Quantization of modified KdV (boson–fermion correspondence)

A. K. Pogrebkov

Steklov Mathematical Institute;
HSE University;
Skolkovo Institute of Science and Technology

NONLINEARITY, NONLOCALITY AND ULTRAMETRICITY
International Conference on the Occasion of
Branko Dragovich 80th Birthday
May 26–30, 2025
Belgrade, Serbia

Introduction

Special quantum fields that first appeared in the literature under the name "massless two-dimensional fermionic fields" are known for decades to be a useful tool of investigation of completely integrable models in quantum and in classical cases. The Coleman–Mandelstam fermionization technique appeared in the context of the quantization of the famous Sine-Gordon equation about fifty years ago. But in spite of original expectations, correspondence of the Sine–Gordon and massive Thirring models did not lead to essential results for either one of these models. Nevertheless, the main observation that the most nonlinear parts of the quantum bosonic Hamiltonians become bilinear when bosonic field is considered as a composition of fermionic ones was confirmed by other examples.

The decay of the degree of monomials of fermionic operators follows from the anticommutator: $\left[\psi(x), \psi(y)\right]_+ = 0$, so that for any normally ordered (with respect to fermions) monomial we have

$$: \ \ldots \psi(x)\psi(x) \ldots \ : \ = 0.$$

Here $\psi(x) = \sqrt{\hbar/(2\pi)} \int dk e^{ikx} \widetilde{\psi}(k)$, where $k \in \mathbb{R}$, and its conjugate, $\psi^*(x)$, denote fermionic field, such that the only nonzero anticommutator relation sounds as

$$[\psi^*(x), \psi(y)]_+ = \hbar\delta(x - y),$$

where annihilation operators are $\widetilde{\psi}(k)\Omega = 0$, when $k < 0$, and $\widetilde{\psi}^*(k)\Omega = 0$, when $k > 0$, where Ω denotes the vacuum. Thus

$$(\Omega, \psi(x)\psi^*(y)\Omega) = (\Omega, \psi^*(x)\psi(y)\Omega) = \frac{-i\hbar}{2\pi(x - y - i0)}.$$

It was discovered by Lieb and Mattis (1962), that

$$\left[:\psi^*\psi:(x), :\psi^*\psi:(y)\right] = \frac{\hbar^2}{2\pi}\delta'(x-y),$$

where δ' denotes derivative of the δ-function. Thus by introducing bosonic field

$$v(x) = \frac{:\psi^*\psi:(x)}{\sqrt{\hbar/(2\pi)}},$$

we get that

$$\left[v(x), v(y)\right] = \hbar\delta'(x-y).$$

It is easy to show that this field admits decomposition

$$v(x) = v(x)^+ + v^-(x),$$

where

$$v(x)^+ = (v^-(x))^*, \qquad v^-(x)\Omega = 0,$$

and we need another version of the Wick ordering, corresponding to bosonic field:

$$\vdots v^2 \vdots(x) = (v(x)^+)^2 + 2v(x)^+v^-(x) + (v^-(x))^2, \text{ etc.}$$

For any real x and y we consider function

$$F(x,y)=\frac{:\psi^*(x+y)\psi(x-y):}{\sqrt{\hbar/(2\pi)}},$$

that generalizes $v(x)$ as

$$F(x,0)=v(x),$$

In 2001 it was proved (AP) that the above bilinear combination of Fermi-fields equals

$$F(x,y)=\frac{\vdots\exp\left(i\sqrt{2\pi/\hbar}\int_{x-y}^{x+y}dx'\,v(x')\right)\vdots-1}{2i\sqrt{2\pi/\hbar}\,y}$$

in the sense of operator-valued distributions with respect to variable x that is smooth, infinitely differentiable function of y. Let us introduce

$$F_n(x)=\left(\sqrt{\frac{\hbar}{2\pi}}\frac{\partial_y}{2i}\right)^n F(x,y)\bigg|_{y=0}.$$

It is easy to check that

$$\forall y: \qquad \int dx\, F(x,y)\Omega = 0,$$

and

$$\forall y, y': \qquad \left[\int dx\, F(x,y), \int dx'\, F(x',y')\right] = 0.$$

Thus, introducing $H_n = \int dx F_n(x)$ we get for any $m, n = 0, 1, \ldots$:

$$H_m\Omega = 0, \text{ and } [H_m, H_n] = 0.$$

Thus we have a hierarchy of integrable evolutions

$$v_{t_m}(x) = i[H_m, v(x)], \qquad m = 0, 1, \ldots,$$

where all H_m are bilinear in terms of fermions.

