p-Adic Quantum Information

Towards a theory of entanglement

Nonlinearity, Nonlocality and Ultrametricity conference

Vincenzo Parisi,
School of advanced studies, university of Camerino

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Introduction

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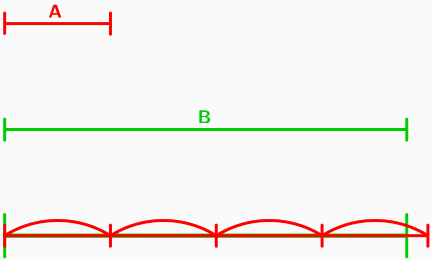

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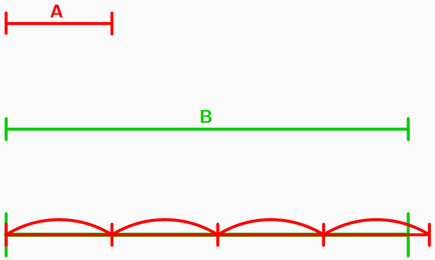

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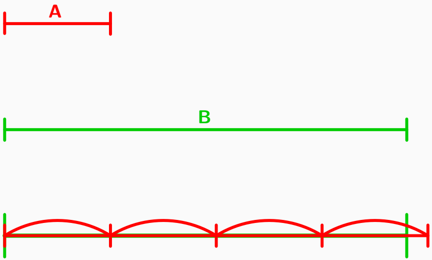

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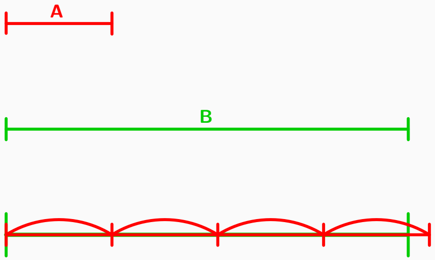

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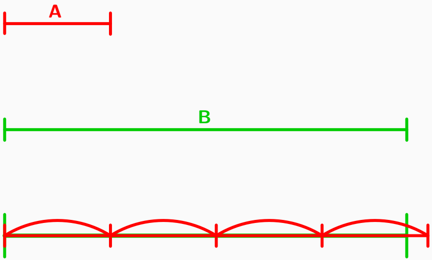

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- A natural description is by means of the field of p-adic numbers.

p-adic quantum mechanics

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- $U(t)$ is the unitary evolution operator induced by a p-adic Hamiltonian.

One can follow two possible approaches:

- Construct a qubit as a two-dimensional state vector in an abstract p-adic Hilbert space.
- Describe a p-adic qubit as a suitable complex representation of $\mathrm{SO}(3,\mathbb{Q}_p)$.

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Quadratic extensions of $\mathbb{Q}_p$

Let $\mu \in \mathbb{Q}_p$ be a non-quadratic element in $\mathbb{Q}_p$ ($\mu \notin (\mathbb{Q}_p^*)^2$). The quadratic extension $\mathbb{Q}_{p,\mu}$ of $\mathbb{Q}_p$, induced by μ, is the field

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The quadratic extensions of $\mathbb{Q}_p$ are classified as follows:

(1) If $p \neq 2$, the non-isomorphic quadratic extensions are given by $\mathbb{Q}_{p,\mu}$, with $\mu \in \{\eta, p, \eta p\}$, where $\eta \in \mathbb{Q}_p$ is a non-quadratic unit ($\eta \notin (\mathbb{Q}_p^*)^2$, and $|\eta|_p = 1$);

(2) if $p = 2$, the non-isomorphic quadratic extensions are given by $\mathbb{Q}_{p,\mu}$, with $\mu \in \{2, 3, 5, 6, 7, 10, 14\}$.

Part I

Mathematical Foundations

Basic Results

p-adic Banach spaces

A p-adic Banach space is a pair $(X, \|\cdot\|)$, where X is a vector space over $\mathbb{Q}_{p,\mu}$, $\|\cdot\| : X \to \mathbb{R}^+$ is a p-adic norm:

(i) $\|x\| = 0 \iff x = 0$;

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$;

(iii) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ (strong triangle inequality),

and X is complete w.r.t. the ultrametric induced by $\|\cdot\|$.

