# *Landscape of Yang Mills Theory Vacuum and Condensation of Magnetic Fluxes in QCD*

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*1. How large is the space of covariantly constant gauge fields*
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*2. Covariantly constant Yang Mills vacuum fields and condensation of magnetic fluxes*
*Phys.Lett.B* 852 (2024) 138612

*3. Landscape of QCD Vacuum*
*Phys.Lett.B* 862 (2025) 139337

4. The problem of strong CP violation in QCD, the $\Theta$ angle

# Jackiw-Rebbi vacuum gauge fields and strong CP violation

It is known that the vacuum state of the Yang Mills theory has a reach topological structure topological effect appeared due to the presence of gauge field configurations that cannot be continuously joined with the identity transformation

Jackiw, Rebbi, 't Hooft, Callan, Dashen, Gross

1976

Vacuum Periodicity in a Yang-Mills Quantum Theory

These flat field configurations $\vec{A}_n(\vec{x})$ have zero potential energies are of the form

$$\vec{A}_n(\vec{x}) = \frac{i}{g} U_n^-(\vec{x}) \nabla U_n(\vec{x}), \qquad U_1(\vec{x}) = \frac{\vec{x}^2 - \lambda^2 - 2i\lambda\vec{\sigma}\vec{x}}{\vec{x}^2 + \lambda^2}, \qquad U_n = U_1^n.$$

$$G_{ij}^a(\vec{A}_n) = 0, \qquad \epsilon(\vec{A}_n) = \frac{1}{4} G_{ij}^a G_{ij}^a = 0$$

$$w = \frac{1}{24\pi^2} \int d\mathbf{r}\, \varepsilon^{ijk}\, \mathrm{tr}(U^{-1}\partial_i U)(U^{-1}\partial_j U)(U^{-1}\partial_k U) = n$$

For gauge functions $U$ $w$ takes on integer values.

# *Jackiw-Rebbi vacuum gauge fields and strong CP violation*

$\vec{A}_1{}'(\vec{x}) = (\frac{1}{2} - \alpha)\vec{A}_1(\vec{x})$ when $\alpha$ is continuously varying from $-\frac{1}{2}$ to $\frac{1}{2}$.

the magnetic energy density $$\epsilon(r, \alpha) = \frac{1}{4} G^a_{ij} G^a_{ij} = \frac{6\lambda^4(1 - 4\alpha^2)}{g^2(r^2 + \lambda^2)^4}$$

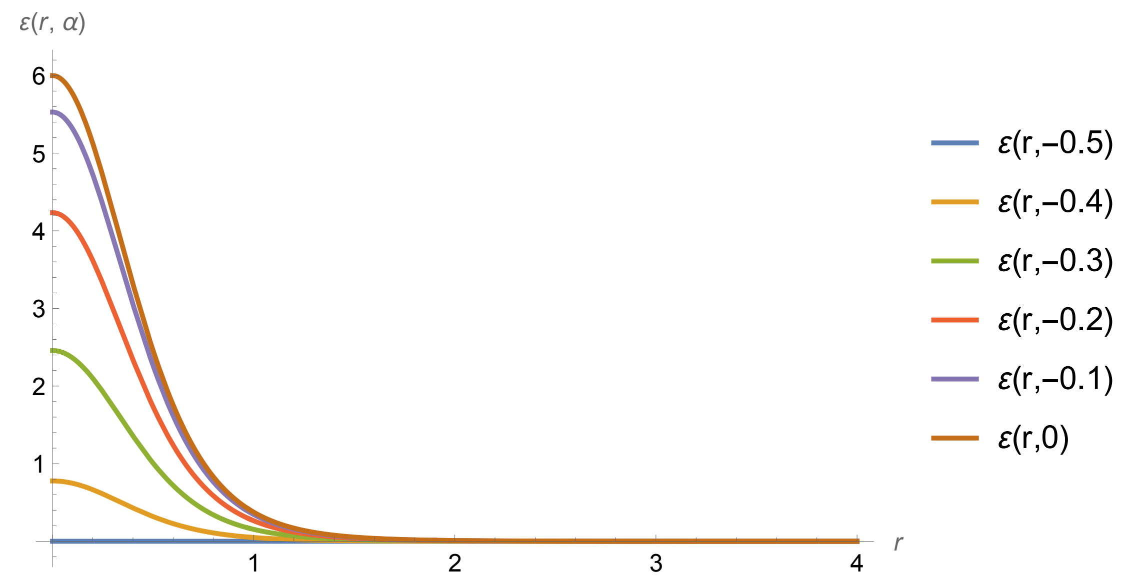
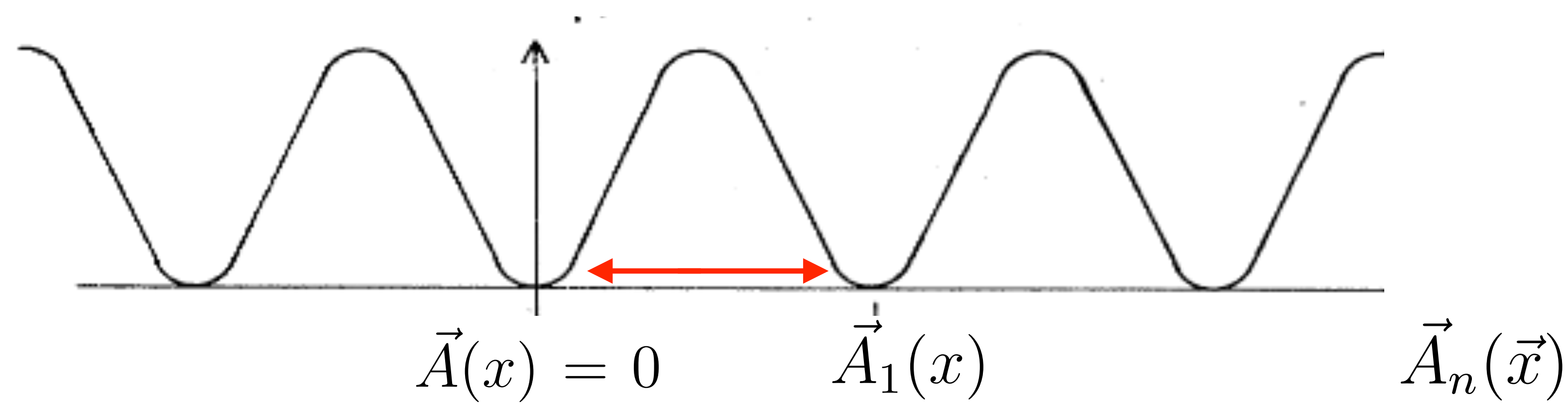



Figure 1: The graph shows the shape of the potential barrier between the Chern-Pontryagin vacua. As $\alpha$ increases, the hight of the barrier increases and reaches its maximum at $\alpha = 0$, then it symmetrically decreases until $\alpha = \frac{1}{2}$, where it again is equal to $\epsilon = 0$.

