mathbb{Q} \to \mathbb{R} | \text{supp}(f) \text{ is left-finite}\}$.
- For $x \in \mathscr{R}$, define

$$\lambda(x) = \begin{cases} \min(\text{supp}(x)) & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}.$$

- Arithmetic on $\mathscr{R}$: Let $x, y \in \mathscr{R}$. For $q \in \mathbb{Q}$, let

$$(x+y)[q] = x[q] + y[q]$$
$$(x \cdot y)[q] = \sum_{q_1+q_2=q} x[q_1] \cdot y[q_2].$$

  Then $x + y \in \mathscr{R}$ and $x \cdot y \in \mathscr{R}$.

Result: $(\mathscr{R}, +, \cdot)$ is a field.

Definition: $\mathscr{C} := \mathscr{R} + i\mathscr{R}$. Then $(\mathscr{C}, +, \cdot)$ is also a field.

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Order in $\mathscr{R}$

- Define the relation $\leq$ on $\mathscr{R} \times \mathscr{R}$ as follows:

$$x \leq y \text{ if } x = y \text{ or } (x \neq y \text{ and } (x-y)[\lambda(x-y)] < 0)\,.$$

- $(\mathscr{R}, +, \cdot, \leq)$ is an ordered field.
- $\mathscr{R}$ is real closed; and hence $\mathscr{C}$ is algebraically closed.
- The map $E : \mathbb{R} \to \mathscr{R}$, given by

$$E(r)[q] = \begin{cases} r & \text{if } q = 0 \\ 0 & \text{else} \end{cases},$$

is an order preserving embedding.
- There are infinitely small and infinitely large elements in $\mathscr{R}$: The number d, given by

$$d[q] = \begin{cases} 1 & \text{if } q = 1 \\ 0 & \text{else} \end{cases},$$

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For $x \in \mathscr{R}$, define

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases};$$

$$|x|_u = \begin{cases} e^{-\lambda(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

For $z = x + iy \in \mathscr{C}$, define

$$|z| = \sqrt{x^2 + y^2};$$

$$\begin{aligned} |z|_u &= \begin{cases} e^{-\lambda(z)} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \\ &= \max\{|x|_u, |y|_u\} \text{ since } \lambda(z) = \min\{\lambda(x), \lambda(y)\}. \end{aligned}$$

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Remarks:

- $|\cdot|$ and $|\cdot|_u$ induce the same topology τ_v on $\mathscr{R}$ (or $\mathscr{C}$). Moreover, $\mathscr{C}$ is topologically isomorphic to $\mathscr{R}^2$ provided with the product topology induced by $|\cdot|$ in $\mathscr{R}$.

- Given $x \neq 0$ in $\mathscr{R}$ (respectively, in $\mathscr{C}$), we can write $\text{supp}(x)$ as a strictly increasing sequence (q_n) that is either finite or otherwise diverges to ∞. Then

$$x = \sum_{n \in \mathbb{N}} x[q_n] d^{q_n},$$

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Uniqueness of $\mathscr{R}$ and $\mathscr{C}$

- $\mathscr{R}$ is the smallest Cauchy complete and real closed non-Archimedean field extension of $\mathbb{R}$.
  - It is small enough so that the $\mathscr{R}$-numbers can be implemented on a computer, thus allowing for computational applications.
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Outline for Section 3

Outer Measure

Definition: Let $A \subset \mathscr{R}$ be given. Then we say that A is outer measurable if

$$\inf \left\{ \sum_{n=1}^{\infty} l(I_n) : I_n\text{'s are intervals and } A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$$

exists in $\mathscr{R}$. If so, we call that number the outer measure of A and denote it by $m_u(A)$

Properties of the Outer Measure

- If $A \subset \mathscr{R}$ is outer measurable then there exists a sequence of sequences of pairwise disjoint intervals $\left(\{I_n^k\}_{n=1}^{\infty}\right)_{k=1}^{\infty}$ such that $\lim\limits_{k\to\infty} \sum\limits_{n=1}^{\infty} l(I_n^k) = m_u(A)$, and for all $k \in \mathbb{N}$, we have that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n^{k+1} \subseteq \bigcup_{n=1}^{\infty} I_n^k.$$

  We say that such a sequence outer-converges to A.

