Holography on Buildings

NONLINEARITY, NONLOCALITY AND ULTRAMETRICITY

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Sarthak Parikh
Indian Institute of Technology Delhi
sarthak@iitd.ac.in

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and work with Elliott Gesteau & Matilde Marcolli, JHEP '22

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REVIEW ARTICLES

p-Adic Mathematical Physics: The First 30 Years*

B. Dragovich[1,2]**, A. Yu. Khrennikov**[3,4]***, **S. V. Kozyrev**[5]****, **I. V. Volovich**[5]*****, **and E. I. Zelenov**[5]

[1] *Institute of Physics, University of Belgrade, 11080 Belgrade, Serbia*

[2] *Mathematical Institute, Serbian Academy of Sciences and Arts, Belgrade, Serbia*

[3] *International Center for Mathematical Modeling in Physics, Engineering, Economics, and Cognitive Science, Linnaeus University, S-35195, Växjö, Sweden*

[4] *National Research University of Information Technologies, Mechanics and Optics (ITMO), St. Petersburg, 197101 Russia*

[5] *Steklov Mathematical Institute of the Russian Academy of Sciences, Gubkina Str. 8, Moscow, 119991 Russia*

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Congratulations on this $2^4 5$th birthday milestone, Branko!

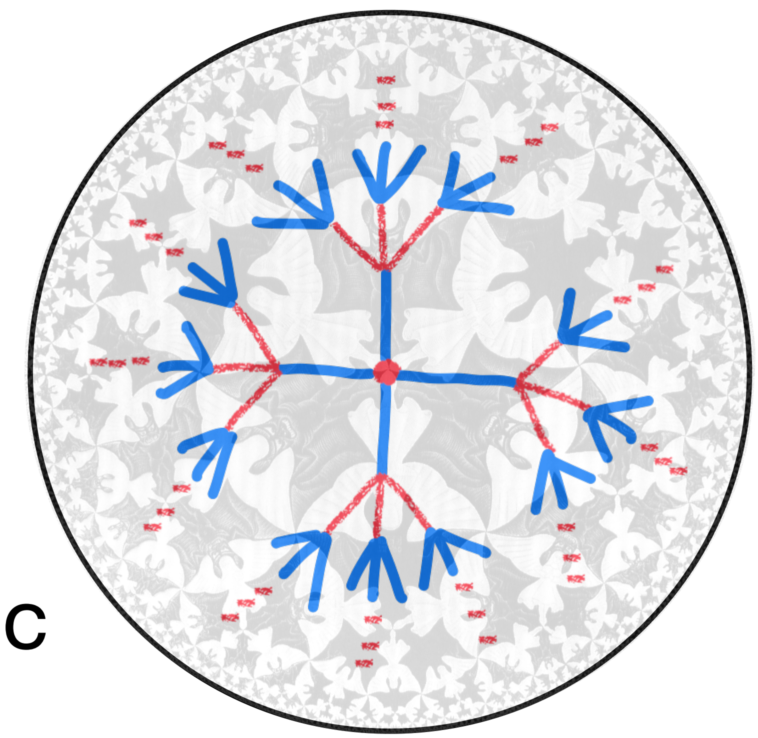
p-adic strings

Volovich '87
Freund, Olson '87
Brekke, Freund, Olson, Witten '89
Zabrodin '89

- Veneziano amplitude: adelic product

$$A_v^{(4)}(s,t,u) := \int_{\mathbb{Q}_v} dx\,|x|_v^{-\alpha(s)-1}\,|1-x|_v^{-\alpha(u)-1}$$

$$A_\infty^{(4)}(s,t,u)\prod_p A_p^{(4)}(s,t,u) = 1$$



- Boundary of worldsheet is real line/p-adic
- Worldsheet is the Upper Half Plane/Bruhat-Tits tree