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Example:

Consider the space of zero-convergent sequences in $\mathbb{Q}_{p,\mu}$

$$c_0(\mathbb{N}, \mathbb{Q}_{p,\mu}) := \{x = \{x_i\}_{i\in\mathbb{N}} \mid x_i \in \mathbb{Q}_{p,\mu}, \ \lim_i |x_i| = 0\}.$$

This is a p-adic Banach space once endowed with the sup-norm

$$\|\cdot\|_\infty : c_0(\mathbb{N}, \mathbb{Q}_{p,\mu}) \ni x \mapsto \|x\|_\infty := \sup_{i\in\mathbb{N}} |x_i|$$

- A finite set of vectors $\{x_1, \ldots, x_n\}$ is norm-orthogonal if

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| = \max_{1 \le i \le n} \left\| x_i \right\|, \quad \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{Q}_{p,\mu}.$$

  A subset $\mathfrak{B} \subset X$ is norm-orthogonal if every finite subset is so, and it is normal if, additionally, $\|x\| = 1$, for every $x \in X$.

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A subset $\mathfrak{B} \subset X$ is norm-orthogonal if every finite subset is so, and it is normal if, additionally, $\|x\| = 1$, for every $x \in X$.

- A normal basis in X is a subset $\mathfrak{B} \equiv \{b_i\}_{i\in I} \subset X \setminus \{0\}$ that is normal and such that, for any $x \in X$,

$$x = \sum_{i\in I} \lambda_i b_i, \quad \{\lambda_i\}_{i\in I} \subset \mathbb{Q}_{p,\mu}, \; \lim_i |\lambda_i| = 0, \quad \|x\| = \sup_{i\in I} |x_i|.$$

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- A p-adic Banach space $(X, \|\cdot\|)$ admits a normal basis $\mathfrak{B} \equiv \{b_i\}_{i \in I}$ iff it is separable, and $\|X\| = |\mathbb{Q}_{p,\mu}|$. Moreover, the map

$$c_0(I, \mathbb{Q}_{p,\mu}) \ni \{x_i\}_{i \in I} \mapsto \sum_{i \in I} x_i b_i \in X,$$

  defines a surjective isometry of $c_0(I, \mathbb{Q}_{p,\mu})$ into X.

p-adic Hilbert spaces

A p-adic Hilbert space is defined as a quadruple $\left\langle \mathcal{H}, \|\cdot\|, \langle \cdot\,,\cdot \rangle, \Phi \equiv \{\phi_i\}_{i \in I} \right\rangle$ where

- $(\mathcal{H}, \|\cdot\|)$ is a p-adic Banach space;
- $\langle \cdot\,,\cdot \rangle$ is a p-adic inner product;
- $\Phi \equiv \{\phi_i\}_{i \in I}$ is an orthonormal basis, i.e.:

$$\|\phi_i\| = 1, \quad \langle \phi_i, \phi_j \rangle = \delta_{ij} \quad \forall i, j \in I, \qquad x = \sum_{i \in I} \langle \phi_i, x \rangle \phi_i, \quad \forall x \in \mathcal{H}.$$

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An orthonormal basis $\Phi \equiv \{\phi_i\}_{i\in I}$ is a normal basis of (mutually) inner-product orthogonal vectors

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Example:

Consider $(c_0(I, \mathbb{Q}_{p,\mu}), \|\cdot\|_\infty)$ and take $\{e_i\}_{i\in I}$

$$e_1 = (1, 0, 0, \ldots),\ \ e_2 = (0, 1, 0, \ldots),\ \ \ldots$$

Define the (canonical) inner product $\langle \cdot\,, \cdot \rangle_c$

$$c_0(I, \mathbb{Q}_{p,\mu}) \times c_0(I, \mathbb{Q}_{p,\mu}) \ni (x, y) \mapsto \langle x\,, y \rangle_c = \sum_i \overline{x_i} y_i \in \mathbb{Q}_{p,\mu},$$

- ✦ $\mathbb{H}(I) \equiv \langle c_0(I, \mathbb{Q}_{p,\mu}), \|\cdot\|_\infty, \langle \cdot\,, \cdot \rangle_c, \{e_i\}_{i\in I} \rangle$ is a p-adic Hilbert space, i.e., the coordinate p-adic Hilbert space

- ✦ The map

$$W_\Phi \colon \mathcal{H} \ni x = \sum_{i\in I} \langle \phi_i\,, x \rangle \phi_i \mapsto \check{x} \equiv \{\langle \phi_i\,, x \rangle\}_{i\in I} \in \mathbb{H}(I)$$

  is a surjective isometry, i.e., an isomorphism of p-adic Hilbert spaces.