# *Jackiw-Rebbi QCD vacuum gauge fields and strong CP violation*

The values of the gauge field although gauge equivalent to $\vec{A}(x) = 0$, are not removed from the integration over the field configurations by gauge fixing procedure because they belong to different topological classes and are separated by potential barriers.




⟷ instanton tunnelling transitions between degenerate zero energy vacua

$$\epsilon(r, \alpha) = \frac{1}{4} G^a_{ij} G^a_{ij} = \frac{6\lambda^4(1 - 4\alpha^2)}{g^2(r^2 + \lambda^2)^4} \tag{5.51}$$

In the quantum theory tunnelling will occur across this barrier and the quantum-mechanical superposition $\Psi_\theta(\vec{A}) = \sum_n e^{in\theta}\psi_n(\vec{A})$ represents the Yang Mills $\theta$ vacuum state The induced Chern-Pontryagin $\theta$-angle term is Lorentz invariant, but breaks the CP invariants, so that the distinct $\theta$ vacuum states correspond to distinct theories

# New exact solution of the sourceless Yang Mills equation

$$A^a_\mu = \frac{1}{g}\begin{cases} (0,0,0) \\ a\Big(\frac{\sin by}{\sqrt{1-(ax)^2}}, -\frac{\cos by}{\sqrt{1-(ax)^2}}, 0\Big) \\ b\sqrt{1-(ax)^2}\Big(-ax\cos by, -ax\sin by, \sqrt{1-(ax)^2}\Big) \\ (0,0,0) \end{cases} \qquad (ax)^2 < 1$$

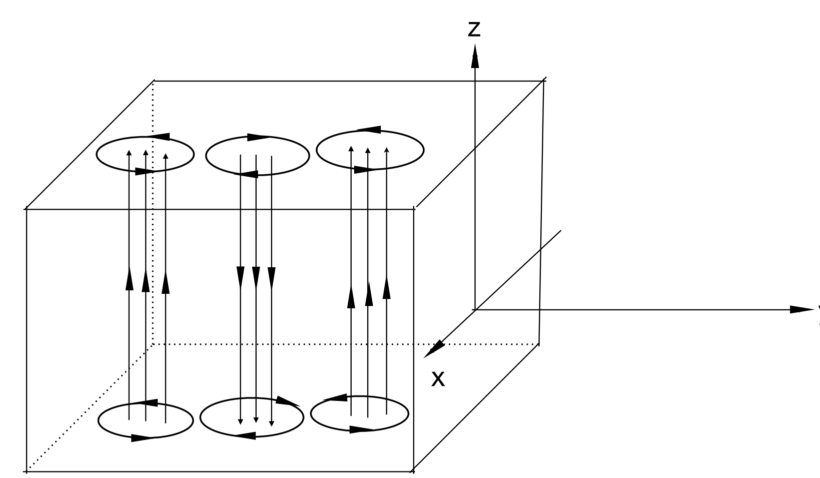
$$A^a_\mu = 0, \qquad (ax)^2 \geq 1,$$

**Field strength** $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\varepsilon^{abc} A^b_\mu A^c_\nu,$

$$G^a_{\mu\nu}(x,y) = \frac{ab}{g}\begin{cases} (0,0,0) \\ \Big(\sqrt{1-\mathrm{a}^2x^2}\cos(\mathrm{b}y), \sqrt{1-\mathrm{a}^2x^2}\sin(\mathrm{b}y), -\mathrm{a}x\Big), \\ (0,0,0), \\ (0,0,0) \end{cases}$$

## Chromo-magnetic energy

$$\epsilon = \frac{1}{4} G^a_{ij} G^a_{ij} = \frac{a^2 b^2}{2g^2}.$$




*non-perturbative magnetic sheet of a finite thickness* $2/|a|$,

in the direction transversal to the sheet the Poynting vector vanishes, $\vec{E}^a \times \vec{H}^a = 0.$

# Conserved current

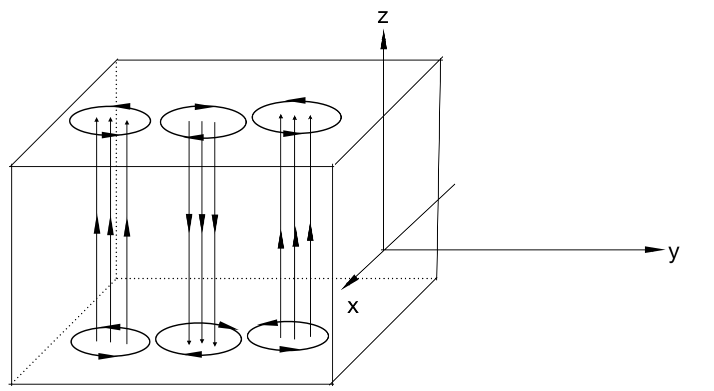
$J^a_\mu = g\epsilon^{abc} A^b_\nu G^c_{\nu\mu}$ on the solutions of the Yang Mills equation $\nabla^{ab}_\mu G^b_{\mu\nu} = 0.$

The non-vanishing components are:

$$J^1_1 = -\frac{ab^2}{g}\sqrt{1-(ax)^2}\cos bz, \qquad J^1_3 = -\frac{a^3bx}{g}\frac{\sin bz}{\sqrt{1-(ax)^2}},$$

$$J^2_1 = \frac{ab^2}{g}\sqrt{1-(ax)^2}\sin bz, \qquad J^2_3 = -\frac{a^3bx}{g}\frac{\cos bz}{\sqrt{1-(ax)^2}},$$

