BV quantization of braided noncommutative field theories

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based on:

M. Dimitrijević Ćirić, G. Giotopoulos, V. Radovanović, R. J. Szabo, *Braided L_∞-Algebras, Braided Field Theory and Noncommutative Gravity*, Letter Math. Phys (2021).

M. Dimitrijević Ćirić, N. Konjik, V. Radovanović, R. J. Szabo, *Braided Quantum Electrodynamics*, JHEP (2023).

Dj. Bogdanović, M. Dimitrijević Ćirić, V. Radovanović, R. J. Szabo, *BV quantization of braided scalar field theory*, Fortsch.Phys. (2024)

M. Dimitrijević Ćirić, B. Nikolić, V. Radovanović, R. J. Szabo,G. Trojani *Braided Scalar Quantum Electrodynamics*, Fortsch.Phys. (2024)

Overview

Motivation

- (Braided) L_∞-algebra and (braided) QFT
- Deformation by a twist

Examples of braided NC QFTs

- Braided ϕ^n
- Braided electrodynamics

Outlook

Motivation

GR and QFT are cornerstones of modern physics, but both theories suffer from singularities. QFT and GR encounter problems small distances. There is no consistent (i. e. renormalizable and unitary) quantum theory of gravity. Modifications of QFT and GR are needed; point-particles or/and space-time structure. Many attempts: String theory, Loop Quantum Gravity, Noncommutative Geometry,..

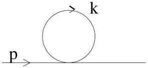
Original motivation: Heisenberg, regularization of divergent electron self-energy. Nowadays: different approaches and different models of NC theories: NC spectral geometry, fuzzy spaces, matrix models, $\star$-product representations...

Motivation

One of the most studied examples: scalar NC QFT with Moyal-Weyl (Groenewold) $\star$-product

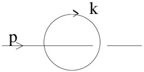
$$S_\star(\phi) = \int \mathrm{d}^4x \Big(\frac{1}{2}\,\phi\,(-\Box - m^2)\,\phi + \frac{\lambda}{4!}\,\phi \star \phi \star \phi \star \phi \Big).$$

Standard quantization: non-planar diagrams and UV/IR mixing [(Minwalla, Van Raamsdonk, Seiberg '99)].



$$\Pi_1(p) \sim \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(p^2-m^2)^2(k^2-m^2)}\ ,$$

Planar diagrams:
usual (quadratic divergent) UV behaviour



$$\Pi_2(p) \sim \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}p\cdot\theta k}}{(p^2-m^2)^2(k^2-m^2)}$$

Non-planar diagrams:
UV convergent, unless $\theta \to 0$ or $p \to 0$

Our approach is based on:

Deformation

Drinfeld twist formalsim: a well defined way to deform a (Hopf) algebra of classical symmetries to a twisted (noncommutative, defomed) Hopf algebra. Module algebras (differential forms, tensors...) are consistently deformed into $\star$-module algebras: noncommutative differential geometry [Aschieri et al. '05...'18].

Construction of NC field theories

L_∞ algebra: Any classical (gauge) field theory described by the corresponding L_∞ algebra [Hohm, Zwiebach '17; Jurco et. al '19]. NC braided field theories can be encoded in a braided L_∞ algebra[Dimitrijević Ćirić, Giotopoulos, Radovanovic, Szabo '21; Giotopoulos, Szabo '22].

Quantization

BV formalism, homological perturbation theory: algebraic techniques for quantization, can be generalized to NC braided field theories [Nguyen, Schenkel, Szabo '21; Dimitrijević Ćirić, Konjik, Radovanović, Szabo '23].

L_∞ algebra and gauge field theories

L_∞-algebra (strong homotopy algebra): generalization of a Lie algebra with higher order brackets.

In physics: higher spin gauge theories with field-dependent gauge parameters [Berends, Burgers, van Dam '85]; Generalized gauge symmetries of closed string field theory [Zwiebach '15]; Any classical field theory with generalized gauge symmetries is determined by an L_∞-algebra, duality with BV-BRST [Hohm, Zwiebach '17; Jurčo et al. '18]. Applications to QFT: correlation functions, amplitudes, double copy [Arvanitakis '19; Jurco et al. '20; Borsten et al. '21]...

NC gauge field theories in the L_∞ setting first discussed in [Blumenhagen et al.'18; Kupriyanov '19]. Homotopy double copy of NC gauge theories discussed in [Szabo, Trojani '23].

L_∞-algebras of ECP gravity [Dimitrijević Ćirić, Giotopoulos, Radovanović, Szabo '20]; Braided NC (quantum) field theories from the baided L_∞-algebra [Dimitrijević Ćirić, Giotopoulos, Radovanović, Szabo '21; Dimitrijević Ćirić, Konjik, Radovanović, Szabo '23].

L_∞ algebra and gauge field theories

$Z-$grading vector space $V = \cdots \oplus V_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus \dots$

- Grading $|V_k| = k$
- n-bracket:

$$\ell_n(\dots, v, v', \dots) = -(-1)^{|v||v'|}\ell_n(\dots, v', v, \dots) \tag{1}$$

- degree $|\ell_n(v_1, \dots, v_n)| = 2 - n + |v_1| + \cdots + |v_n|$
- infinity many homotopy relations:

$$\ell_1(\ell_1(v)) = 0 \ ,$$

$$\ell_1(\ell_2(v_1, v_2)) = \ell_2(\ell_1(v_1), v_2) + (-1)^{|v_1|}\,\ell_2(v_1, \ell_1(v_2)) \ ,$$

$$\begin{aligned}
&\ell_2(\ell_2(v_1, v_2), v_3) - (-1)^{|v_2|\,|v_3|}\,\ell_2(\ell_2(v_1, v_3), v_2) \\
&\quad + (-1)^{(|v_2|+|v_3|)\,|v_1|}\,\ell_2(\ell_2(v_2, v_3), v_1) \\
&= -\ell_3(\ell_1(v_1), v_2, v_3) - (-1)^{|v_1|}\,\ell_3(v_1, \ell_1(v_2), v_3) \\
&\quad - (-1)^{|v_1|+|v_2|}\,\ell_3(v_1, v_2, \ell_1(v_3)) - \ell_1(\ell_3(v_1, v_2, v_3)) \ , \\
&\dots
\end{aligned}$$

L_∞ algebra and gauge field theories

L_∞ algebra is homotopy deformation of graded differential Lie algebra.

- cyclic paring $\langle -, - \rangle : V \otimes V \to R$

$$\langle v_0, \ell_n(v_1, v_2, \dots, v_n)\rangle = \pm \langle v_n, \ell_n(v_0, v_1, \dots, v_{n-1})\rangle$$

Example

- V_0-gauge parameters (ghosts), V_1-fields, V_2-EoM, V_3-Noether ids.

$$\delta_\rho A = \ell_1(\rho) + \ell_2(\rho, A) - \frac{1}{2}\ell_3(\rho, A, A) + \dots \quad (2)$$

$$F_A = \ell_1(A) - \frac{1}{2}\ell_2(A, A) - \frac{1}{3!}\ell_3(A, A, A) + \dots = 0 \quad (3)$$

$$S = \frac{1}{2}\langle A, \ell_1(A)\rangle_- \frac{1}{3!}\langle A, \ell_2^\star(A, A)\rangle + \dots \quad (4)$$

$$\delta_\rho S = \langle F_A, \delta_\rho A\rangle = -\langle \mathrm{d}F_A, \rho\rangle \quad (5)$$

L_∞ algebra and gauge field theories

- Gauge transformations

$$\delta_\rho A = d\rho - [\rho, A] \Rightarrow \ell_1(\rho) = \partial_\mu \rho, \;\; \ell_2(\rho, A) = -[\rho, A] \qquad (6)$$