- If A, B and C are outer measurable sets in $\mathscr{R}$ such that $A \subseteq B \cup C$ then $m_u(A) \leq m_u(B) + m_u(C)$.

- If $A, B \subset \mathscr{R}$ are outer measurable with $m_u(B) = 0$ then, for any subset $C \subseteq B$, we have that $m_u(C) = 0$ and $m_u(A \setminus C) = m_u(A)$.

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Intervals and the Outer Measure

- If $A \subset \mathscr{R}$ is outer measurable and if $\{I_n\}_{n=1}^N$ are intervals in $\mathscr{R}$ then $A \cap \left(\bigcup_{n=1}^N I_n\right) = \bigcup_{n=1}^N (A \cap I_n)$ and $A \cap \left(\bigcup_{n=1}^N I_n\right)^c = A \setminus \bigcup_{n=1}^N I_n$ are outer measurable. Moreover, if $\{I_n\}_{n=1}^N$ are pairwise disjoint then

$$m_u\left(A \cap \left(\bigcup_{n=1}^N I_n\right)\right) = \sum_{n=1}^N m_u(A \cap I_n)$$

- If $A \subset \mathscr{R}$ is outer measurable and if $\{J_n\}_{n=1}^\infty$ is a sequence of pairwise disjoint intervals in $\mathscr{R}$ with $\lim_{n\to\infty} l(J_n) = 0$ then $A \cap \left(\bigcup_{n=1}^\infty J_n\right)$ is outer measurable and

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$$m_u\left(A \cup \left(\bigcup_{n=1}^{N} I_n\right)\right) = m_u(A) + \sum_{n=1}^{N} l(I_n)$$
- If $A \subset \mathscr{R}$ is outer measurable and if $\{I_n\}_{n=1}^{\infty}$ is a collection of intervals in $\mathscr{R}$ such that $\lim_{n\to\infty} l(I_n) = 0$ then $A \cup \left(\bigcup_{n=1}^{\infty} I_n\right)$ is outer measurable. Moreover, if A and the I_n's are pairwise disjoint then
$$m_u\left(A \cup \left(\bigcup_{n=1}^{\infty} I_n\right)\right) = m_u(A) + \sum_{n=1}^{\infty} l(I_n).$$
- If X is a dense subset of an interval I in $\mathscr{R}$ then X is outer measurable and $m_u(X) = l(I)$.

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Outline for Section 4

The Measure

Definition: Let $A \subset \mathscr{R}$ be an outer measurable set. Then we say that A is measurable if for every other outer measurable set $B \subset \mathscr{R}$ both $A \cap B$ and $A^c \cap B$ are outer measurable and

$$m_u(B) = m_u(A \cap B) + m_u(A^c \cap B).$$

In this case, we define the measure of A to be $m(A) := m_u(A)$.

Properties of the Measure

- If $A, B \subset \mathcal{R}$ are measurable then $A \cap B, A \cup B, A \cap B^c$ are measurable, with

$$m(A \cup B) = m(A) + m(B) - m(A \cap B).$$

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- For each $n \in \mathbb{N}$, let $A_n \subset \mathscr{R}$ be measurable.
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- For each $n \in \mathbb{N}$, let $A_n \subset \mathcal{R}$ be measurable.
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Outline for Section 5

Definition: Let $A \subseteq \mathscr{R}$ be measurable and let $f : A \to \mathscr{R}$ be bounded. We say that f is a simple function on A if for all $\epsilon > 0$ there exist some collection of mutually disjoint intervals $\{I_n\}_{n=1}^{\infty}$ (which we will call interval cover) and a bounded function $\hat{f} : \bigcup_{n=1}^{\infty} I_n \to \mathscr{R}$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$, $\sum_{n=1}^{\infty} l(I_n) - m(A) < \epsilon$, $\hat{f}$ is analytic on each I_n and for all $x \in A$, $f(x) = \hat{f}(x)$. We call such a function a simple extension of f over $\bigcup_{n=1}^{\infty} I_n$.

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Integral of a Simple Function

Theorem: Let $A \subseteq \mathscr{R}$ be measurable and let $f : A \to \mathscr{R}$ be simple. Then, the limit

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} \int_{I_n^k} \hat{f}\, dx$$

exists and is both independent of the choice of the simple extension $\hat{f}$ of f and of the sequence of intervals $\{I_n^k\}$ that converges to A; i.e. $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right) = m(A)$.