$$J^3_1 = 0, \qquad J^3_3 = \frac{a^2b}{g}.$$

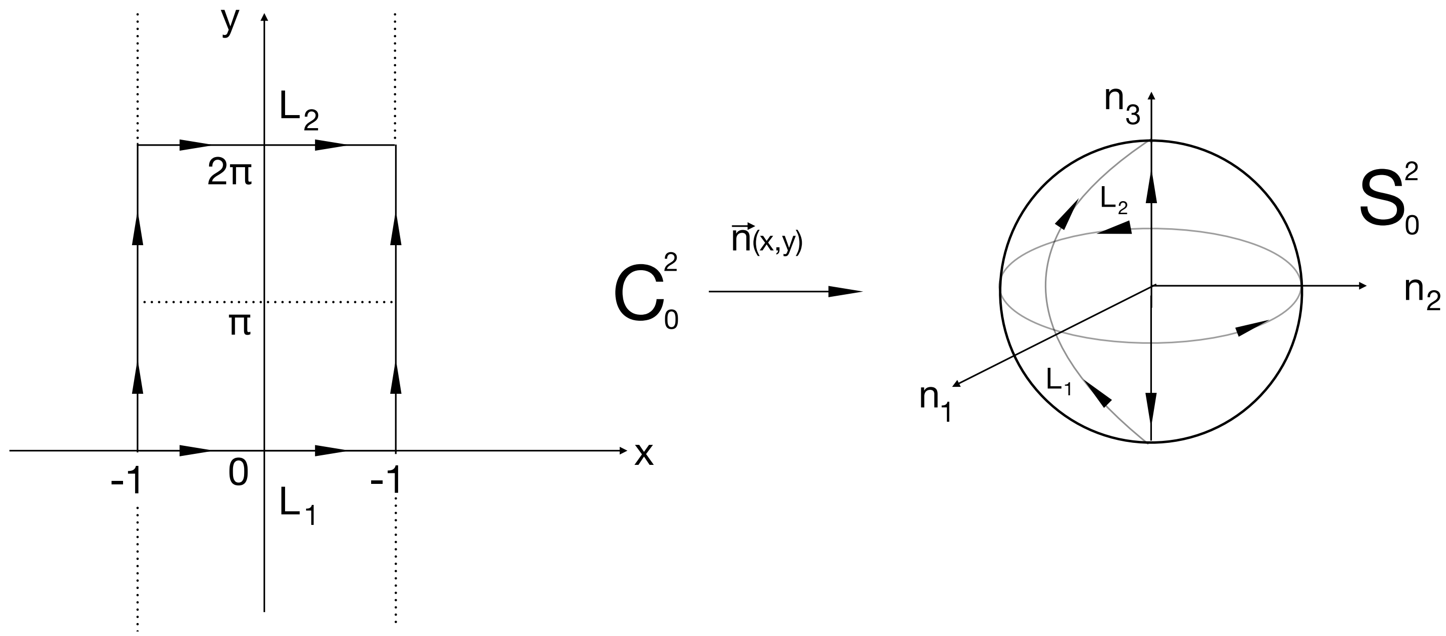



$$\partial_\mu J^a_\mu = \partial_x J^a_1 + \partial_y J^a_2 = 0$$

The current vorticity $\omega^a_i = \epsilon_{ijk}\partial_j J^a_k$

$$\omega^a_3 = \frac{1}{g}\frac{(ab-gH)(a^2+b^2(1-a^2x^2)^2)}{(1-(ax)^2)^{3/2}}\Big(\cos by, \sin by, 0\Big), \quad (ax)^2 < 1$$

# Chromo-magnetic Fluxes




The magnetic flux $$W(L) = \frac{1}{2} Tr P \exp\left(ig \oint_L A_k dx^k\right) = \cos\left(\frac{1}{2}\; g\; \Phi\right)$$

is equal to $\Phi = \frac{4\pi}{g}$ when a closed loop $L$ is surrounding any oriented magnetic flux tube

$$\Phi = \frac{4\pi}{g}$$

This solution is similar to the superposition of the Nielsen-Olesen magnetic flux tubes

# *Exact hyperbolic solution of the sourceless Yang Mills equation*

$$A_\mu^a = \frac{1}{g}\begin{cases} (0,0,0) \\ \left(\frac{\mathrm{a}\left(\mathrm{sech}^2(\mathrm{a}x)\cos\left(\mathrm{b}y\cosh^2(\mathrm{a}x)\right)+2\mathrm{b}y\tanh^2(\mathrm{a}x)\sin\left(\mathrm{b}y\cosh^2(\mathrm{a}x)\right)\right)}{\mathrm{sech}(\mathrm{a}x)},\right. \\ \left.\frac{2\mathrm{ab}y\tanh^2(\mathrm{a}x)\cos\left(\mathrm{b}y\cosh^2(\mathrm{a}x)\right)-\mathrm{a}\,\mathrm{sech}^2(\mathrm{a}x)\sin\left(\mathrm{b}y\cosh^2(\mathrm{a}x)\right)}{\mathrm{sech}(\mathrm{a}x)}, -2\mathrm{ab}y\tanh(\mathrm{a}x)\right) \\ \left(\mathrm{b}\sinh(\mathrm{a}x)\cosh(\mathrm{a}x)\mathrm{sech}(\mathrm{a}x)\sin\left(\mathrm{b}y\cosh^2(\mathrm{a}x)\right),\right. \\ \left.\mathrm{b}\sinh(\mathrm{a}x)\cosh(\mathrm{a}x)\mathrm{sech}(\mathrm{a}x)\cos\left(\mathrm{b}y\cosh^2(\mathrm{a}x)\right), -\mathrm{b}\right) \\ (0,0,0) \end{cases}$$

*Field strength* $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\varepsilon^{abc}A_\mu^b A_\nu^c,$

$$G_{\mu\nu}^a(x,y) = \frac{ab}{g}\begin{cases} (0,0,0), \\ \left(\frac{\sin(by\cosh^2(ax))}{\cosh(ax)}, \frac{\cos(by\cosh^2(ax))}{\cosh(ax)}, \frac{\sin(ax)}{\cosh(ax)}\right), \\ (0,0,0), \\ (0,0,0) \end{cases}$$

*Chromo-magnetic energy*

$$\epsilon = \frac{1}{4}G_{ij}^a G_{ij}^a = \frac{a^2b^2}{2g^2}.$$

# How large is the class of covariantly constant vacuum gauge fields

*how large is the class of covariantly constant gauge fields* defined by the equation

$$\nabla^{ab}_{\rho} G^{b}_{\mu\nu} = 0.$$

Savvidy, Duff, Brown, Ramon-Medrano

1977

$\nabla^{ab}_{\mu} G^{b}_{\mu\nu} = 0.$ By taking covariant derivative $\nabla^{ca}_{\lambda}$ $\quad [\nabla_{\lambda}, \nabla_{\rho}]^{ab} G^{b}_{\mu\nu} = 0.$

$$[G_{\lambda\rho}, G_{\mu\nu}] = 0.$$

the field strength tensor factorises into the product of Lorentz tensor $G_{\mu\nu}(x)$ and colour unit vector $n^a(x)$,

$$G^{a}_{\mu\nu}(x) = G_{\mu\nu}(x) n^{a}(x).$$

The solution has the following form

$$A^{a}_{\mu} = -\frac{1}{2} F_{\mu\nu} x_{\nu} n^{a},$$

## Consider the Ansatz

$$A_\mu^a = B_\mu n^a + \frac{1}{g}\varepsilon^{abc} n^b \partial_\mu n^c,$$

$$n^a n^a = 1,\ n^a \partial_\mu n^a = 0.$$

Cho, Faddeev, Niemi

$$\nabla_\mu^{ab} n^b = \partial_\mu n^a - g\varepsilon^{abc} A_\mu^b n^c = 0,$$

and therefore $[\nabla_\mu, \nabla_\nu]^{ab} n^b = 0.$

$$[\nabla_\mu, \nabla_\nu]^{ab} n^b = -g\varepsilon^{acb} G_{\mu\nu}^c n^b = 0.$$