- From $[\delta_1, \delta_2]A = \delta_{[\rho_1,\rho_2]}A$ follows

$$\ell_2(\rho_1, \rho_2) = -[\rho_1, \rho_2] \; .$$

- EoM

$$\begin{aligned} \ell_1(A) &= \Box A_\nu - \partial_\nu(\partial^\mu A_\mu) \\ \ell_2(A, A) &= -2\partial^\mu [A_\mu, A_\nu] + 2[A^\mu, \partial_\mu A_\nu - \partial_\nu A_\mu] \\ \ell_3(A, A, A) &= -6[A^\mu, [A_\mu, A_\nu]] \end{aligned}$$

NC geometry via the twist deformation

Start from a symmetry algebra g and its universal covering algebra Ug. Then define a twist operator $\mathcal{F}$ as:

-an invertible element of $Ug \otimes Ug$
-fulfills the 2-cocycle condition (ensures the associativity of the $\star$-product).

$$\mathcal{F} \otimes 1(\Delta \otimes \mathrm{id})\mathcal{F} = 1 \otimes \mathcal{F}(\mathrm{id} \otimes \Delta)\mathcal{F}.$$

-normalization $(\mathrm{id} \otimes \epsilon)\mathcal{F} = (\epsilon \otimes \mathrm{id})\mathcal{F} = 1 \otimes 1$.

Braiding (noncommutativity): controled by the R-matrix
$\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1} = \mathcal{F}^{-2} = \mathrm{R}^k \otimes \mathrm{R}_k$; triangular $\mathcal{R}_{21} = \mathcal{R}^{-1} = \mathrm{R}_k \otimes \mathrm{R}^k$.

$$\text{Symmetry Hopf algebra } Ug \overset{\mathcal{F}}{\to} \text{Twisted symmetry Hopf algebra } Ug^{\mathcal{F}}$$

$$\text{Module algebra } \mathcal{A} \overset{\mathcal{F}}{\to} \star \text{ module algebra } \mathcal{A}_\star$$

$$a, b \in \mathcal{A},\ a \cdot b \in \mathcal{A} \overset{\mathcal{F}}{\to} a \star b = \cdot \circ \mathcal{F}^{-1}(a \otimes b) = \mathrm{R}_k(b) \star \mathrm{R}^k(a).$$

Well known example: Moyal-Weyl twist $\mathcal{F} = e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma}$

$$f \star g\,(x) = \cdot \circ \mathcal{F}^{-1}(f \otimes g)$$
$$= f \cdot g + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho f) \cdot (\partial_\sigma g) + \mathcal{O}(\theta^2) = \mathrm{R}_k g \star \mathrm{R}^k f \neq g \star f.$$

Associative, noncommutative: $\mathcal{R}^{-1} = \mathrm{R}_k \otimes \mathrm{R}^k$ encodes the noncommutativity.

Braided L_∞-algebra

Generalization of a quantum Lie algebra [Woronowicz '89; Majid '94].

- $\mathbb{Z}$-graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V_k$. For an irreducible gauge theory:

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3 .$$

- deformed brackets: $\ell_n^\star : \bigotimes^n V \to V$

$$\ell_n^\star(v_1 \otimes \cdots \otimes v_n) = \ell_n(v_1 \otimes_\star \cdots \otimes_\star v_n),$$

with $v \otimes_\star v' := \mathcal{F}^{-1}(v \otimes v') = \bar{\mathrm{f}}^\alpha(v) \otimes \bar{\mathrm{f}}_\alpha(v')$ for $v, v' \in V$. The brackets are braided graded antisymmetric!

$$\ell_n^\star(\dots, v, v', \dots) = -(-1)^{|v|\,|v'|}\,\ell_n^\star\big(\dots, \mathrm{R}_k(v'), \mathrm{R}^k(v), \dots\big)\ .$$

Example: in a non-Abelian gauge theory $\ell_2(\rho, A) = i[\rho, A]$ is deformed to

$$\begin{aligned}\ell_2^\star(\rho, A) =& i[\bar{\mathrm{f}}^k(\rho), \bar{\mathrm{f}}_k(A)] = i[\rho, A]_\star = -i[\mathrm{R}_k(A), \mathrm{R}^k(\rho)]_\star \\ =& i\rho^a \star A^b [T^a, T^b] = i f^{abc} \rho^a \star A^b\, T^c .\end{aligned}$$

The braided commutator closes in the corresponding Lie algebra! This is different from $[\rho \stackrel{\star}{,} A] = \rho \star A - A \star \rho = -[A \stackrel{\star}{,} \rho]$!

Braided L_∞-algebra

- braided homotopy relations:

$$\ell_1^\star\big(\ell_1^\star(v)\big) = 0 \ ,$$

$$\ell_1^\star\big(\ell_2^\star(v_1, v_2)\big) = \ell_2^\star\big(\ell_1^\star(v_1), v_2\big) + (-1)^{|v_1|}\, \ell_2^\star\big(v_1, \ell_1^\star(v_2)\big) \ ,$$

$$\begin{aligned} &\ell_2^\star\big(\ell_2^\star(v_1, v_2), v_3\big) - (-1)^{|v_2|\,|v_3|}\, \ell_2^\star\big(\ell_2^\star(v_1, \mathrm{R}_k(v_3)), \mathrm{R}^k(v_2)\big) \\ &\quad + (-1)^{(|v_2|+|v_3|)\,|v_1|}\, \ell_2^\star\big(\ell_2^\star(\mathrm{R}_k(v_2), \mathrm{R}_j(v_3)), \mathrm{R}^j\mathrm{R}^k(v_1)\big) \\ &= -\ell_3^\star\big(\ell_1^\star(v_1), v_2, v_3\big) - (-1)^{|v_1|}\, \ell_3^\star\big(v_1, \ell_1^\star(v_2), v_3\big) \\ &\quad - (-1)^{|v_1|+|v_2|}\, \ell_3^\star\big(v_1, v_2, \ell_1^\star(v_3)\big) - \ell_1^\star\big(\ell_3^\star(v_1, v_2, v_3)\big) \ , \\ &\dots \end{aligned}$$

- Strict cyclic pairing:

$$\langle v_2, v_1\rangle_\star = \langle\ ,\ \rangle \circ \mathcal{F}^{-1}(v_2 \otimes v_1) = \langle \mathrm{R}_k(v_1), \mathrm{R}^k(v_2)\rangle_\star = \langle v_1, v_2\rangle_\star,$$
$$\langle v_0, \ell_n^\star(v_1, v_2, \dots, v_n)\rangle_\star = \langle R\dots R(v_n), \ell_n^\star(R(v_0), R(v_1), \dots, R(v_{n-1}))\rangle_\star.$$

Strict cyclicity enables a well defined variational principle. Twist operator fulfilling this is a compatible Drinfel'd twists. It define a strictly cyclic braided L_∞-algebra.

Homotopy retract and perturbation lema

$(V, \partial)-$ chain complex

$$\begin{array}{ccccccc}
V^0 & \underset{h_1}{\overset{\partial_0}{\rightleftarrows}} & V^1 & \underset{h_2}{\overset{\partial_1}{\rightleftarrows}} & V^2 & \underset{h_3}{\overset{\partial_2}{\rightleftarrows}} & V^3 \\
 & & p_1 \downarrow \uparrow i_1 & & p_2 \downarrow \uparrow i_2 & & \\
H^0 & \xrightarrow{0} & H^1 & \xrightarrow{0} & H^2 & \xrightarrow{0} & H^3
\end{array}$$

$$\partial_n \circ \partial_{n-1} = 0 \tag{7}$$

Cohomology of degree n =closed /exact vectors

$$\mathrm{H}^n = \frac{\mathrm{Ker}\partial_\mathrm{n}}{\mathrm{Im}\partial_{n-1}} \tag{8}$$

Degree $|\mathrm{h}| = -1, \ |\partial| = 1, \ |\mathrm{p}| = |\mathrm{i}| = 0$

h is contracting homotopy, p is projection, i is inclusion.