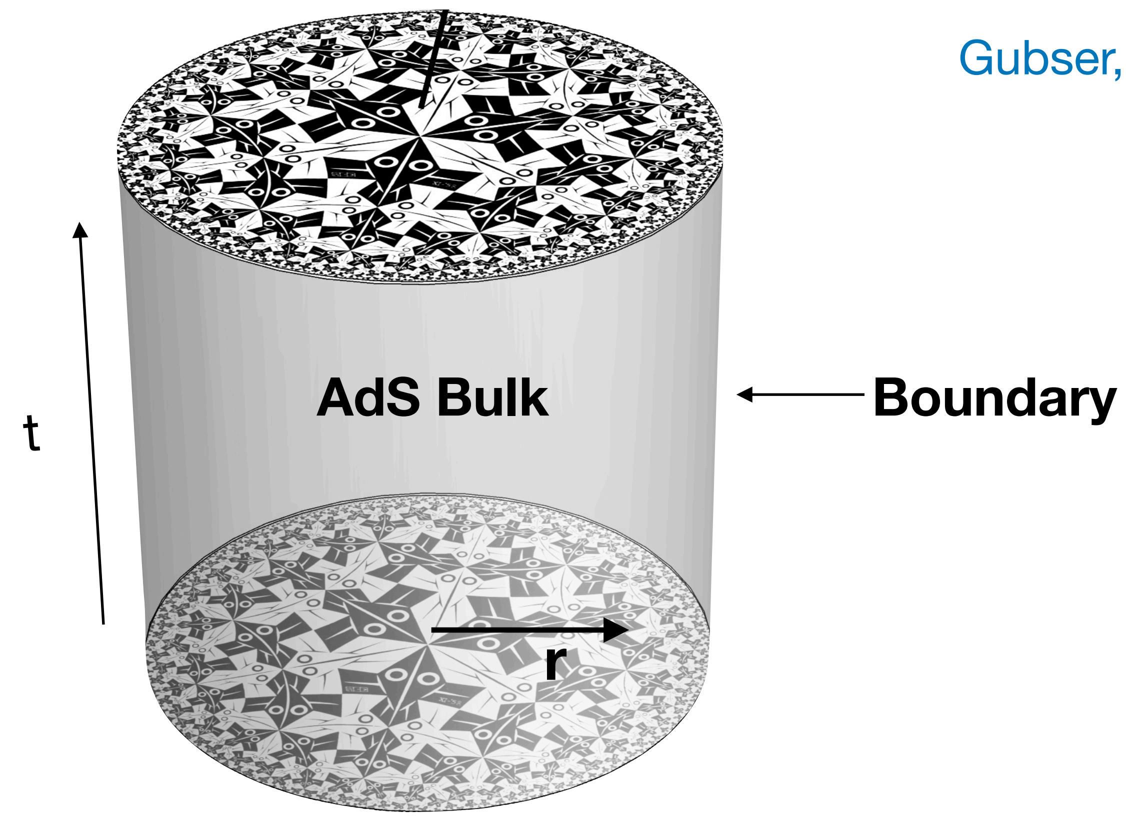
Hyperbolic disk vs Bruhat-Tits tree

- X = G/K where G = SL(2,$\mathbb{R}$) and K = SO(2,$\mathbb{R}$)
- ∂X = $\mathbb{P}^1(\mathbb{R})$
- vol(B(R)) $\sim \exp(R/L)$ where L = radius of curvature

- X = G/K where G = PGL(2,$\mathbb{Q}_p$) and K = PGL(2,$\mathbb{Z}_p$)
- ∂X = $\mathbb{P}^1(\mathbb{Q}_p)$
- vol(B(R)) $\sim \exp(R/L_p)$ where $L_p = 1/\log p$

Anti de-Sitter/CFT correspondence

't Hooft '93
Susskind '95
Maldacena '97
Gubser, Klebanov, Polyakov '98
Witten '98



Dictionary:

$$Z_{\text{gravity}_{d+1}}[\phi_0] = \left\langle \exp\left(\int d^d x\, \phi_0(x) \mathcal{O}(x) \right) \right\rangle_{\text{CFT}_d}$$

p-adic AdS/CFT

Gubser, Knaute, SP, Samberg, Witaszczyk. CMP '17
Heydeman, Marcolli, Saberi, Stoica. ATMP '18
Gubser, SP. PRD '17

- Adelic product formula for 3-point function/OPE coefficients of dual CFT

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle_\infty \prod_{\text{primes } p} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle_p$$