It follows that the field strength tensor factorises

$$G_{\mu\nu}^a = \left(F_{\mu\nu} + \frac{1}{g} S_{\mu\nu}\right) n^a,$$

where

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \qquad S_{\mu\nu} = \varepsilon^{abc} n^a \partial_\mu n^b \partial_\nu n^c.$$

$$\partial_\rho \left(F_{\mu\nu} + \frac{1}{g} S_{\mu\nu}\right) = 0,$$

# General solution

$$A_\mu^a = B_\mu(x) n^a(x) + \frac{1}{g} \varepsilon^{abc} n^b(x) \partial_\mu n^c(x).$$

$$B_\mu = -\frac{1}{2} F_{\mu\nu} x_\nu$$

$$n^a(\vec{x}) = \{\sin f(X) \cos\left(\frac{Y}{f'(X)\sin(f(X))}\right),\ \sin(f(X))\sin\left(\frac{Y}{f'(X)\sin(f(X))}\right),\ \cos(f(X))\}.$$

$$X = a_1 x + a_2 y + a_3 z + a_0 t, \qquad Y = b_1 x + b_2 y + b_3 z + b_0 t,$$

The equation defines the general solution which depends on an arbitrary function $f(X)$.

Our aim is to describe the moduli space of the covariantly constant gauge fields defined by the equations and investigate their physical properties.

# Properties of General solution

The square of the field strength tensor is

$$\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} = \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{a_\mu F_{\mu\nu} b_\nu}{g} + \frac{a^2 b^2 - (a\cdot b)^2}{2g^2}.$$

where $a_\mu$ and $b_\nu$ are arbitrary constant Lorentz vectors

The magnetic energy density can be represented in the following form:

$$\epsilon = \frac{\vec{H}^2}{2} - \frac{1}{g}\vec{H}\cdot(\vec{a}\times\vec{b}) + \frac{1}{2g^2}(\vec{a}\times\vec{b})^2.$$

Let us consider the solution when $B_\mu = F_{\mu\nu} = 0$, so that

$$A^a_\mu = \frac{1}{g}\varepsilon^{abc} n^b \partial_\mu n^c, \qquad \epsilon = \frac{1}{2g^2}(\vec{a}\times\vec{b})^2.$$

# Properties of General solution

Let us considering the vectors $a_\mu = (0, a, 0, 0)$ and $b_\nu = (0, 0, b, 0)$, so that $\theta(x) = f(ax)$, $\phi(x, y) = by/f^{'}(ax)\sin f(ax)$. The gauge field will take the following form:

$$A^a_\mu(x, y) = \frac{1}{g}\begin{cases} (0,0,0) \\ a\Big(by\frac{\cos^2 f}{\sin f}\cos(\frac{by}{f'\sin f}) - f'\sin(\frac{by}{f'\sin f}) + by\frac{f''}{f'^2}\cos(f)\cos(\frac{by}{f'\sin f}), \\ by\frac{\cos^2}{\sin f}\sin(\frac{by}{f'\sin f}) + f'\cos(\frac{by}{f'\sin f}) + by\frac{f''}{f'^2}\cos(f)\sin(\frac{by}{f'\sin f}), \\ -by(\cos(f) + \frac{f''}{f'^2}\sin(f))\Big) \\ \frac{b}{f'}\Big( - \cos(f)\cos(\frac{by}{f'\sin f}), - \cos(f)\sin(\frac{by}{f'\sin f}),\ \sin f\Big), \\ (0,0,0) \end{cases}$$

where the derivatives are over the whole argument $ax$. One can verify explicitly that it is a solution of the Yang Mills equation.

the energy density of the chromomagnetic field is a space time constant

$$\epsilon = \frac{1}{4}G^a_{ij}G^a_{ij} = \frac{a^2b^2}{2g^2}.$$

# Properties of General solution

The non-vanishing components of the conserved current $J^a_\mu = g\epsilon^{abc} A^b_\nu G^c_{\nu\mu}$ are[5]

$$J^a_1 = \frac{ab^2}{gf'}\Big( \sin(\frac{by}{\sin f}), -\cos(\frac{by}{\sin f}), 0\Big);$$

$$J^1_2 = \frac{a^2 b}{g}\Big( f' \cos f \cos(\frac{by}{\sin f}) + by \cot f \sin(\frac{by}{\sin f}) + by \frac{f''}{f'^2} \sin(\frac{bz}{\sin f})\Big),$$

$$J^2_2 = \frac{a^2 b}{g}\Big( f' \cos f \sin(\frac{by}{\sin f}) - by \cot f \cos(\frac{by}{\sin f}) - by \frac{f''}{f'^2} \cos(\frac{by}{\sin f})\Big),$$

$$J^3_2 = -\frac{a^2 b}{g} f' \sin f.$$

$$\partial_\mu J^a_\mu = \partial_x J^a_1 + \partial_y J^a_2 = 0.$$

[5] This current is conserved on the solutions of the Yang Mills equation $\nabla^{ab}_\mu G^b_{\mu\nu} = 0$.

# Properties of particular solution

Considering $\theta(X) = \arcsin(\sqrt{1-(a\cdot x)^2})$

"chromomagnetic flux sheet" solution

$$n^a(x) = \{\sqrt{1-(a\cdot x)^2}\cos(b\cdot x),\ \sqrt{1-(a\cdot x)^2}\sin(b\cdot x),\ (a\cdot x)\},$$

where $\vec{a} = (a,0,0)$, $\vec{b} = (0,b,0)$.

which represents a *non-perturbative magnetic sheet of a finite thickness* $2/|a|$,

and the corresponding gauge field has the following form:

$$A^a_\mu = \frac{1}{g}\begin{cases} (0,0,0) \\ a\left(\frac{\sin by}{\sqrt{1-(ax)^2}}, -\frac{\cos by}{\sqrt{1-(ax)^2}}, 0\right) \\ b\sqrt{1-(ax)^2}\Big(-ax\cos by, -ax\sin by, \sqrt{1-(ax)^2} \\ (0,0,0) \end{cases} \qquad (ax)^2 < 1$$

$$A^a_\mu = 0, \qquad (ax)^2 \geq 1,$$

# Comparison with the 't Hooft Polyakov monopole solution

The electromagnetic field strength is defined by 't Hooft as

$$G_{\mu\nu} = n^a G^a_{\mu\nu} + \frac{1}{g}\epsilon^{abc} n^a \nabla_\mu n^b \nabla_\nu n^c \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + \underline{\frac{1}{g}\epsilon^{abc} n^a \partial_\mu n^b \partial_\nu n^c}, \qquad n^a = \frac{\phi^a}{|\phi|}, \qquad (1.1)$$

where $\nabla_\mu n^a = \partial_\mu n^a - g\epsilon^{abc} A^b_\mu n^c$, $A_\mu = A^a_\mu n^a$ and $n^a$ is a unit colour vector. It reduces to $G_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$ in the space regions where the scalar field is in the third direction $n_a = (0,0,1)$ and the Abelian field $A_\mu$ does not have Dirac string singularities.