Homotopy retract and perturbation lema

Decompose $\mathsf{i} \circ \mathsf{p}$ into Hodge-Kodaira relation

$$\mathsf{h}_{k+1} \circ \partial_k + \partial_{k-1} \circ \mathsf{h}_k = \mathsf{i}_k \circ \mathsf{p}_k - \mathrm{id}_V, \tag{9}$$

Stronger set of conditions:

$$\mathsf{p}_k \circ \mathsf{i}_k = \mathrm{id}_H, \tag{10}$$

$$\mathsf{h}^2 = 0, \quad \mathsf{h} \circ \mathsf{i} = 0 \tag{11}$$

(9)- Hodge-Kodaira relation; (10)-deformation retract; (11)-strong deformation retract

Scalar field theory

$$\mathsf{H}^1 = \mathrm{Ker}\ell_1 = \{\phi | (\Box + m^2)\phi = 0\}; \quad \mathsf{H}^2 = \mathrm{Coker}(l_1)$$

$$\ell_1 \circ \mathsf{h} = -(\Box + m^2) \circ h = \mathrm{id}_{V_2} \to \mathsf{h} = \frac{1}{\Box + m^2} = -\mathsf{G}(x - y)$$

Homotopy retract and perturbation lema

Homological Perturbation lema (Crinic 94)
The diferential l_1 is perturbed to the diferential $\tilde{Q} = \ell_1 + \boldsymbol{\delta}, \quad \tilde{Q}^2 = 0$, where δ is a small perturbation

$$\tilde{\mathsf{p}} = \mathsf{p} + \mathsf{p}_{\boldsymbol{\delta}} = \mathsf{p} + \mathsf{p}(1 - \boldsymbol{\delta}\mathsf{h})^{-1}\boldsymbol{\delta}\mathsf{h}$$
$$\tilde{\partial} = \partial + \mathsf{p}(1 - \boldsymbol{\delta}\mathsf{h})^{-1}\mathsf{i}$$
$$\tilde{\mathsf{i}} = \mathsf{i} + \mathsf{hp}(1 - \boldsymbol{\delta}\mathsf{h})^{-1}\boldsymbol{\delta}\mathsf{i}$$
$$\tilde{\mathsf{h}} = \mathsf{h} + \mathsf{h}(1 - \boldsymbol{\delta}\mathsf{h})^{-1}\boldsymbol{\delta}\mathsf{h}$$

Homotopy retract and perturbation lema

The observables live in (braided) symmetric algebra

$$\mathrm{Sym}_{\mathcal{R}} V[2]$$

- Projection: $\mathrm{P}(1) = 1, \; \mathrm{P}(\varphi_1 \odot \cdots \odot \varphi_n) = 0$
- Extended contracting homotopy H

$$\mathrm{H}(1) = 0 \tag{12}$$

$$\mathrm{H}(\varphi_1 \odot \cdots \odot \varphi_n) = \frac{1}{n} \sum_{i=0}^{n} \pm \varphi_1 \odot \cdots \mathrm{h}(\varphi_i) \odot \varphi_{i+1} \cdots \varphi_n \tag{13}$$

- Quantum differential $Q_\delta = \ell_1 + \boldsymbol{\delta}$

Braided homological perturbation theory

How do we calculate correlation functions? We use the (braided) homological perturbation lemma.

- On L_∞ algebra $V[2]$: propagators h define (braided) strong deformation retracts:

$$\mathsf{h} \circlearrowright (V, \ell_1) \underset{\mathsf{p}}{\overset{\iota}{\leftrightarrows}} (H^\bullet(V), 0)$$

- This can be extended to the symmetric algebra $\mathrm{Sym}_{\mathcal{R}} V[2]$. In particular $\mathsf{h} \to \mathsf{H}$:

$$\mathsf{H} \circlearrowright (\mathrm{Sym}(V[2]), \ell_1) \underset{\mathsf{P}}{\overset{\mathsf{I}}{\leftrightarrows}} (\mathrm{Sym}(H^\bullet(V[2])), 0)$$

- A preturbation δ defines a new (braided) strong deformation retract

$$\tilde{\mathsf{H}} \circlearrowright (\mathrm{Sym}(V[2]), Q_{\mathrm{BV}}) \underset{\tilde{\mathsf{P}}}{\overset{\tilde{\mathsf{I}}}{\leftrightarrows}} (\mathrm{Sym}(H^\bullet(V[2])), \tilde{\delta})$$

Braided homological perturbation theory

Braided homological perturbation lema defines the perturbed projection map $\tilde{P} = \mathrm{P} + \mathrm{P}_{\boldsymbol{\delta}}$ with

$$\mathrm{P}_{\boldsymbol{\delta}} = \mathrm{P}\left(\mathrm{id}_{\mathrm{Sym}_{\mathcal{R}} V[2]} - \boldsymbol{\delta}\,\mathrm{H}\right)^{-1} \boldsymbol{\delta}\,\mathrm{H} \ .$$

When there is no NC deformation, this gives the path integral [Doubek, Jurčo, Pulmann '17].

The new projection $\mathrm{P}_{\boldsymbol{\delta}}$ with $\boldsymbol{\delta} = i\hbar\Delta_{\mathrm{BV}} + \{\mathcal{S}^{\star}_{\mathrm{int}}, \ \}_{\star}$ gives correlation functions for the braided QFT:

$$\begin{aligned} G^{\star}_{n}(x_1, \dots, x_n) &= \langle 0|\mathrm{T}[\phi(x_1) \star \cdots \star \phi(x_n)]|0\rangle_{\star} := \mathrm{P}_{\boldsymbol{\delta}}(\delta_{x_1} \odot_{\star} \cdots \odot_{\star} \delta_{x_n}) \\ &= \sum_{m=1}^{\infty} \mathrm{P}\big((\boldsymbol{\delta}\,\mathrm{H})^{m}(\delta_{x_1} \odot_{\star} \cdots \odot_{\star} \delta_{x_n})\big) \ , \\ G^{\star}_{n}(p_1, \dots, p_n) &= \sum_{m=1}^{\infty} \mathrm{P}\big((\boldsymbol{\delta}\,\mathrm{H})^{m}(\mathrm{e}^{k_1} \odot_{\star} \cdots \odot_{\star} \mathrm{e}^{k_n}\big). \end{aligned}$$

where $\delta_{x_a}(x) := \delta(x - x_a)$ are Dirac distributions supported at the insertion points x_a of the physical field $\phi \in V^1$ and $\mathrm{e}^k = e^{ik\cdot x} \in V_2$ is the basis in momentum space.

The braided BV laplacian Δ_{BV} encodes the braided Wick theorem and the interaction action $\mathcal{S}^{\star}_{\mathrm{int}}$ encodes interaction (vertices).

BV quantization: basic ideas

Classical BV formalism: quantization of (higher) gauge theories, models with open gauge symmetry algebras...

Braided BV formalism developed in [Nguyen, Schenkel, Szabo '21], following [Costello, Gwilliam '16)] and [Jurco et al. '19].