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- ✦ $\|x\| \neq \sqrt{\langle x\,,x \rangle}$. There exist isotropic vectors $x \in \mathcal{H}$, i.e., $x \in \mathcal{H}$, $x \neq 0$ such that

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- $\mathcal{H}$ is not self-dual, i.e., $\mathcal{H} \neq \mathcal{H}'$. The dual $\mathcal{H}'$ can be identified with $\ell^\infty(I, \mathbb{Q}_{p,\mu})$

$$\ell^\infty(I, \mathbb{Q}_{p,\mu}) \coloneqq \{\xi = \{\xi_i\}_{i\in I} \mid \xi_i \in \mathbb{Q}_{p,\mu},\ \xi \text{ bounded sequence in } \mathbb{Q}_{p,\mu}\}.$$

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- There is no surjective isometry between $\mathcal{H}$ and $\mathcal{H}''$, i.e., $\mathcal{H}$ is not reflexive. Instead, $\mathcal{H}$ is pseudoreflexive, i.e., the map

$$\mathcal{I}_{\mathcal{H}} \colon \mathcal{H} \ni \psi \mapsto \Big(\mathcal{H}' \ni \phi' \mapsto \phi'(\psi) \in \mathbb{Q}_{p,\mu}\Big) \in \mathcal{H}''$$

is an isometry (not surjective) of $\mathcal{H}$ into its bidual $\mathcal{H}''$

Linear operators in a p-adic Hilbert space

Bounded and adjointable operators

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- For every orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$,

$$\mathcal{B}(\mathcal{H}) = \big\{\mathrm{op}_\Phi(A_{mn}) \mid \lim_m A_{mn} = 0,\ \forall n, \text{ and } \sup_{m,n} |A_{mn}| < \infty\big\},$$

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- If $A \colon \mathcal{H} \to \mathcal{H}$ is bounded, then $A = \mathrm{op}_\Phi(\langle \phi_m , A\phi_n \rangle) \equiv \mathrm{op}_\Phi(A_{mn})$,

  $$A = \textstyle\sum_m \sum_n A_{mn} \langle \phi_n , \cdot \rangle \phi_m,$$

  where $\big(\langle \phi_n , \cdot \rangle \phi_m\big)\psi := \langle \phi_n , \psi \rangle \phi_m$, with the series converging w.r.t. the strong operator topology.

Since $\mathcal{H} \ncong \mathcal{H}'$, not all bounded operators admits a Hilbert space adjoint. A is adjointable — i.e., $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ — iff

$$\sup_{m,n} |A_{mn}| < \infty, \ \ \lim_m A_{mn} = 0, \ \forall n \in \mathbb{N}, \ \lim_n A_{mn} = 0, \ \forall m \in \mathbb{N}.$$

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In this case, $A^* = \mathrm{op}_\Phi(A^*_{mn}) = \mathrm{op}_\Phi(\overline{A_{nm}})$, i.e.,

$$A^* = \sum_m \sum_n A^*_{mn} \langle \phi_n, \cdot \rangle \phi_m = \sum_m \sum_n \overline{\langle \phi_n, A\phi_n \rangle} \langle \phi_n, \cdot \rangle \phi_m.$$

Since $\mathcal{H} \not\cong \mathcal{H}'$, not all bounded operators admits a Hilbert space adjoint. A is adjointable — i.e., $A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ — iff

$$\sup_{m,n} |A_{mn}| < \infty, \ \ \lim_m A_{mn} = 0, \ \forall n \in \mathbb{N}, \ \lim_n A_{mn} = 0, \ \forall m \in \mathbb{N}.$$

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The Trace class of a p-adic Hilbert space

The set of trace class operators is defined, for $(T_{mn}) \in \mathrm{M}_\infty(\mathbb{Q}_{p,\mu})$, as

$$\mathcal{T}(\mathcal{H})_\Phi := \left\{ T = \mathrm{op}_\Phi(T_{mn}) \mid \lim_{m+n} T_{mn} = \lim_{m+n} \langle \phi_m, T\phi_n \rangle = 0 \right\}.$$