the expression of the topologically conserved current is

$$K_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\partial_\nu G_{\lambda\rho} = \frac{1}{2g}\epsilon_{\mu\nu\lambda\rho}\underline{\epsilon^{abc}\partial_\nu n^a \partial_\lambda n^b \partial_\rho n^c}\ , \qquad \partial_\mu K_\mu = 0.$$

The 't Hooft-Polyakov solution has the following form:

$$\phi^a = u(r) n^a, \qquad A^a_i = \epsilon^{aij} n^j a(r)$$

and has the following asymptotic properties

$$u(0) = 0, \qquad a(0) = 0, \quad u(r) \underset{r\to\infty}{\rightarrow} \frac{m}{\lambda}, \qquad a(r) \underset{r\to\infty}{\rightarrow} -\frac{1}{gr}$$

# Comparison with the 't Hooft Polyakov monopole solution

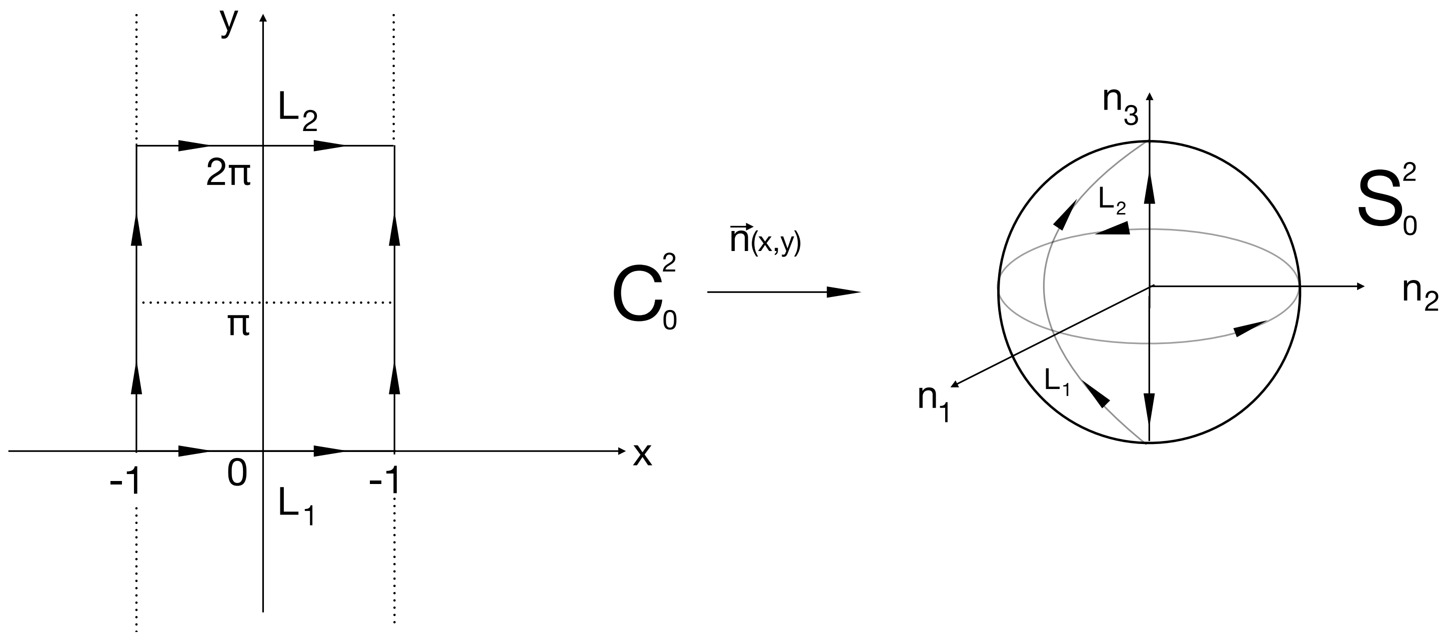
The scalar field $\phi^a$ vanishes at $x^a = 0$ and the corresponding topological density $K_0(x)$ vanishes everywhere expect for $x^a = 0$ where it has singularity $K_0 = \frac{4\pi}{g}\delta^3(\vec{x})$, which contributes to the topological charge and is equal to the winding number of the map $n^a(x)$:

$$g_m = \int_{R^3} d^3x K_0 = \frac{1}{2g}\int_{S^2} d^2\sigma_i \epsilon_{ijk}\epsilon^{abc} n^a \partial_j n^b \partial_k n^c = \frac{4\pi}{g}.$$

induces a magnetic flux of a single monopole:

$$H_i = \frac{x_i}{gr^3}, \qquad g_m = \int H_i dS_i = \frac{4\pi}{g}.$$

# Properties of General solution - Magnetic Fluxes




$$g_m = \int d^3 x K_0 = \frac{1}{2g} \int d^2 \sigma_i \epsilon_{ijk} \epsilon^{abc} n^a \partial_j n^b \partial_k n^c = \frac{4\pi}{g}$$

$$g_m(k) = \frac{1}{g} \int_{-\frac{1}{a}}^{\frac{1}{a}} dax \int_{\frac{2\pi}{b}k}^{\frac{2\pi}{b}(k+1)} dby = \frac{4\pi}{g}.$$

# *Landscape of Yang Mills theory vacuum*

$$\hat{A}^a_\mu = w(\frac{1}{2}-\alpha)A^a_\mu + w(\frac{1}{2}+\alpha)A'^a_\mu.$$

$$\epsilon(x,\alpha) = \frac{a^2b^2}{2g^2}\Big((2-w_-)^2w_-^2 + w_+^2 + 2(2-w_-)w_-(1+w_-)w_+\cos f(ax) + \frac{w_-^2w_+^2}{\sin^2 f(ax)}\Big),$$

where $w_- \equiv w(\frac{1}{2}-\alpha)$ and $w_+ \equiv w(\frac{1}{2}+\alpha)$.