Free scalar field theory

$$\boldsymbol{\delta} = -i\hbar\Delta_{BV}$$

The braided BV Laplacian $\Delta_{\rm BV}$

$$\begin{aligned}
\Delta_{\rm BV}(1) &= 0, \quad \Delta_{\rm BV}(\phi_1) = 0, \quad \Delta_{\rm BV}(\phi_1 \odot_\star \phi_2) = \pm\langle\phi_1, \phi_2\rangle_\star \ , \\
\Delta_{\rm BV}(a_1 \odot_\star a_2) &= (\Delta_{\rm BV} a_1) \odot_\star a_2 + (-1)^{|a_1|} a_1 \odot_\star (\Delta_{\rm BV} a_2) + \{a_1, a_2\}_\star , \\
\ell_1^\star \Delta_{\rm BV} + \Delta_{\rm BV} \ell_1^\star &= 0, \quad \Delta_{\rm BV}^2 = 0, \quad \Delta_{\rm BV}(\mathcal{S}^\star_{\rm int}) = 0.
\end{aligned}$$

Green functions:

$$\begin{aligned}
G_2(p_1, p_2) &= \mathrm{P}(\boldsymbol{\delta}\,\mathrm{H}(\mathrm{e}^{k_1} \odot \mathrm{e}^{k_2})) = \frac{i\hbar}{k_1^2 - m^2}\delta(k_1 + k_2) \\
G_4(p_1, p_2, p_3, p_4) &= \mathrm{P}((\boldsymbol{\delta}\,\mathrm{H})^2(\mathrm{e}^{k_1} \odot \mathrm{e}^{k_2} \odot \mathrm{e}^{k_3} \odot \mathrm{e}^{k_4}) \\
&= G_2(p_1, p_2)G_2(p_3, p_4) + e^{ip_3\theta p_2} G_2(p_1, p_3)G_2(p_2, p_4) + G_2(p_1, p_4)G_2(p_2, p_3) \ . \quad (14)
\end{aligned}$$

BV quantization: basic ideas

Interacting field theory

$$\delta = -i\hbar\Delta_{BV} - \{\mathcal{S}^\star_{\rm int},\ \}_\star$$

- (braided, cyclic) L_∞ algebra $(V, \ell^\star_n, \langle\ ,\ \rangle_\star)$ is extended to the braided symmetric algebra $\mathrm{Sym}_{\mathcal{R}} V[2]$

$$\begin{aligned} v_1 \odot_\star (v_2) &= (-1)^{|v_1||v_2|} \mathrm{R}_k(v_2) \odot_\star \mathrm{R}^k(v_1) \\ \ell^\star_1(a_1 \otimes v_1) &= \pm a_1 \otimes \ell^\star_1(v_1)\ , \\ \ell^\star_2(a_1 \otimes v_1, a_2 \otimes v_2) &= \pm\big(a_1 \odot_\star \mathrm{R}_k(a_2)\big) \otimes \ell^\star_2\big(\mathrm{R}^k(v_1), v_2\big)\ , \\ &\cdots \\ \langle\!\langle a_1 \otimes v_1, a_2 \otimes v_2\rangle\!\rangle_\star &= \pm\,\big(a_1 \odot_\star \mathrm{R}_k(a_2)\big)\,\langle \mathrm{R}^k(v_1), v_2\rangle_\star\ , \end{aligned}$$

  for $a_1, a_2 \in \mathrm{Sym}_{\mathcal{R}} V[2]$, $v_1, v_2 \in V$.

- The braided BV action $\mathcal{S}^\star_{\rm BV} \in \mathrm{Sym}_{\mathcal{R}} V[2]$ is defined as

$$\mathcal{S}^\star_{\rm BV} = \frac{1}{2}\langle\!\langle \boldsymbol{\xi}\,,\, \ell^\star_1(\boldsymbol{\xi})\rangle\!\rangle_\star - \frac{1}{3!}\langle\!\langle \boldsymbol{\xi}\,,\, \ell^\star_2(\boldsymbol{\xi},\boldsymbol{\xi})\rangle\!\rangle_\star + \cdots = \mathcal{S}^\star_{(0)} + \mathcal{S}^\star_{\rm int},$$

  with contracted coordinate functions $\boldsymbol{\xi} \in (\mathrm{Sym}_{\mathcal{R}} V[2]) \otimes V$; constructed using the basis in V, τ_k, and the corresponding dual (via pairing) basis τ^k: $\boldsymbol{\xi} = \sum_k \tau_k \otimes \tau^k$.

BV quantization: basic ideas

- $\mathcal{S}^{\star}_{\rm BV}$ satisies the classical master equation

$$\{\mathcal{S}^{\star}_{\rm BV},\mathcal{S}^{\star}_{\rm BV}\}_{\star}=0,$$

the braided antibracket $\{\,,\}_{\star}:\mathrm{Sym}_{\mathcal{R}}V[2]\otimes\mathrm{Sym}_{\mathcal{R}}V[2]\to\mathrm{Sym}_{\mathcal{R}}V[2]$

$$\{a,b\}_{\star}=(-1)^{|a||b|}\{\mathrm{R}_k b,\mathrm{R}^k a\}_{\star},\quad \{\phi_1,\phi_2\}_{\star}=\pm\langle\phi_1,\phi_2\rangle_{\star},\ \ \phi_{1,2}\in V[2]$$
$$\{a,b\odot_{\star}c\}_{\star}=\{a,b\}_{\star}\odot_{\star}\ c+(-1)^{|b|(|a|+1)}\,\mathrm{R}_k b\odot_{\star}\{\mathrm{R}^k a,c\}_{\star}.$$

- The operator $Q=\ell_1^{\star}+\{\mathcal{S}^{\star}_{\rm int},\ \}_{\star}$ satisfies $Q^2=0$ and

$$Q\{\phi_1,\,\phi_2\}_{\star}=\{Q\phi_1,\,\phi_2\}_{\star}+(-1)^{|a_1|}\{\phi_1,Q\phi_2\}_{\star}.$$

The algebra of classical observables: $(\mathrm{Sym}_{\mathcal{R}}V[2],Q,\{\,,\,\}_{\star})$.

- The (braided) algebra of quantum observables: $(\mathrm{Sym}_{\mathcal{R}}V[2]),Q_{\rm BV},\{\,,\,\}_{\star})$ with

$$Q_{\rm BV}=\ell_1^{\star}+\{\mathcal{S}^{\star}_{\rm int},\ \}_{\star}+i\hbar\Delta_{\rm BV}.$$

These properties enable $Q^2_{\rm BV}=0$!

(An attempt of) Dictionary

Classical field theory: $\rightarrow$ $L_\infty : (V, \ell_n^\star, \langle\, , \,\rangle_\star)$

[Hohm, Zwiebach '17; Jurčo et al. '18; MDC, Giotopoulos, Radovanović, Szabo '21]

fields φ^A, antifields φ_A^+ $\rightarrow$ $V = \cdots \oplus \underbrace{V_0}_{\text{ghosts } \rho} \oplus \underbrace{V_1}_{\text{fields } A} \oplus \underbrace{V_2}_{A^+} \oplus \underbrace{V_3}_{\rho^+} \oplus \ldots$

Gauge transformations $\rightarrow$ $\delta_\rho^\star A = \ell_1^\star(\rho) + \ell_2^\star(\rho, A) - \frac{1}{2}\ell_3^\star(\rho, A, A) + \ldots$

EoM $\rightarrow$ $F_A^\star = \ell_1^\star(A) - \frac{1}{2}\ell_2^\star(A, A) - \frac{1}{3!}\ell_3^\star(A, A, A) + \cdots = 0$

Action S $\rightarrow$ $S_\star = \frac{1}{2}\langle A, \ell_1^\star(A)\rangle_\star - \frac{1}{3!}\langle A, \ell_2^\star(A, A)\rangle_\star + \ldots$

Classical observables: usually polinomial in fields $\rightarrow$ $(V, \ell_n^\star, \langle\, , \,\rangle_\star)$ extended to $\mathrm{Sym}_{\mathcal{R}} V[2]$

$$v_1 \odot_\star (v_2) = (-1)^{|v_1||v_2|} \mathrm{R}_k(v_2) \odot_\star \mathrm{R}^k(v_1)$$

For example:

$$\ell_2^{\star\, \mathrm{ext}}(a_1 \otimes v_1, a_2 \otimes v_2) = \pm\big(a_1 \odot_\star \mathrm{R}_k(a_2)\big) \otimes \ell_2^\star\big(\mathrm{R}^k(v_1), v_2\big)$$

$$\langle\!\langle a_1 \otimes v_1, a_2 \otimes v_2 \rangle\!\rangle_\star = \pm\big(a_1 \odot_\star \mathrm{R}_k(a_2)\big)\, \langle \mathrm{R}^k(v_1), v_2\rangle_\star$$

for $a_1, a_2 \in \mathrm{Sym}_{\mathcal{R}} V[2]$, $v_1, v_2 \in V$.