While in terms of bosons:

$$H_2 = \int dx \left\{ \frac{1}{3} :v^3:(x) - \frac{\hbar v_{xx}(x)}{24\pi} \right\},$$
$$H_3 = \int dx \left\{ \frac{1}{4} :v^4:(x) - \frac{\hbar}{8\pi} :v(x) v_{xx}(x): \right\},$$
$$H_4 = \int dx \left\{ \frac{1}{5} :v^5:(x) - \frac{\hbar}{4\pi} :v^2(x) v_{xx}(x): \right\}, \text{ etc.}$$

Thus we generate infinite hierarchy of equations

$$v_{t_2}(x) = \partial_x :v^2:(x),$$
$$v_{t_3}(x) = \partial_x \left(:v^3:(x) - \frac{\hbar}{4\pi} v_{xx}(x) \right),$$
$$v_{t_4}(x) = \partial_x \left(:v^4:(x) - \frac{\hbar}{\pi} :v v_{xx}:(x) - \frac{\hbar}{2\pi} :v_x^2:(x) \right), \ldots .$$

This equations are close to the equations of the KdV hierarchy, but some terms are missed. The only exception is given by t_3-evolution, where we have a special case of the modified KdV equation. In the classical situation this equation sounds as

$$v_t = v^2 v_x + \alpha v_{xxx}$$

on the real function $v(t,x)$. It has the Lax pair, $L_t = [L, A]$, with operator

$$L = \begin{pmatrix} \partial_x - 2iz^2 - iv^2 & 2izv + v_x \\ 2izv - v_x & \partial_x + 2iz^2 + iv^2 \end{pmatrix}$$

where z is a spectral parameter. This equation is Hamiltonian, $v_t = -\{H, v\}$, with respect to the Poisson bracket

$$\{v(x), v(y)\} = \delta'(x-y),$$

where Hamiltonian H equals

$$H = \int dx \Big\{ \frac{1}{4} v^4(x) + \alpha v_x^2(x) \Big\},$$

In the quantum case we had above:

$$v_t(x) = i[H, v(x)],$$
$$H = \int dx \left\{ \frac{1}{4} \vdots v^4 \vdots (x) - \frac{\hbar}{8\pi} \vdots v(x) v_{xx}(x) \vdots \right\},$$
$$v_t(x) = \partial_x \left(\vdots v^3 \vdots (x) - \frac{\hbar}{4\pi} v_{xx}(x) \right),$$

so that we have a special version of the mKdV equation with

$$\alpha = -\frac{\hbar}{4\pi}.$$

It is unclear if this equation must be considered as the dispersionless one, or not. But in the contrast to the classical dispersionless case it has global solution.

Indeed, taking into account that Hamiltonian H is a bilinear function of ψ we get the time dependence of $\psi(x)$ as $\psi_t = i[H, \psi]$, we get

$$\psi_t(t,x) = -\frac{\hbar}{2\pi}\partial_x^3\psi(t,x),$$

so that under time evolution

$$\psi(x) = \sqrt{\hbar/(2\pi)}\int dk\, \exp\{ikx\}\widetilde{\psi}(k) \quad \Rightarrow$$

$$\psi(t,x) = \sqrt{\hbar/(2\pi)}\int dk\, \exp\left\{ikx + \frac{i\hbar k^3 t}{2\pi}\right\}\widetilde{\psi}(k)$$