# Landscape of Yang Mills theory vacuum

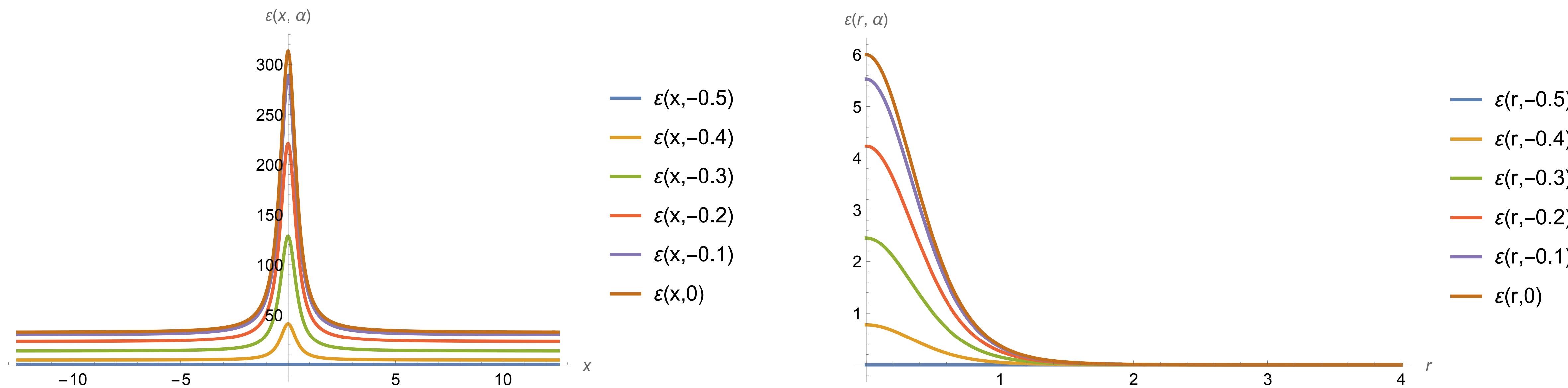
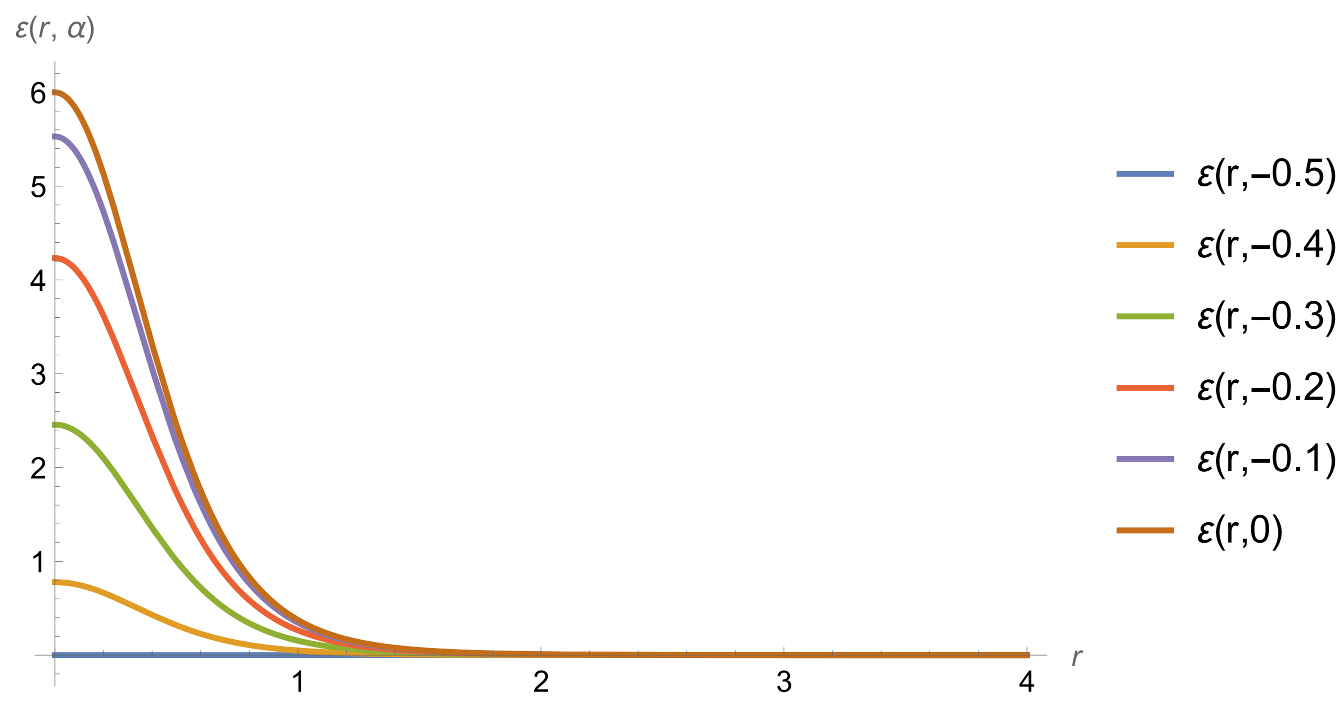



Figure 4: The l.h.s. graph shows the shape of the barrier $\epsilon(x,\alpha)$ when $\alpha$ parameter changes in the interval $[-\frac{1}{2},0]$. At $\alpha=-\frac{1}{2}$ the energy density is equal to $\epsilon=1/2$ $(a=b=g=1)$. As $\alpha$ increases, the hight of the barrier increases and reaches its maximum at $\alpha=0$, then it symmetrically decreases until $\alpha=\frac{1}{2}$, where it again is equal to $\epsilon=1/2$. The r.h.s graph shows the shape of the potential barrier between the Chern-Pontryagin vacua

$$\vec{A}_n(\vec{x})=\frac{i}{g}U_n^{-}(\vec{x})\nabla U_n(\vec{x}),\qquad U_1(\vec{x})=\frac{\vec{x}^2-\lambda^2-2i\lambda\vec{\sigma}\vec{x}}{\vec{x}^2+\lambda^2},\qquad U_n=U_1^n.$$

$$\epsilon(r,\alpha)=\frac{1}{4}G_{ij}^aG_{ij}^a=\frac{6\lambda^4(1-4\alpha^2)}{g^2(r^2+\lambda^2)^4}$$

# *Chern-Pontryagin  Θ-angle and breaking the CP invariants*




$$\epsilon(r,\alpha) = \frac{1}{4} G^a_{ij} G^a_{ij} = \frac{6\lambda^4(1-4\alpha^2)}{g^2(r^2+\lambda^2)^4} \tag{5.51}$$

In the quantum theory tunnelling will occur across this barrier and the quantum-mechanical superposition $\Psi_\theta(\vec{A}) = \sum_n e^{in\theta}\psi_n(\vec{A})$ represents the Yang Mills $\theta$ vacuum state The induced Chern-Pontryagin $\theta$-angle term is Lorentz invariant, but breaks the CP invariants, so that the distinct $\theta$ vacuum states correspond to distinct theories

# Properties of General solution nonzero H

The square of the field strength tensor is

$$\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} = \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{a_\mu F_{\mu\nu} b_\nu}{g} + \frac{a^2b^2 - (a\cdot b)^2}{2g^2}.$$

where $a_\mu$ and $b_\nu$ are arbitrary constant Lorentz vectors

The magnetic energy density can be represented in the following form:

$$\epsilon = \frac{\vec{H}^2}{2} - \frac{1}{g}\vec{H}\cdot(\vec{a}\times\vec{b}) + \frac{1}{2g^2}(\vec{a}\times\vec{b})^2 = \frac{1}{2g^2}(g\vec{H} - \vec{a}\times\vec{b})^2.$$