Classical action S_{BV} $\rightarrow$ Braided BV action $\mathcal{S}^{\star}_{\rm BV} \in \mathrm{Sym}_{\mathcal{R}} V[2]$

[... Weinberg '96; Gomis et al '94]

$$S_{BV} = S + \varphi^{+}_{A}(\dots) + \dots$$

$$\mathcal{S}^{\star}_{\rm BV} = \tfrac{1}{2}\langle\!\langle \boldsymbol{\xi}\,,\, \ell_1^{\star\, ext}(\boldsymbol{\xi})\rangle\!\rangle_{\star} - \tfrac{1}{3!}\langle\!\langle \boldsymbol{\xi}\,,\, \ell_2^{\star\, ext}(\boldsymbol{\xi},\boldsymbol{\xi})\rangle\!\rangle_{\star} + \dots$$

$$= \mathcal{S}^{\star}_{(0)} + \mathcal{S}^{\star}_{\rm int}, \text{ with}$$

$\boldsymbol{\xi} = \sum_k \tau_k \otimes \tau^k$; τ_k basis in V, τ^k dual basis.

Classical master equation

$$\{S_{BV}, S_{BV}\} = 0 \quad \rightarrow \quad \{\mathcal{S}^{\star}_{\rm BV}, \mathcal{S}^{\star}_{\rm BV}\}_{\star} = 0$$

$\{\,,\,\} \sim \sum_A \left(\frac{\delta}{\delta\varphi^A}\frac{\delta}{\delta\varphi^+_A} - \frac{\delta}{\delta\varphi^+_A}\frac{\delta}{\delta\varphi^A}\right)$ $\rightarrow$ the braided antibracket $\{\,,\,\}_{\star}$

$$\{a,b\}_{\star} = (-1)^{|a||b|}\{\mathrm{R}_k b, \mathrm{R}^k a\}_{\star} \qquad \{\phi_1,\phi_2\}_{\star} = \pm\langle\phi_1,\phi_2\rangle_{\star}$$

$$\{a, b\odot_{\star} c\}_{\star} = \{a,b\}_{\star}\odot_{\star} c + (-1)^{|b|(|a|+1)}\,\mathrm{R}_k b \odot_{\star} \{\mathrm{R}^k a, c\}_{\star}$$

$$Q\varphi^A = \{S_{BV}, \varphi\},\; Q^2 = 0 \quad \rightarrow \quad Q = \ell_1^{\star\, ext} + \{\mathcal{S}^{\star}_{\rm int},\,\}_{\star},\; Q^2 = 0$$

Classical observables: cohomology of Q $\rightarrow$ Algebra of classical observables $(\mathrm{Sym}_{\mathcal{R}} V[2], Q, \{\,,\,\}_{\star})$

Quantization: Correlation functions (generating functional $Z[J]$) do not depend on the choice of the gauge fixing fermion Ψ and $\varphi^+_A \sim \frac{\delta\Psi}{\delta\varphi^A}$.

Quantum master equation: $\rightarrow$ Braided quantum master equation?

$$\frac{1}{2}\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} - \mathrm{i}\hbar\,\Delta_{BVcl} S_{\mathrm{BV}} = 0$$

BV Laplacian Δ_{BVcl} $\rightarrow$ Braided BV Laplacian Δ_{BV}

$$\Delta_{BVcl} \sim \sum_A \frac{\delta^2}{\delta\varphi^A \delta\varphi^+_A} \quad \rightarrow \quad \Delta_{\mathrm{BV}}(\phi_1 \odot_\star \phi_2) = \pm\langle\phi_1, \phi_2\rangle_\star$$

$$\Delta_{\mathrm{BV}}(a_1 \odot_\star a_2) = (\Delta_{\mathrm{BV}} a_1) \odot_\star a_2 + (-1)^{|a_1|} a_1 \odot_\star (\Delta_{\mathrm{BV}} a_2) + \{a_1, a_2\}_\star$$

...

$$Q_{\mathrm{BV}} = \ell_1^{\star\, ext} + \{\mathcal{S}^\star_{\mathrm{int}}, \ \}_\star + \mathrm{i}\hbar\Delta_{\mathrm{BV}} \text{ and } Q^2_{\mathrm{BV}} = 0$$

Braided algebra of quantum observables:

$$(\mathrm{Sym}_{\mathcal{R}} V[2]), Q_{\mathrm{BV}}, \{\ ,\ \}_\star)$$

Correlation functions $\to$ Braided correlation functions

from $Z[J] \sim \int \mathcal{D}\phi e^{\mathrm{i}(S+\int \mathrm{d}^4 x J\phi)}$ $\to$ Braided homological perturbation lemma

[Nguyen, Schenkel, Szabo '21]

[Costello, Gwilliam '16; Jurco et al. '19; Doubek et al. '17]

$G_2^\star(x_1, x_2) = -\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_1)}\Big|_{J=0}$ $\to$ $G_n^\star(x_1, \dots, x_n) =$

... $:= \sum_{m=1}^{\infty} \mathrm{P}\big((\boldsymbol{\delta}\,\mathrm{H})^m(\delta_{x_1} \odot_\star \cdots \odot_\star \delta_{x_n})\big)$

$G_n^\star(p_1, \dots, p_n) = \sum_{m=1}^{\infty} \mathrm{P}\big((\boldsymbol{\delta}\,\mathrm{H})^m(\mathrm{e}^{k_1} \odot_\star \cdots \odot_\star \mathrm{e}^{k_n}\big)$

$\boldsymbol{\delta} = i\hbar\Delta_{\mathrm{BV}} + \{\mathcal{S}^\star_{\mathrm{int}}, \ \}_\star$

Δ_{BV} the braided Wick theorem, $\mathcal{S}^\star_{\mathrm{int}}$ interaction (vertices)

H : extension of propagators h from $(V, \ell_n^\star, \langle\ ,\ \rangle_\star)$ to $\mathrm{Sym}_{\mathcal{R}} V[2]$

$\delta_{x_a}(x)$ and e^k coordinate and momentum basis of fields.

Example I: Braided ϕ^3

Braided scalar field theory: 4D Minkowski space-time, Moyal-Weyl twist and a real massive scalar field ϕ.

Classical theory is given by the graided vector space $V = V_1 \oplus V_2$ with $V_1 = V_2 = \Omega^0(\mathbb{R}^{1,3})$ and

$$\ell_1(\phi) = -(\Box + m^2)\phi, \;\; \ell_2(\phi_1, \phi_2) = -\lambda\phi_1 \cdot \phi_2,$$
$$\langle \phi, \phi^+ \rangle = \int \mathrm{d}^4x \; \phi \; \phi^+,$$
$$S(\phi) = \frac{1}{2}\langle \phi, \ell_1(\phi)\rangle - \frac{1}{3!}\langle \phi, \ell_2(\phi, \phi)\rangle = \int \mathrm{d}^4x \; \Big(\frac{1}{2}\,\phi\,(-\Box - m^2)\,\phi - \frac{\lambda}{3!}\phi^3\Big).$$

Braided NC scalar field theory: the same vector space V with

$$\ell_1^\star(\phi) = -(\Box + m^2)\phi, \quad \ell_2^\star(\phi_1, \phi_2) = -\lambda\phi_1 \star \phi_2$$
$$\langle \phi, \phi^+ \rangle_\star = \int \mathrm{d}^4x \; \phi \star \phi^+,$$
$$S_\star(\phi) = \frac{1}{2}\,\langle \phi, \ell_1(\phi)\rangle_\star - \frac{1}{3!}\,\langle \phi, \ell_2^\star(\phi, \phi)\rangle_\star = \int \mathrm{d}^4x \Big(\frac{1}{2}\,\phi\,(-\Box - m^2)\,\phi + \frac{\lambda}{3!}\,\phi_\star^3\Big).$$

At the classical level, this action is the same as in the usual $\phi_\star^n$ theory! Differences appear at the quantum level. Global symmetries (Lorentz...) discussed in [Giotopoulos, Szabo '22].