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- $\mathcal{T}(\mathcal{H})$ is a two-sided $*$-ideal in $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$;
- $\mathcal{T}(\mathcal{H}) = \mathcal{C}(\mathcal{H})_{\mathrm{ad}} := \mathcal{C}(\mathcal{H}) \cap \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ is a norm-closed subset of $\mathcal{C}(\mathcal{H})$; any trace class operator T can be expressed as

  $$T = \textstyle\sum_{j\in J} \lambda_i \langle f_j, \cdot \rangle e_j,$$

  with the series converging w.r.t. the norm-topology. This gives the singular-value decomposition of $T \in \mathcal{T}(\mathcal{H})$.

On $\mathcal{T}(\mathcal{H})$ we can define the Hermitian sesquilinear form

$$\mathcal{T}(\mathcal{H}) \times \mathcal{T}(\mathcal{H}) \ni (S, T) \mapsto \langle S, T \rangle_{\mathcal{T}(\mathcal{H})} := \mathrm{tr}(S^* T) \in \mathbb{Q}_{p,\mu}$$

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✦ For any given orthonormal basis $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, we can construct the family of matrix operators $\{{}^{jk}E^{\Phi}\}_{j,k\in\mathbb{N}}$ as

$$ {}^{jk}E^{\Phi} := \mathrm{op}_{\Phi}({}^{jk}E^{\Phi}_{mn}) = \langle \phi_j, \cdot \rangle \phi_k, \quad \text{where } {}^{jk}E^{\Phi}_{mn} = \delta_{jm}\delta_{kn}.$$

In particular, we have: $\langle {}^{jk}E^{\Phi}, T \rangle_{\mathcal{T}(\mathcal{H})} = T_{jk}$, and $\{{}^{jk}E^{\Phi}\}_{j,k\in\mathbb{N}}$ is an o.n.b. in $\mathcal{T}(\mathcal{H})$.

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- $(\mathcal{T}(\mathcal{H}), \langle \cdot , \cdot \rangle_{\mathcal{T}(\mathcal{H})})$ is an inner product p-adic Banach space. For every o.n.b. $\Phi \equiv \{\phi_m\}_{m\in\mathbb{N}}$ in $\mathcal{H}$, $(\mathcal{T}(\mathcal{H}), \|\cdot\|, \langle \cdot , \cdot \rangle_{\mathcal{T}(\mathcal{H})}, \{{}^{jk}E^{\Phi}\}_{j,k\in\mathbb{N}})$ is the p-adic Hilbert-Schmidt space.

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Tensor product of p-adic Hilbert spaces

✦ Define the projective norm over $\mathcal{H}\hat{\otimes}\mathcal{K}$

$$\mathcal{H}\hat{\otimes}\mathcal{K} \ni u \mapsto \|u\|_{\pi} := \inf\left\{ \max_{1\leq i\leq n} \|x_i\|_{\mathcal{H}} \|y_i\|_{\mathcal{K}} \mid u = \sum_{i=1}^{n} x_i \hat{\otimes} y_i \right\}.$$

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- If $\Phi \equiv \{\phi_i\}_{i\in\mathbb{N}}$ and $\Psi \equiv \{\psi_j\}_{j\in\mathbb{N}}$ are two o.n.b. in $\mathcal{H}$ and $\mathcal{K}$, then $\Pi \equiv \{\phi_i\hat{\otimes}\psi_j\}_{i,j\in\mathbb{N}\times\mathbb{N}}$ is an o.n.b. in $\mathcal{H} \otimes_\pi \mathcal{K}$. Thus, $\langle \mathcal{H} \otimes_\pi \mathcal{K}, \|\cdot\|_\pi, \langle\,\cdot\,,\,\cdot\,\rangle_\pi, \Pi\rangle$ is a p-adic Hilbert space.