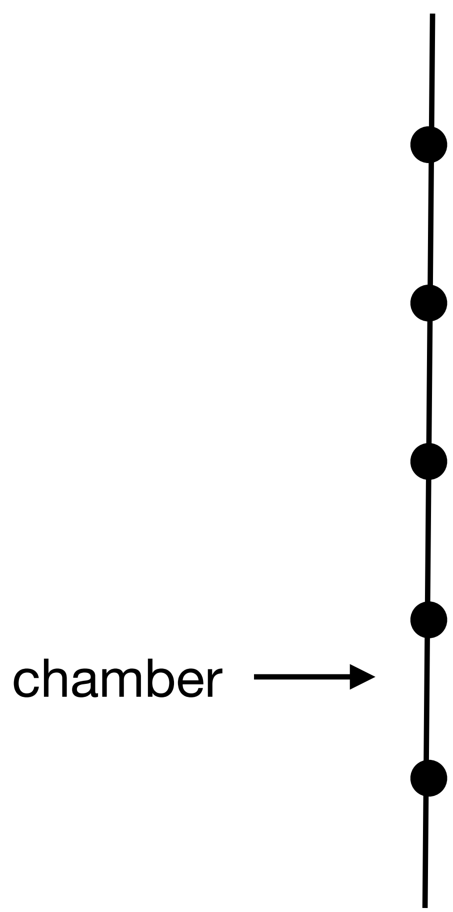
$$= \frac{\zeta^*(2\Delta_{12,3})\zeta^*(2\Delta_{23,1})\zeta^*(2\Delta_{31,2})\zeta^*(\Delta_{123,}-1)}{2\,\zeta^*(2\Delta_1)\zeta^*(2\Delta_2)\zeta^*(2\Delta_3)}$$

$$\zeta^*(s) = \pi^{-s/2}\Gamma_{\text{Euler}}(s/2)\,\zeta(s)$$

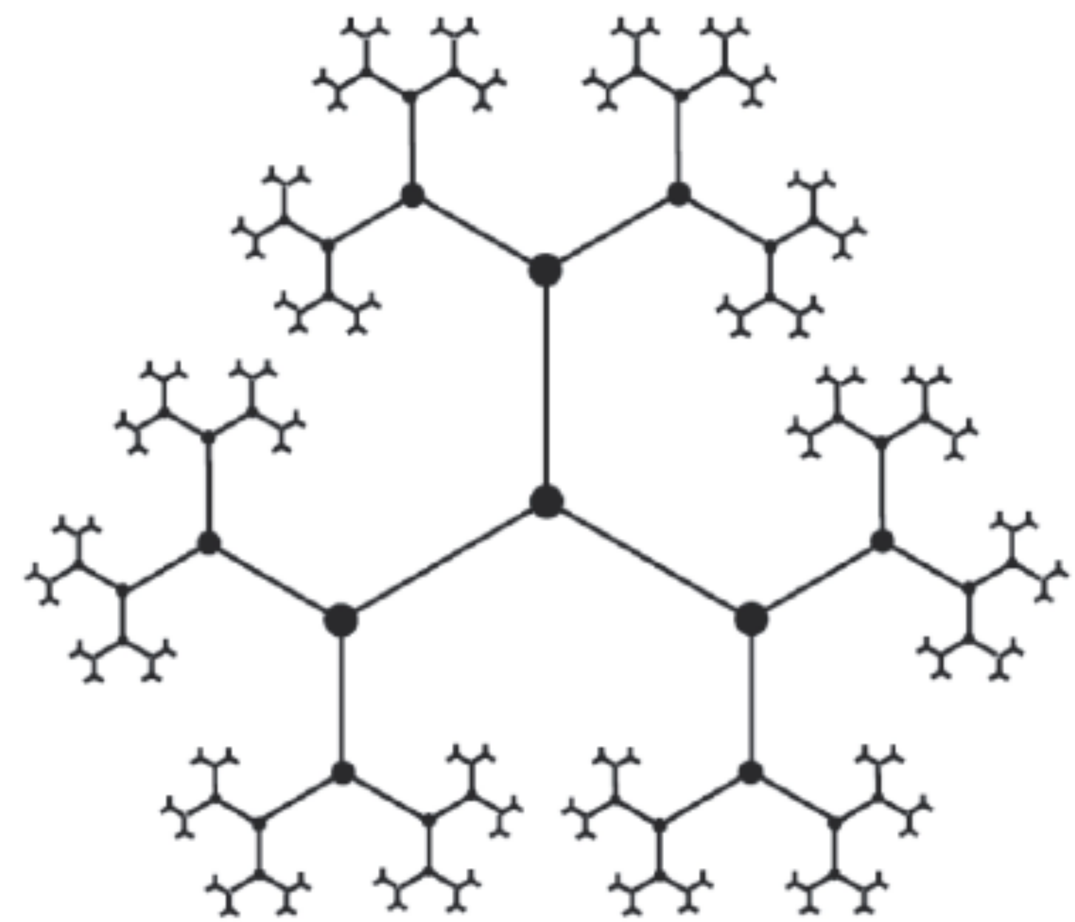
- $\mathbb{Q}_p/\mathbb{R}$-independent expressions for anomalous dimensions of operators