BV quantization: braided ϕ^3

Momentum space basis in V: $\mathrm{e}_k = e^{-\mathrm{i}kx} \in V_1$ and $\mathrm{e}^k = e^{\mathrm{i}kx} \in V_2$.

We extend the braided L_∞-algebra of Example I to the space of observables $SymV[2]$. Contracted coordinate functions $\boldsymbol{\xi} \in SymV[2] \otimes V$

$$\boldsymbol{\xi} = \int_k \left(\mathrm{e}_k \otimes \mathrm{e}^k + \mathrm{e}^k \otimes \mathrm{e}_k\right)$$

define the interaction action

$$S^\star_{\rm int} = -\frac{1}{6}\langle\!\langle \xi, \ell_2^{\star\,\mathrm{ext}}(\xi,\xi)\rangle\!\rangle_\star = \int_k V(k_1,k_2,k_3) e^{k_1} \odot_\star e^{k_2} \odot_\star e^{k_3},$$

where $V(k_1,k_2,k_3) = \frac{\lambda}{3!} e^{\frac{i}{2}\sum_{a<b} k_a\cdot\theta k_b} (2\pi)^D \delta^D(k_1+k_2+k_3)$.

Braided symmetry of the vertex:

$$V(k_2,k_1,k_3) = e^{-ik_1\cdot\theta k_2} V(k_1,k_2,k_3).$$

BV quantization: braided ϕ^3, tree level

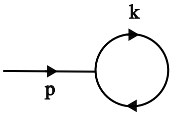
Tree level

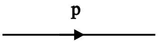
$$G_1^\star(p_1)^{(0)} = \mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H}\{\mathcal{S}_{\mathrm{int}},\mathsf{H}(\mathrm{e}^{p_1})\} \sim \mathrm{i}\,\hbar(-\frac{\lambda}{3!})\frac{\delta(p_1)}{p_1^2-m^2}\int_k \frac{1}{k^2-m^2}$$

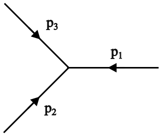
$$G_2^\star(p_1,p_2)^{(0)} = \mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}},\mathsf{H}\big(\mathrm{e}^{p_1}\odot_\star \mathrm{e}^{p_2}\big) = (2\pi)^d\,\frac{\delta(p_1+p_2)}{p_1^2-m^2}\,,$$

$$G_3^\star(p_1,p_2,p_3)^{(0)} = \Big((\mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H})^2\{\mathcal{S}_{\mathrm{int}},\mathsf{H}\quad\} + \mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H}\{\mathcal{S}_{\mathrm{int}},\mathsf{H}(\mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H}(\quad)\}\Big)(\mathrm{e}^{p_1}\odot_\star \mathrm{e}^{p_2}\odot_\star \mathrm{e}^{p_3}))$$

$$\sim (\mathrm{i}\,\hbar)^2(-\frac{\lambda}{3!})\left(\frac{\delta(p_1)}{p_1^2-m^2}\,\frac{\delta(p_2+p_3)}{p_2^2-m^2}\int_k\frac{1}{k^2-m^2} + \frac{\delta(p_1+p_2+p_3)\,e^{-\frac{\mathrm{i}}{2}\sum_{a<b} p_a\cdot\theta\,p_b}}{(p_1^2-m^2)(p_2^2-m^2)(p_3^2-m^2)}\right).$$





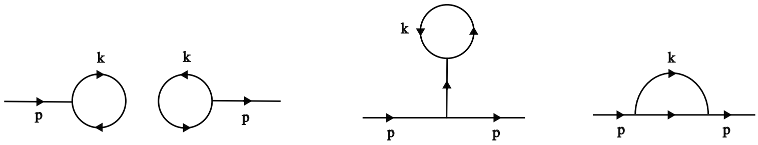


BV quantization: braided ϕ^3, 1-loop

2-point function at 1-loop:

$$G_2^\star(p_1,p_2)^{(1)} = (\mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H})^2\,\{\mathcal{S}_{\mathrm{int}},\mathsf{H}\,\{\mathcal{S}_{\mathrm{int}},\mathsf{H}\,(\mathrm{e}^{p_1}\odot_\star \mathrm{e}^{p_2})\}_\star\}_\star$$
$$+\,\mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H}\,\{\mathcal{S}_{\mathrm{int}},\mathsf{H}\,((\mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H})\,\{\mathcal{S}_{\mathrm{int}},\mathsf{H}\,(\mathrm{e}^{p_1}\odot_\star \mathrm{e}^{p_2})\}_\star)\}_\star$$

$$= -\frac{(\hbar\,\lambda)^2}{4}\,\frac{(2\pi)^6\,\delta(p_1)}{p_1^2-m^2}\,\Big[\int_k\frac{1}{k^2-m^2}\Big]^2\,\frac{(2\pi)^6\,\delta(p_2)}{p_2^2-m^2}$$
$$-\frac{(\hbar\,\lambda)^2}{2}\,\frac{(2\pi)^6\,\delta(p_1+p_2)}{(p_1^2-m^2)\,(p_2^2-m^2)}\,\int_k\frac{1}{(0-m^2)\,(k^2-m^2)}$$
$$-\frac{(\hbar\,\lambda)^2}{2}\,\frac{(2\pi)^6\,\delta(p_1+p_2)}{(p_1^2-m^2)\,(p_2^2-m^2)}\,\int_k\frac{1}{(k^2-m^2)\,\big((p_1-k)^2-m^2\big)}\ .$$



No NC contributions, no nonplanar diagrams and no UV/IR mixing at 1-loop!

Consistent with [Oeckel '00].

BV quantization: braided ϕ^3, 1-loop

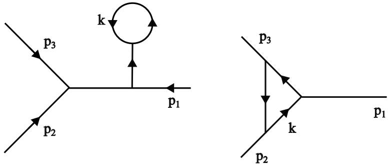
3-point function at 1-loop:

$$G_3^\star(p_1,p_2,p_3)^{(1)} = \mathsf{P}\left((\mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H} + \{\mathcal{S}_{\mathrm{int}}, -\}_\star\,\mathsf{H})^6\,(\mathrm{e}^{p_1}\odot_\star \mathrm{e}^{p_2}\odot_\star \mathrm{e}^{p_3})\right)$$

$$= \cdots + (\mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H})^3\Big(\{\mathcal{S}_{\mathrm{int}}, \mathsf{H}\{\mathcal{S}_{\mathrm{int}}, \mathsf{H}\{\mathcal{S}_{\mathrm{int}}, \mathsf{H}(\mathrm{e}^{p_1}\odot_\star \mathrm{e}^{p_2}\odot_\star \mathrm{e}^{p_3})\}_\star\}_\star\}_\star\Big)$$

$$\sim \cdots + (\mathrm{i}\,\hbar)^3\Big(-\frac{\lambda}{3!}\Big)^3 \frac{\delta(p_1+p_2+p_3)}{(p_1^2-m^2)(p_2^2-m^2)(p_3^2-m^2)}\, e^{-\frac{\mathrm{i}}{2}\sum_{a<b} p_a\cdot\theta\, p_b}$$

$$\Big\{\frac{1}{((p_2+p_3)^2-m^2)}\frac{1}{(0-m^2)}\int_k \frac{1}{(k^2-m^2)} + \int_k \frac{1}{(k^2-m^2)((p_3-k)^2-m^2)((p_1+k)^2-m^2)}\Big\}.$$



NC contribution appears as a phase factor in external momneta. No UV/IR mixing!