The minimum of $\epsilon$ is realised when

$$g\vec{H}_{vac} = \vec{a}\times\vec{b}, \qquad \epsilon(g\vec{H}_{vac}) = 0$$

# Properties of General solution nonzero H

When the Abelian part $B_\mu n^a$ of the gauge potential is also present then the potential barrier will take the following form

$$\epsilon(H,x,y,\alpha) = \frac{a^2b^2}{32g^2}\Big((12-8\alpha+16\alpha^2+32\alpha^3-8(1+4\alpha^2)\frac{gH}{ab}\ )(1-\frac{gH}{ab})+ \\ +2(1-4\alpha^2)((2\frac{gH}{ab}-3)^2-4\alpha^2)\cos f(ax)+\frac{(1-4\alpha^2)^2}{\sin^2 f(ax)}+\frac{(1-4\alpha^2)^2 g^2H^2y^2}{f'_x(ax)^2}\Big).$$

At $H=0$ it reduces to the previous expression $\epsilon(H=0,x,y,\alpha)=\epsilon(x,\alpha)$

At $\alpha=\pm 1/2$ we have

$$\epsilon(H,x,y,\pm 1/2) = \frac{1}{2g^2}(gH-ab)^2.$$

At the minimum $gH_{min}=ab$ the initial and final configurations $\alpha=\pm 1/2$ have magnetic energy density equal to zero

$$\epsilon(H_{min},x,y,\pm 1/2)=0.$$

# *Potential barriers between vacuum solutions*

$$A_i^a = -\tfrac{1}{2}F_{ij}x_j n^a + \tfrac{1}{g}\varepsilon^{abc}n^b\partial_i n^c,\ A_0^a = 0,$$

$$\epsilon = \frac{1}{4}G_{ij}^a G_{ij}^a \quad = \quad \frac{(g\vec{H} - \vec{a}\times\vec{b})^2}{2g^2}.$$

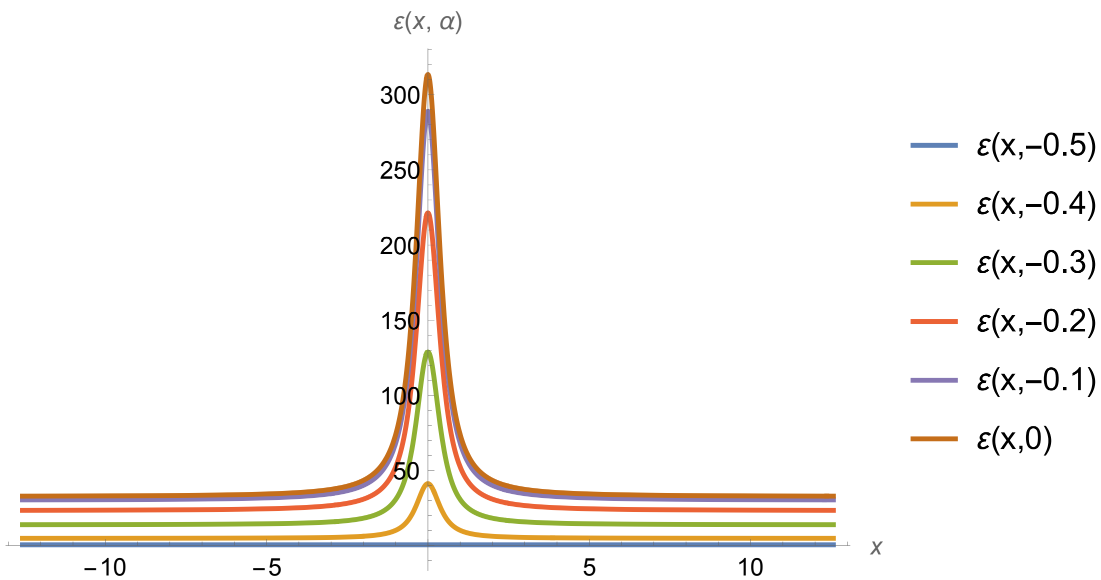
the zero energy density $\epsilon$ is realised when

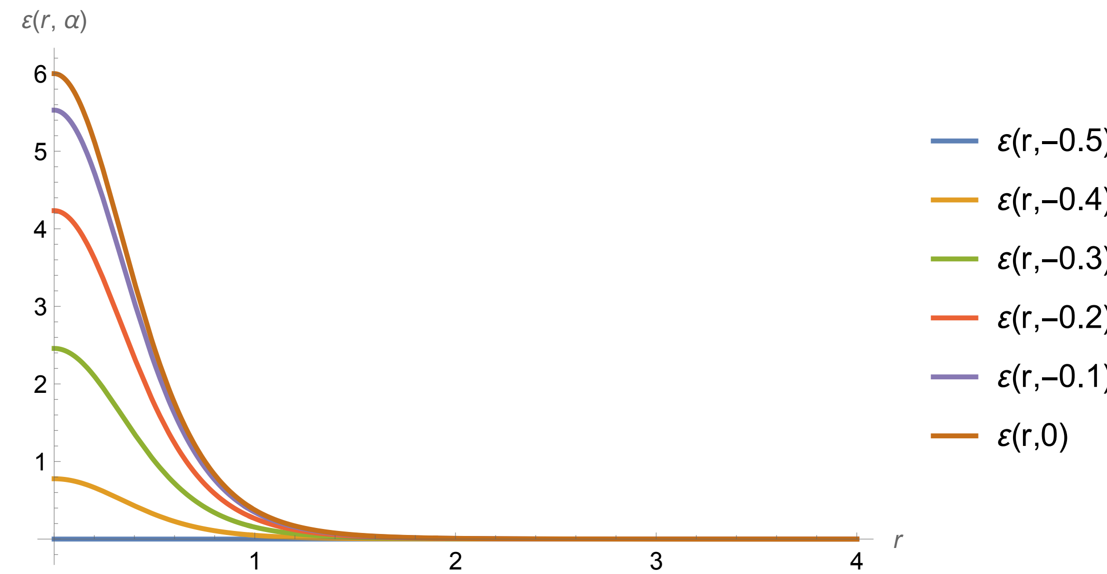
$$g\vec{H}_{vac} = \vec{a}\times\vec{b}, \qquad \epsilon(g\vec{H}_{vac}) = 0.$$

$$\vec{A}_{g\vec{H}=\vec{a}\times\vec{b}} \;=\; -\frac{i}{g}U^{-}\vec{\nabla}U$$

$$\epsilon(gH = ab, x, y, \alpha) \;=\; (1-4\alpha^2)^2\frac{a^2b^2}{32g^2}\Big(2\cos f(ax) + \frac{1}{\sin^2 f(ax)} + \frac{b^2y^2}{f_x'(ax)^2}\Big).$$

*A possible tunnelling transition between superfluxon flat configurations and the flat Jackiw-Rabbi configurations will wash out the CP violating Θ angle to zero dynamically restoring CP symmetry*




We do not know yet whether there exist the instanton-like transitions that would induce a tunnelling between vacuum configurations with nonzero Pontryagin index and the "superfluxon" vacuum configurations. A possible tunnelling transition between superfluxon flat configurations and the flat configurations with non-vanishing Chern-Pontryagin index will wash out the CP violating $\theta$ angle to zero, dynamically restoring CP symmetry.