Definition: Let $A \subseteq \mathscr{R}$ be measurable and let $f : A \to \mathscr{R}$ be simple. We define

$$\int_A f\, dx := \lim_{k\to\infty} \sum_{n=1}^{\infty} \int_{I_n^k} \hat{f}\, dx$$

where $\hat{f}$ is a simple extension of f and $\{I_n^k\}$ is a sequence of interval coverings of A that converges to A.

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Outline for Section 6

Definition: Let $A \subseteq \mathscr{R}$ be measurable. We say that a collection $\{A_n\}_{n=1}^{\infty}$ of mutually disjoint measurable sets is a partition of A if $A = \bigcup_{n=1}^{\infty} A_n$ and $m(A_n) \underset{n\to\infty}{\to} 0$.

Definition: Let $A \subseteq \mathscr{R}$ be measurable and let $f : A \to \mathscr{R}$ be a function. We say that f is measurable if for every $\epsilon > 0$ there exists a partition $\{A_n\}_{n=1}^{\infty}$ of A and two collections of simple functions $\{i_n : A_n \to \mathscr{R}\}_{n=1}^{\infty}$, $\{s_n : A_n \to \mathscr{R}\}_{n=1}^{\infty}$ such that $i_n \le f \le s_n$ for each n, the series $\sum_{n=1}^{\infty} \int_{A_n} |s_n|\, dx$ and $\sum_{n=1}^{\infty} \int_{A_n} |i_n|\, dx$ both converge in $\mathscr{R}$ and

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Definition: Let $A \subseteq \mathscr{R}$ be measurable and let $f : A \to \mathscr{R}$ be a function. We say that f is measurable if for every $\epsilon > 0$ there exists a partition $\{A_n\}_{n=1}^{\infty}$ of A and two collections of simple functions $\{i_n : A_n \to \mathscr{R}\}_{n=1}^{\infty}$, $\{s_n : A_n \to \mathscr{R}\}_{n=1}^{\infty}$ such that $i_n \leq f \leq s_n$ for each n, the series $\sum_{n=1}^{\infty} \int_{A_n} |s_n|\, dx$ and $\sum_{n=1}^{\infty} \int_{A_n} |i_n|\, dx$ both converge in $\mathscr{R}$ and

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Properties of Measurable Functions

- If $A \subseteq \mathscr{R}$ and $f : A \to \mathscr{R}$ are measurable then f is locally bounded.
- If $A \subseteq \mathscr{R}$ and $f, g : A \to \mathscr{R}$ are measurable, and if $\alpha \in \mathscr{R}$ then $\alpha f + g$ is measurable on A.
- If $B \subseteq A \subseteq \mathscr{R}$ and $f : A \to \mathscr{R}$ are measurable then the restriction of f to B is measurable.
- If $A, B \subseteq \mathscr{R}$ are measurable and $f : A \cup B \to \mathscr{R}$ is a function then f is measurable on $A \cup B$ if and only if it is measurable on A and B.

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- If $A \subseteq \mathscr{R}$ and $f : A \to \mathscr{R}$ are measurable then $|f|$ is measurable.
- If $A \subseteq \mathscr{R}$ and $f, g : A \to \mathscr{R}$ are measurable then the functions $\min\{f, g\}$ and $\max\{f, g\}$ are measurable.
- If $A \subseteq \mathscr{R}$ is measurable and $f : A \to \mathscr{R}$ is a function then f is measurable if and only if $f_+ := \max\{f, 0\}$ and $f_- := \max\{-f, 0\}$ are measurable.
- If $A \subseteq \mathscr{R}$ is measurable and $f, g : A \to \mathscr{R}$ are measurable with g bounded on A then $f \cdot g$ is measurable.