BV quantization: braided ϕ^3, 2-loops

Preliminary results on 2-point function at 2-loops:

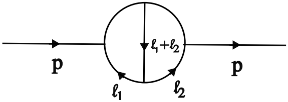
$$G_2^\star(p_1,p_2)^{(1)} = \mathsf{P}\left((\mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H} + \{\mathcal{S}_{\mathrm{int}}, -\}_\star\,\mathsf{H})^7\,(\mathrm{e}^{p_1} \odot_\star \mathrm{e}^{p_2})\right).$$

Lots of disconnected and reducible diagrams, also 1PI diagrams. Examples include:

$$(\mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}\,\mathsf{H})^3\{\mathcal{S}_{\mathrm{int}},\mathsf{H}\{\mathcal{S}_{\mathrm{int}},\mathsf{H}\{\mathcal{S}_{\mathrm{int}},\mathsf{H}\{\mathcal{S}_{\mathrm{int}},\mathsf{H}(\mathrm{e}^{p_1}\odot_\star \mathrm{e}^{p_2})\}_\star\}_\star\}_\star\}_\star$$

$$\sim \cdots + (\mathrm{i}\,\hbar)^3(-\frac{\lambda}{3!})^4\,\frac{\delta(p_1+p_2)}{(p_1^2-m^2)(p_2^2-m^2)}$$

$$\int_{l_1}\int_{l_2}\frac{1}{(l_1^2-m^2)(l_2^2-m^2)((l_1+l_2)^2-m^2)((p_1+l_1)^2-m^2)((p_2+l_2)^2-m^2)}.$$



No NC contributions, no nonplanar diagrams and no UV/IR mixing in the 2-point function at 2-loops. Consistent with [Oeckel '00]. Maybe boring? Any non-trivial NC contributions in scattering amplitudes, Schwinger-Dyson equations?

Schwinger Dyson equations

SD equations are quantum analogous of the classical equations of motion.

$$\mathsf{P}_\delta = \mathsf{P}\frac{1}{1-\boldsymbol{\delta}\circ\mathsf{H}}\circ\boldsymbol{\delta}\circ\mathsf{H} = \tilde{\mathsf{P}}\circ\boldsymbol{\delta}\circ\mathsf{H}\ , \tag{15}$$

Using the identity (15) we can derive the Schwinger-Dyson equations. From (15) it follows

$$\mathsf{P}_\delta(\ell_1(\mathsf{e}_{-k_0})\odot_\star \mathsf{e}^{k_1}\cdots\odot_\star \mathsf{e}^{k_n}) = (\mathsf{P}+\mathsf{P}_\delta)\boldsymbol{\delta}\mathsf{H}(\ell_1(\mathsf{e}_{-k_0})\odot_\star \mathsf{e}^{k_1}\odot_\star\cdots\odot_\star \mathsf{e}^{k_n})\ . \tag{16}$$

(16) is our main equation from which we derive the Schwinger-Dyson equations.

Schwinger Dyson equations

Case $n = 3$

$$(k_0^2 - m^2)\langle\Omega|T\tilde{\phi}(k_0)\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)|\Omega\rangle =$$
$$-\,\mathrm{i}\hbar(2\pi)^4\Big(\delta(k_0+k_1)\langle\Omega|T\tilde{\phi}(k_2)\tilde{\phi}(k_3))|\Omega\rangle + \delta(k_0+k_2)\mathrm{e}^{\mathrm{i}k_2\cdot\theta k_1}\langle\Omega|T\tilde{\phi}(k_1)\tilde{\phi}(k_3)|\Omega\rangle$$
$$+\,\delta(k_0+k_3)\langle\Omega|T\tilde{\phi}(k_1)\tilde{\phi}(k_4)|\Omega\rangle\Big)\ . \tag{17}$$

This is braided Schwinger-Dyson equation in the free braided scalar field theory.

Example II: Braided electrodynamics

4D Minkowski space-time, Moyal-Weyl twist, massive spinor field ψ, $U(1)$ gauge field A_μ. An example of L_∞ algebra with gauge and matter fields. More examples discussed in [Gomes et al. '20].

The braided L_∞ algebra of spinor electrodynamics:

$$\mathcal{A} = \begin{pmatrix} \bar\psi \\ \psi \\ A_\mu \end{pmatrix}, \quad F_{\mathcal{A}} = \begin{pmatrix} F_{\bar\psi} \\ F_\psi \\ (F_A)_\mu \end{pmatrix}, \quad \ell_1^\star(\rho) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{e}\partial_\mu\rho \end{pmatrix}, \quad \ell_2^\star(\rho, \mathcal{A}) = \begin{pmatrix} -i\mathrm{R}_k(\bar\psi)\star\mathrm{R}^k(\rho) \\ i\rho\star\psi \\ i[\rho, A]_\star = 0 \end{pmatrix},$$

$$\ell_1^\star(F_{\mathcal{A}}^\star) = \partial_\mu(F_A^\star)^\mu, \quad \ell_2^\star(\mathcal{A}, F_{\mathcal{A}}^\star) = -ie\big(\bar\psi\star F_{\bar\psi} - \mathrm{R}_k(F_\psi)\star\mathrm{R}^k(\psi)\big),$$

$$\ell_1^\star(\mathcal{A}) = \begin{pmatrix} i\gamma^\mu\partial_\mu\psi \\ -i\gamma^\mu\partial_\mu\bar\psi \\ -\partial_\mu\partial_\nu A^\nu + \partial_\nu\partial^\nu A_\mu \end{pmatrix}, \quad \ell_2^\star(\mathcal{A}_1, \mathcal{A}_2) = -\frac{e}{2}\begin{pmatrix} \gamma^\mu A_{1\,\mu}\star\psi_2 + \mathrm{R}_k\gamma^\mu A_{2\,\mu}\star\mathrm{R}^k\psi_1 \\ \bar\psi_1\star\gamma^\mu A_{2\,\mu} + \mathrm{R}_k\bar\psi_2\star\gamma^\mu\mathrm{R}^k A_{1\,\mu} \\ \bar\psi_1\gamma^\mu\star\psi_2 + \mathrm{R}_j\bar\psi_2\gamma^\mu\star\mathrm{R}^j\psi_1 \end{pmatrix}.$$

Braided action

$$S = \int \mathrm{d}^4x\,\Big\{-\frac{1}{4}F^{\mu\nu}\star F_{\mu\nu} + \bar\psi\star i\gamma^\mu\partial_\mu\psi + \frac{e}{2}\Big(\bar\psi\star A_\mu\gamma^\mu\star\psi + \bar\psi\star\mathrm{R}_k(A_\mu)\gamma^\mu\star\mathrm{R}^k(\psi)\Big)\Big\},$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Braided equations of motion

$$F^{\star}_{\bar{\psi}} = i\gamma^{\mu}(\partial_{\mu}\psi - i\frac{e}{2}\big(A_{\mu}\star\psi + \mathrm{R}_k(A_{\mu})\star\mathrm{R}^k(\psi)\big) = 0,$$

$$(F^{\star}_{A})^{\mu} = \partial_{\mu}F^{\mu\nu} + \frac{e}{2}\big(\bar{\psi}\star\gamma^{\nu}\psi + \mathrm{R}_k\bar{\psi}\star\gamma^{\nu}\mathrm{R}^k\psi\big) = 0.$$

Where do we see the braided gauge symmetry? II Noether identity:

$$\partial_{\mu}(F^{\star}_{A})^{\mu} + \frac{e}{2}\partial_{\mu}\big(\bar{\psi}\gamma^{\mu}\star\psi + \mathrm{R}_k(\bar{\psi})\gamma^{\mu}\star\mathrm{R}^k(\psi)\big) = 0.$$

If $F^{\star}_{A} = 0$, then the metter current $\bar{\psi}\star\gamma^{\mu}\star\psi + \mathrm{R}_k(\bar{\psi})\gamma^{\mu}\star\mathrm{R}^k(\psi)$ is conserved. The corresponding conserved charge:

$$Q^{\star} = e\int \mathrm{d}^3\vec{x}\,\big(\psi^{\dagger}\star\psi + \mathrm{R}_k(\psi^{\dagger})\star\mathrm{R}^k(\psi)\big).$$

Nontrivial contribution if $\theta^{0j}\neq 0$. Consequences...