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- The map $\mathcal{J}\colon \mathcal{T}(\mathcal{H},\mathcal{K}) \to \mathcal{H}\otimes_\pi\mathcal{K}$ defined as

$$\mathcal{T}(\mathcal{H},\mathcal{K}) \ni T \mapsto \mathcal{J}(T) := \textstyle\sum_{i,j\in\mathbb{N}} \langle\psi_j, T\phi_i\rangle\, \phi_i\hat{\otimes}\psi_j, \quad \mathcal{J}(\langle\phi_i,\cdot\rangle\psi_j) = \left(J_\Phi\phi_i\right)\hat{\otimes}\psi_j,$$

  is an isomorphism of the p-adic Hilbert space $\mathcal{H}\otimes_\pi\mathcal{K}$ into $\mathcal{T}(\mathcal{H},\mathcal{K})$.

Part II

p-Adic Quantum Mechanics

Algebraic formulation

Algebraic formulation of quantum mechanics

- The (bounded) observables of the system form the self-adjoint part $\mathfrak{A}_{\mathrm{sa}}$ of an abstract non-commutative unital C^*-algebra $\mathfrak{A}$.

- The set of states, $\mathfrak{S}(\mathfrak{A})$, is a convex subset of the (complex) Banach space of bounded functionals on $\mathfrak{A}$. A state ω of $\mathfrak{A}$ satisfies

$$\omega(A^*A) \geq 0, \quad \forall A \in \mathfrak{A}, \quad \omega(\mathrm{Id}) = 1.$$

- The pairing between sates and observables is provided by the evaluation map:

$$\mathfrak{A}_{\mathrm{sa}} \times \mathfrak{S}(\mathfrak{A}) \ni (A, \omega) \mapsto \omega(A) \in \mathbb{R};$$

  $\omega(A)$ gives the expectation value of the observable A in the state ω

- The GNS construction allows us to recover the usual description of quantum mechanics in terms of density operators.

Definition

An algebraic state for a p-adic system is a functional $\omega_p \colon \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}$ such that:

(i) $\|\omega_p\| := \sup_{\|A\|=1} |\omega_p(A)| < \infty$;

(ii) $\omega_p(\mathrm{Id}) = 1$;

(iii) $\omega_p(A^*) = \overline{\omega_p(A)}, \quad \forall A \in \mathcal{B}_{\mathrm{ad}}(\mathcal{H})$.

We denote the set of p-adic algebraic states by $\mathcal{S}(\mathcal{B}_{\mathrm{ad}}(\mathcal{H}))$.

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Conditions (i)–(iii) are related to the following observations:

- ✦ The set of possible experimental outcomes is bounded.
- ✦ Since the possible experimental outcomes are p-adic numbers, we must use a p-adic model of probability: (ii) and (iii) assure that we can always construct a p-adic probability distribution.
- ✦ The field $\mathbb{Q}_{p,\mu}$ is not ordered. Hence, the positivity condition for states need not be required.

p-Adic trace induced states

Consider the linear functional ω_p on $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, defined, for $\rho \in \mathcal{T}(\mathcal{H})$, as

$$\omega_p \equiv \mathrm{tr}((\,\cdot\,)\rho) \colon \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \ni B \mapsto \omega_p(B) \equiv \mathrm{tr}(B\rho) \in \mathbb{Q}_{p,\mu}.$$

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- For ω_p to be an algebraic state of $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ it must be true that

$$\omega_p(\mathrm{Id}) = 1 \iff \mathrm{tr}(\rho) = 1, \quad \omega_p(B^*) = \overline{\omega_p(B)} \iff \rho = \rho^*.$$

- We define the set of statistical operators

$$\mathcal{T}_{\mathrm{st}}(\mathcal{H}) := \{S \in \mathcal{T}(\mathcal{H}) \mid S = S^*,\ \mathrm{tr}(S) = 1\},$$

for every $S \in \mathcal{T}_{\mathrm{st}}(\mathcal{H})$, $\mathrm{tr}((\,\cdot\,)S)\colon \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}$ is a state for $\mathcal{H}$.

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$$\mathcal{T}_{\mathrm{st}}(\mathcal{H}) := \{S \in \mathcal{T}(\mathcal{H}) \mid S = S^*,\ \mathrm{tr}(S) = 1\},$$

  for every $S \in \mathcal{T}_{\mathrm{st}}(\mathcal{H})$, $\mathrm{tr}((\,\cdot\,)S) \colon \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}$ is a state for $\mathcal{H}$.