Gubser, Jepsen, SP, Trundy. JHEP '17

Example of Building



An apartment of the Building



Subset of the building

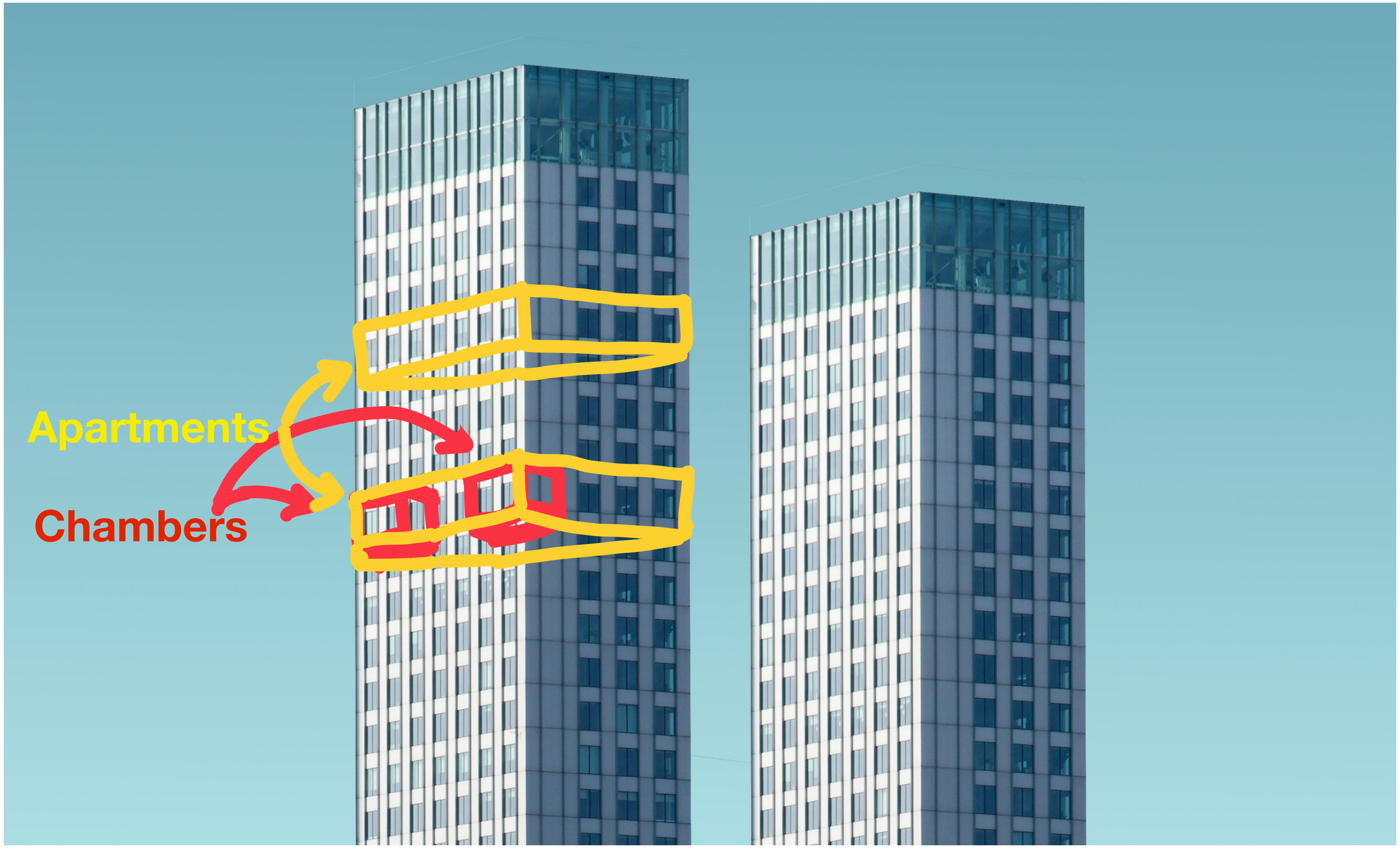