Now operator $F(t,x,y)$ equals to

$$F(t,x,y) = \sqrt{\frac{2\pi}{\hbar}} : \psi^*(t,x+y)\psi(t,x-y): \ .$$

and taking $v(t,x) = F(t,x,0)$ into account we derive global solution

$$v(t,x)=\sqrt{\frac{\hbar}{2\pi}}\iint dkdp\, e^{i(k+p)x+i\hbar(k^3+p^3)t/(2\pi)}:\widetilde{\psi}^*(-k)\widetilde{\psi}(p):\,,$$

where due to the above

$$:\widetilde{\psi}^*(-k)\widetilde{\psi}(p):\;=\frac{1}{2i\pi\hbar}\iint dxdy\, e^{-ikx-ipy}\times \\ \times\frac{\vdots\exp\left\{i\sqrt{2\pi\hbar}\int_y^x d\xi v(\xi)\right\}\vdots-1}{x-y},$$

that is independent of t. Inserting this equality in the previous one we derive the global solution of the quantum version of the mKdV equation:

$$v(t,x) = \frac{-i}{\sqrt{(2\pi)^2\hbar}} \iiiint dk dp dx' dy' e^{ik(x-x')+ip(x-y')+i\hbar(k^3+p^3)t/(2\pi)} \times$$

$$\times \frac{\vdots \exp\left\{ i\sqrt{2\pi\hbar} \int_{y'}^{x'} d\xi v(\xi) \right\} \vdots - 1}{x' - y'},$$

Solution of the initial problem for the (dispersionless!) mKdV equation can be written in the parametric form as

$$x = s - 3tv^2(s), \qquad v(t,x) = v(s),$$

where $v(x)$ is initial data and s is defined by the first equality. This solution is known to describe overturn of the front, so the initial problem has no global solution. It is easy to see that in the quantum case the global solution in the limit $\hbar \to 0$ takes the limit

$$v(t,x) = \int dp \left[\theta\left(v(x + 3tp^2) - p \right) - \theta(-p) \right],$$

that coincides with the above before the overturn of the front.

Expectation values.

Massless fermions have following (nonzero) expectation values:

$$\big(\Omega, \widetilde{\psi}(p)\widetilde{\psi}^*(-k)\Omega\big) = \theta(k)\delta(k+p),$$
$$\big(\Omega, \widetilde{\psi}^*(-k)\widetilde{\psi}(p)\Omega\big) = \theta(p)\delta(k+p),$$

so that for t-dpendence we have

$$\big(\Omega, \psi^*(t_1, x_1)\psi(t_2, x_2)\big) = \big(\Omega, \psi(t_1, x_1)\psi^*(t_2, x_2)\big) = D(t_1 - t_2, x_1 - x_2),$$

where

$$D(t, x) = \frac{\hbar}{2\pi}\int_0^\infty dk e^{-ikx - i\hbar k^3 t/(2\pi)}.$$

Now relation

$$v(t, x) = \sqrt{\frac{2\pi}{\hbar}} : \psi^*(t, x)\psi(t, x) : \; .$$

provides calculation of expectation values:

$$\big(\Omega, v(t_1, x_1) \cdots v(t_n, x_n)\Omega\big). \qquad (1)$$

In particular,

$$(\Omega, v(t_1, x_1)v(t_2, x_2)\Omega) = D_{12}^2.$$

where we denoted $D(t_i - t_j, x_i - x_j) = D_{ij}$. Here we have a kind of the standard coupling, but calculation for the higher powers is more involved. Say,

$$\begin{aligned}
&(\Omega, v(t_1, x_1)v(t_2, x_2)v(t_3, x_3)\Omega) = 2D_{12}D_{13}D_{23}, \\
&(\Omega, v(t_1, x_1)v(t_2, x_2)v(t_3, x_3)v(t_4, x_4)\Omega) = \\
&\quad = D_{12}^2 D_{34}^2 + D_{13}^2 D_{24}^2 + D_{14}^2 D_{24}^2 \\
&\quad + 2D_{12}D_{13}D_{24}D_{34} + 2D_{13}D_{14}D_{23}D_{24}.
\end{aligned}$$

These vacuum expectation values demonstrate composite character of the field v. Field obeys nonlinear evolution equation, but in terms of fermions it is linearised. Nevertheless, fermions never appear in interaction. Their existence is manifested only in the loop structure of the expectation values above. Thus we have here model of confinement of the fermions.

Best wishes to
Branko Dragovich!!!