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Properties of Measurable Functions

- If $A \subseteq \mathcal{R}$ and $f : A \to \mathcal{R}$ are measurable then the set

$$U_{f,A} := \left\{ \sum_{n=1}^{\infty} \int_{A_n} s_n \, dx : \{A_n\}_{n=1}^{\infty} \text{ is a partition of } A, \; s_n \text{ simple on } A_n, \right.$$

$$\left. \text{the series } \sum_{n=1}^{\infty} \int_{A_n} |s_n| \, dx \text{ converges in } \mathcal{R} \text{ and } f \leq s_n \right\}$$

has an infimum. Also, the set

$$L_{f,A} := \left\{ \sum_{n=1}^{\infty} \int_{A_n} i_n \, dx : \{A_n\}_{n=1}^{\infty} \text{ is a partition of } A, \; i_n \text{ simple on } A_n, \right.$$

$$\left. \text{the series } \sum_{n=1}^{\infty} \int_{A_n} |i_n| \, dx \text{ converges in } \mathcal{R} \text{ and } i_n \leq f \right\}$$

has a supremum. Moreover,

$$\inf(U_{f,A}) = \sup(L_{f,A}).$$

Outline for Section 7

The integral

Definition: Let $A \subseteq \mathscr{R}$ and $f : A \to \mathscr{R}$ be measurable. We define the integral of f over A to be

$$\int_A f\,dx := \inf(U_{f,A}) = \sup(L_{f,A}).$$

Properties of the Integral

- If $A \subseteq \mathscr{R}$ is measurable and if $\alpha \in \mathscr{R}$ then $\int_A \alpha\, dx = \alpha m(A)$.
- Linearity: If $A \subseteq \mathscr{R}$ and $f, g : A \to \mathscr{R}$ are measurable and if $\alpha \in \mathscr{R}$ then
$$\int_A (\alpha f + g)\, dx = \alpha \int_A f\, dx + \int_A g\, dx.$$
- If $A \subseteq \mathscr{R}$ is measurable with $m(A) = 0$ and $f : A \to \mathscr{R}$ is a function then f is measurable on A and $\int_A f\, dx = 0$.
- If $A \subseteq \mathscr{R}$ is measurable and $f, g : A \to \mathscr{R}$ are functions such that $f = g$ a.e. in A then, f is measurable on A if and only if g is measurable on A, in which case, we have $\int_A f\, dx = \int_A g\, dx$.
- If $A \subseteq \mathscr{R}$ and $f : A \to \mathscr{R}$ are measurable with $f \geq 0$ a.e. on A then $\int_A f\, dx \geq 0$. Moreover, $\int_A f\, dx = 0$ if and only if $f = 0$ a.e. on A.
- If $A \subseteq \mathscr{R}$ and $f : A \to \mathscr{R}$ are measurable with $|f| \leq M$ a.e. on A then $\left|\int_A f\, dx\right| \leq Mm(A)$.

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Properties of the Integral

- Additivity: If $A, B \subseteq \mathscr{R}$ are measurable with $B \subseteq A$ and if $f : A \to \mathscr{R}$ is measurable then

$$\int_A f\,dx = \int_B f\,dx + \int_{A \setminus B} f\,dx.$$

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- Countable Additivity: If $A \subseteq \mathscr{R}$ and $f : A \to \mathscr{R}$ are measurable and if $\{A_n\}_{n=1}^{\infty}$ is a partition of A then

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Properties of the Integral

Theorem: If $A \subseteq \mathscr{R}$ is measurable and $f : A \to \mathscr{R}$ is a function then f is measurable on A if and only if there exists some partition $\{A_n\}_{n=1}^{\infty}$ of A such that f is measurable and bounded on each A_n and the series

$$\sum_{n=1}^{\infty} \int_{A_n} |f| \, dx$$

converges. Furthermore,

$$\int_A f \, dx = \sum_{n=1}^{\infty} \int_{A_n} f \, dx.$$

Classical Theorems

Fundamental Theorem of Calculus: If $f : [a,b] \to \mathscr{R}$ is a measurable function that is continuous at $c \in [a,b]$ then $F : [a,b] \to \mathscr{R}$ given by

$$F(x) := \int_a^x f(t)\,dt$$

is differentiable at c and has derivative $F'(c) = f(c)$.

Uniform Convergence Theorem: If $(f_n : A \to \mathscr{R})_{n\in\mathbb{N}}$ is a sequence of measurable functions that converges uniformly to a function $f : A \to \mathscr{R}$ then f is measurable on A. Furthermore,

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Thank you for your attention!