BV quantization

Braided QED is an Abelian gauge theory, ghost decouple, no photon self-interactions. However, the photon-fermion interaction is different compared to the $\star$-electrodynamics.

As before, we extend the L_∞-algebra to the space of observables $Sym V[2]$.

Momentum space basis: $A \sim \mathtt{v}^\mu\, \mathtt{e}_k$, $\psi \sim \mathtt{u}^s\, \mathtt{e}_k$, $\bar{\psi} \sim \bar{\mathtt{u}}^s\, \mathtt{e}_k$, $A^+ \sim \mathtt{v}_\mu\, \mathtt{e}^k$, $\psi^+ \sim \mathtt{u}_s\, \mathtt{e}^k$, $\bar{\psi}^+ \sim \bar{\mathtt{u}}_s\, \mathtt{e}^k$.

Contracted coordinate functions $\boldsymbol{\xi} \in Sym V[2] \otimes V$

$$\boldsymbol{\xi} = \int_k \left(\mathtt{v}_\mu\, \mathtt{e}^k \otimes \mathtt{v}^\mu\, \mathtt{e}_k + \bar{\mathtt{u}}_s\, \mathtt{e}^k \otimes \mathtt{u}^s\, \mathtt{e}_k + \mathtt{u}_s\, \mathtt{e}^k \otimes \bar{\mathtt{u}}^s\, \mathtt{e}_k \right) \ .$$

Interaction action

$$\mathcal{S}_{\rm int} = -\frac{1}{6} \langle\!\langle \xi, \ell_2^{\star\, ext}(\xi,\xi) \rangle\!\rangle_\star = \int_{k_1,k_2,k_3} V^{\mu;s,s'}(k_1,k_2,k_3)\ \mathtt{v}_\mu\, \mathtt{e}^{k_1} \odot_\star \mathtt{u}_s\, \mathtt{e}^{k_2} \odot_\star \bar{\mathtt{u}}_{s'}\, \mathtt{e}^{k_3} \ ,$$

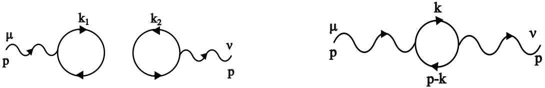
$$V^{\mu;s,s'}(k_1,k_2,k_3) = -e\, \bar{\mathtt{u}}^s\, \gamma^\mu\, \mathtt{u}^{s'}\, \mathrm{e}^{\frac{\mathrm{i}}{2} \sum_{a<b} k_a\cdot\theta\, k_b}\ (2\pi)^4\, \delta(k_1+k_2+k_3)$$

$$= \mathrm{e}^{\frac{\mathrm{i}}{2}\, k_1\cdot\theta\, k_2}\, V^{\mu;s,s'}(k_2,k_1,k_3), \quad \text{braided symmetric.}$$

BV quantization: braided QED, 1-loop

2-point functions at 1-loop

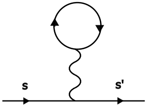
$$G^{\star}_{A_\mu,A_\nu}(p_1,p_2)^{(1)} = (i\,\hbar\,\Delta_{\rm BV}\,\mathsf{H})^2\,\big\{\mathcal{S}_{\rm int}\,,\mathsf{H}\,\big\{\mathcal{S}_{\rm int}\,,\mathsf{H}\big(\mathrm{v}_\mu\,\mathsf{e}^{p_1}\odot_\star \mathrm{v}_\nu\,\mathsf{e}^{p_2}\big)\big\}_\star\big\}_\star$$
$$+\,i\,\hbar\,\Delta_{\rm BV}\,\mathsf{H}\,\big\{\mathcal{S}_{\rm int}\,,\mathsf{H}\,(i\,\hbar\,\Delta_{\rm BV}\,\mathsf{H})\big\{\mathcal{S}_{\rm int}\,,\mathsf{H}\big(\mathrm{v}_\mu\,\mathsf{e}^{p_1}\odot_\star \mathrm{v}_\nu\,\mathsf{e}^{p_2}\big)\big\}_\star\big\}_\star\,.$$

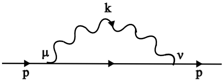


Vacuum polarization at 1-loop:

$$\frac{\mathrm{i}}{\hbar}\,\Pi^{\mu\nu}_{\star 2}(p) = e^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{Tr}\big((\not p-\not k-m)\,\gamma^\mu\,(\not k+m)\,\gamma^\nu\big)}{\big((p-k)^2-m^2\big)\,\big(k^2-m^2\big)}\ .$$

$$G^{\star}_{\psi_{s_1},\bar{\psi}_{s_2}}(p_1,p_2)^{(1)} = (i\,\hbar\,\Delta_{\rm BV}\,\mathsf{H})^2\,\big\{\mathcal{S}_{\rm int}\,,\mathsf{H}\,\big\{\mathcal{S}_{\rm int}\,,\mathsf{H}\big(\bar{\mathrm{u}}_{s_1}\,\mathrm{e}^{p_1}\odot_\star\mathrm{u}_{s_2}\,\mathrm{e}^{p_2}\big)\big\}_\star\big\}_\star$$
$$+\,i\,\hbar\,\Delta_{\rm BV}\,\mathsf{H}\,\big\{\mathcal{S}_{\rm int}\,,\mathsf{H}\,(i\,\hbar\,\Delta_{\rm BV}\,\mathsf{H})\big\{\mathcal{S}_{\rm int}\,,\mathsf{H}\big(\bar{\mathrm{u}}_{s_1}\,\mathrm{e}^{p_1}\odot_\star\mathrm{u}_{s_2}\,\mathrm{e}^{p_2}\big)\big\}_\star\big\}_\star\,.$$





Fermion self-energy at 1-loop:

$$\frac{\mathrm{i}}{\hbar}\,\Sigma_{\star 2}(p) = e^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\gamma^\mu\,(\not{p}-\not{k}-m)\,\gamma_\mu}{k^2\,\big((p-k)^2-m^2\big)}\ .$$

No NC corrections, no UV/IR mixing in 2-point functions at 1-loop! 3-point function at 1-loop needs to be calculated. Any non-trivial NC contributions in scattering amplitudes for physical processes?

Outlook

- We deformed the L_∞-algebra to a braided L_∞-algebra.

  -well defined way to construct a braided L_∞-algebra starting from the classical one.

  -enables constructions of new NC field theories (unexpected deformations, different from the "naive" expectations).

- Quantization, so far

  -no non-planar diagram, no UV/IR mixing in NC braided QFT.

  -renormalization in braided ϕ^3 needs to be studied carefully.

  -scattering amplitudes, measurable effects?

- Future work

  -better understanding of braided symmetries and classical braided field theories, new solutions of the classical equations (in gravity).

  -better understanding of braided QFT, BV quantization of braided NC Yang-Mills.