## *Landscape of Yang Mills theory vacuum*

The existence of an even larger class of covariantly constant gauge fields described above pointed out to the fact that the Yang-Mills vacuum has even higher degeneracy of vacuum field configutations. Each covariantly constant gauge field configuration on its own contains a rich diversity of emergent nonperturbative structures, and it is a challenging problem to investigate possible tunneling transitions between these highly degenerate states and to calculate the vacuum polarisation induced by the new class of covariantly constant gauge fields.

# Landscape of QCD Vacuum

## RIEMANNIAN GEOMETRY & TENSOR CALCULUS

Tensor calculations in the Riemannian Geometry and General Relativity - EDCRGTC

This package was developed by the late colleague and friend Dr. Sotirios Bonanos.

The package can be downloaded here: http://www.inp.demokritos.gr/~sbonano/

```
In[∘]:= << EDCRGTCcode.m
```

```
In[∘]:= f[a_, b_, c_] := Signature[{a, b, c}];
        fabc = Table[f[a, b, c], {a, 1, 3}, {b, 1, 3}, {c, 1, 3}];
```

The unit colour vector n

```
In[∘]:= n := {Sin[f[a1 x]] Cos[b2 y / (f'[a1 x] Sin[f[a1 x]])],
          Sin[f[a1 x]] Sin[b2 y / (f'[a1 x] Sin[f[a1 x]])], Cos[f[a1 x]]};
```

```
A0 = 1/g FullSimplify[
    Contract[Outer[Times, Contract[Outer[Times, fabc, n], {2, 4}], n0], {2, 3}]] ;
A1 = 1/g FullSimplify[
    Contract[Outer[Times, Contract[Outer[Times, fabc, n], {2, 4}], n1], {2, 3}]] ;
A2 = 1/g FullSimplify[
    Contract[Outer[Times, Contract[Outer[Times, fabc, n], {2, 4}], n2], {2, 3}]];
A3 = 1/g FullSimplify[
    Contract[Outer[Times, Contract[Outer[Times, fabc, n], {2, 4}], n3], {2, 3}]] ;

F12 = FullSimplify[D[A2, x] - D[A1, y] -
   g Contract[Outer[Times, Contract[Outer[Times, fabc, A1], {2, 4}], A2], {2, 3}]]
F13 = FullSimplify[D[A3, x] - D[A1, z] -
   g Contract[Outer[Times, Contract[Outer[Times, fabc, A1], {2, 4}], A3], {2, 3}]]
F23 = FullSimplify[D[A3, y] - D[A2, z] -
   g Contract[Outer[Times, Contract[Outer[Times, fabc, A2], {2, 4}], A3], {2, 3}]]
F01 = D[A1, t] - D[A0, x];
F02 = D[A2, t] - D[A0, y];
F03 = D[A3, t] - D[A0, x];
ε = 1/4 × 2 FullSimplify[F12.F12 + F13.F13 + F23.F23]
```

## Yang Mills equation

```
In[ ]:= FullSimplify[ D[F12, x ] - g Contract[
            Outer[Times, Contract[Outer[Times, fabc, A1], {2, 4}], F12], {2, 3}] ]
        FullSimplify[ D[F13, x ] - g Contract[
            Outer[Times, Contract[Outer[Times, fabc, A1], {2, 4}], F13], {2, 3}] ]
        FullSimplify[ D[F23, x ] - g Contract[
            Outer[Times, Contract[Outer[Times, fabc, A1], {2, 4}], F23], {2, 3}] ]
```

```
t[ ]=
        {0, 0, 0}
t[ ]=
        {0, 0, 0}
t[ ]=
        {0, 0, 0}
```

## The potential barrier between A and A' fields

In[∘]:= 
$$K0 = A0\, w\left[\frac{1}{2} - \alpha\right];$$
$$K1 = A1\, w\left[\frac{1}{2} - \alpha\right] + w\left[\frac{1}{2} + \alpha\right] L1;$$
$$K2 = A2\, w\left[\frac{1}{2} - \alpha\right] + w\left[\frac{1}{2} + \alpha\right] L2;$$
$$K3 = A3\, w\left[\frac{1}{2} - \alpha\right] + w\left[\frac{1}{2} + \alpha\right] L3;$$

In[∘]:=
```
K12 = FullSimplify[D[K2, x] - D[K1, y] - g Contract[
      Outer[Times, Contract[Outer[Times, fabc, K1], {2, 4}], K2], {2, 3}]];
K13 = FullSimplify[D[K3, x] - D[K1, z] - g Contract[
      Outer[Times, Contract[Outer[Times, fabc, K1], {2, 4}], K3], {2, 3}]];
K23 = FullSimplify[D[K3, y] - D[K2, z] - g Contract[
      Outer[Times, Contract[Outer[Times, fabc, K2], {2, 4}], K3], {2, 3}]];
K01 = FullSimplify[D[K1, t] - D[K0, x] - g Contract[
      Outer[Times, Contract[Outer[Times, fabc, K0], {2, 4}], K1], {2, 3}]];
K02 = D[K2, t] - D[K0, y] -
   g Contract[Outer[Times, Contract[Outer[Times, fabc, K0], {2, 4}], K2], {2, 3}];
K03 = D[K3, t] - D[K0, x] -
   g Contract[Outer[Times, Contract[Outer[Times, fabc, K0], {2, 4}], K3], {2, 3}];
```
$$\frac{1}{4} \times 2\, \text{FullSimplify}[K12.K12 + K13.K13 + K23.K23]$$

Out[∘]=
$$\frac{1}{2 g^2} a1^2 b2^2 \left(\left(-2 + w\left[\frac{1}{2} - \alpha\right]\right)^2 w\left[\frac{1}{2} - \alpha\right]^2 - 2\, \text{Cos}[f[a1\, x]] \left(-2 + w\left[\frac{1}{2} - \alpha\right]\right) w\left[\frac{1}{2} - \alpha\right] \left(1 + w\left[\frac{1}{2} - \alpha\right]\right) w\left[\frac{1}{2} + \alpha\right] + \left(1 + \text{Cot}[f[a1\, x]]^2\, w\left[\frac{1}{2} - \alpha\right]^2\right) w\left[\frac{1}{2} + \alpha\right]^2\right)$$

*Happy birthday dear Brako !*