- $\mathcal{T}_{\mathrm{st}}(\mathcal{H})$ is a closed $\mathbb{Q}_p$-affine subset of $\mathcal{T}(\mathcal{H})$; for every $T \in \mathcal{T}_{\mathrm{st}}(\mathcal{H})$

$$\mathcal{T}_{\mathrm{st}}(\mathcal{H}) = T + \mathcal{T}(\mathcal{H})_0,$$

  where $\mathcal{T}(\mathcal{H})_0 := \{S \in \mathcal{T}(\mathcal{H}) \mid S = S^*,\ \ \mathrm{tr}(S) = 0\}$.

p-Adic trace induced states

Consider the linear functional ω_p on $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$, defined, for $\rho \in \mathcal{T}(\mathcal{H})$, as

$$\omega_p \equiv \mathrm{tr}((\,\cdot\,)\rho)\colon \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \ni B \mapsto \omega_p(B) \equiv \mathrm{tr}(B\rho) \in \mathbb{Q}_{p,\mu}.$$

- For ω_p to be an algebraic state of $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$ it must be true that

$$\omega_p(\mathrm{Id}) = 1 \iff \mathrm{tr}(\rho) = 1, \quad \omega_p(B^*) = \overline{\omega_p(B)} \iff \rho = \rho^*.$$

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- The map defined as

$$\tau_{\mathcal{H}}\colon \mathcal{T}_{\mathrm{st}}(\mathcal{H}) \ni S \mapsto \big(\mathrm{tr}((\,\cdot\,)S)\colon \mathcal{B}_{\mathrm{ad}}(\mathcal{H}) \to \mathbb{Q}_{p,\mu}\big) \in \mathcal{S}(\mathcal{B}_{\mathrm{ad}}(\mathcal{H}))$$

is a continuous $\mathbb{Q}_p$-affine injection of $\mathcal{T}_{\mathrm{st}}(\mathcal{H})$ into $\mathcal{S}(\mathcal{B}_{\mathrm{ad}}(\mathcal{H}))$.

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Define the set

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- A statistical operator $S \in \mathcal{T}_{\rm st}(\mathcal{H})$ is in $\mathcal{D}(\mathcal{H})$ iff

$$S = \sum_{j \in J} \lambda_j \langle f_j, \cdot \rangle e_j, \quad \max_{j \in J} |\lambda_j| = 1, \quad \sum_{j \in J} \lambda_j \langle f_j \, , e_j \rangle = 1,$$

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  where $\{e_j\}_{j \in J}$ — or $\{f_j\}_{j \in J}$ — is a normalized norm-orthogonal system in $\mathcal{H}$.
- For a two-dimensional Hilbert space, $\dim(\mathcal{H}) = 2$, we can define p-adic qubits

  $$\mathcal{D}(\mathcal{H}) := \left\{\rho \in \mathrm{M}_2(\mathbb{Q}_{p,\mu}) \mid \rho = \frac{1}{2}\Big(\mathrm{Id}_2 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3\Big),\ (x_i)_{i=1}^3 \in \mathbb{Q}_p^3,\ \|\rho\| = 1\right\},$$

  $$\sigma_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 0 & \sqrt{\mu} \\ -\sqrt{\mu} & 0 \end{pmatrix}.$$

Statistical interpretation of the theory

p-Adic probability theory

Definition

A (discrete) p-adic probability distribution is a countable set $\{\pi_i\}_{i \in I} \subset \mathbb{Q}_p$ such that $\sum_{i \in I} \pi_i = 1$[2].

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- The collection of all probability distributions indexed by I can be identified with a $\mathbb{Q}_p$-affine subset of $c_0(I, \mathbb{Q}_{p,\mu})$:

$$\varpi(I, \mathbb{Q}_{p,\mu}) := \Big\{ \{\pi_i\}_{i\in I} \in c_0(I, \mathbb{Q}_{p,\mu}) \mid \pi_i \in \mathbb{Q}_p,\ \forall i \in I,\ \sum_{i\in I} \pi_i = 1 \Big\}.$$

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p-Adic observables

Definition

A (discrete) selfadjoint-operator-valued measure (SOVM) is a norm-bounded countable family $\{A_i\}_{i\in I} \subset \mathcal{B}_{\mathrm{sa}}(\mathcal{H})$ such that $\sum_{i\in I} A_i = Id$ (with the series converging in the weak operator topology). A SOVM is said to be contractive if $\|A_i\| \leq 1$ for all $i \in I$.

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A SOVM always induces a p-adic probability distribution. Let $\{A_i\}_{i\in I} \subset \mathcal{B}_{sa}(\mathcal{H})$ be a SOVM. Then

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- If $\Omega \in \mathcal{S}(\mathcal{B}_{\mathrm{ad}}(\mathcal{H}))$ is a trace induced state, i.e., $\Omega = \tau_{\mathcal{H}}(\rho)$, for some $\rho \in \mathcal{T}_{\mathrm{st}}(\mathcal{H})$, the sequence $\{\Omega(A_i)\}_{i\in I}$ defines a p-adic probability distribution in $\varpi(I, \mathbb{Q}_{p,\mu})$.

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- If Ω is a density state, i.e., $\Omega = \tau_{\mathcal{H}}(\rho)$ for some $\rho \in \mathcal{D}(\mathcal{H})$, and if $\{A_i\}_{i\in I}$ is contractive, the sequence $\{\Omega(A_i)\}_{i\in I}$ is contained in the probability simplex $\upsilon_0(I, \mathbb{Q}_{p,\mu})$.

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A (discrete) selfadjoint-operator-valued measure (SOVM) is a norm-bounded countable family $\{A_i\}_{i\in I} \subset \mathcal{B}_{\rm sa}(\mathcal{H})$ such that $\sum_{i\in I} A_i = Id$ (with the series converging in the weak operator topology). A SOVM is said to be contractive if $\|A_i\| \leq 1$ for all $i \in I$.

A SOVM always induces a p-adic probability distribution. Let $\{A_i\}_{i\in I} \subset \mathcal{B}_{sa}(\mathcal{H})$ be a SOVM. Then

- If $\Omega \in \mathcal{S}(\mathcal{B}_{\rm ad}(\mathcal{H}))$ is a trace induced state, i.e., $\Omega = \tau_{\mathcal{H}}(\rho)$, for some $\rho \in \mathcal{T}_{\rm st}(\mathcal{H})$, the sequence $\{\Omega(A_i)\}_{i\in I}$ defines a p-adic probability distribution in $\varpi(I, \mathbb{Q}_{p,\mu})$.
- If Ω is a density state, i.e., $\Omega = \tau_{\mathcal{H}}(\rho)$ for some $\rho \in \mathcal{D}(\mathcal{H})$, and if $\{A_i\}_{i\in I}$ is contractive, the sequence $\{\Omega(A_i)\}_{i\in I}$ is contained in the probability simplex $\upsilon_0(I, \mathbb{Q}_{p,\mu})$.
- Consider the family of Hermitian operators $\mathcal{M} = \{M_i\}_{i=1}^5$,

$$M_1 = \mathrm{Id}_2, \quad M_2 = -\sigma_1, \quad M_3 = -\sigma_2, \quad M_4 = -\sigma_3, \quad M_5 = \begin{pmatrix} 1 & 1+\sqrt{\mu} \\ 1-\sqrt{\mu} & -1 \end{pmatrix}.$$

  Then, $\sum_{i=1}^5 M_i = \mathrm{Id}_2$, $\{\mathrm{tr}(\rho M_i)\}_{i=1}^5 = \{1, -x_1, -x_2, \mu x_3, x_1 + x_2 - \mu x_3\}$ is a p-adic probability distribution.

Conclusions

Main results and outlook

Main results

- p-adic Hilbert space $\mathcal{H}$ over $\mathbb{Q}_{p,\mu}$;
- Characterization of
  - Bounded operators $\mathcal{B}(\mathcal{H})$.
  - Adjointable operators $\mathcal{B}_{\mathrm{ad}}(\mathcal{H})$.
  - Trace class operators $\mathcal{T}(\mathcal{H})$.
- Characterization of p-adic states:
  - p-adic algebraic states.
  - p-adic statistical and density operators.
- Definition of SOVMs as a suitable description of p-adic observables.

Outlook

- Symmetry transformations:
  - Maps which preserve the affine structure of the state space.
- Orthogonal projections:
  - Logic structure of a p-adic quantum system.
- Tensor product of p-adic Hilbert spaces:
  - Characterization of separable and entangled states.
- Dynamical maps and dynamical semigroups:
  - p-adic quantum channels and instruments.

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