Photo by Simone Hutsch on Unsplash

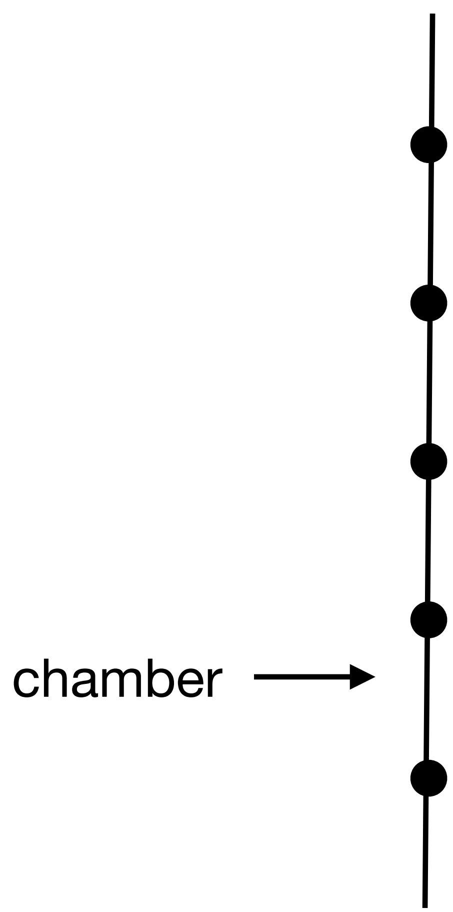
Buildings

in everyday life

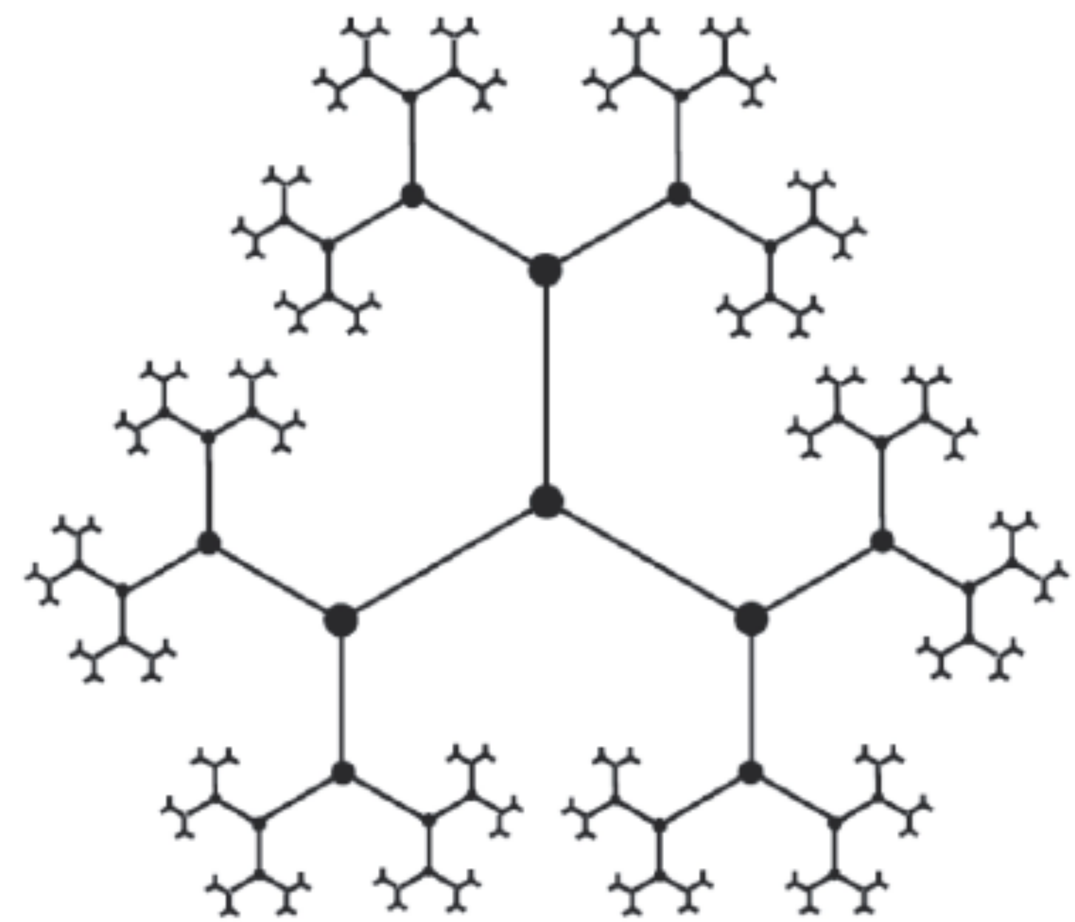
Buildings

- A **building** is a simplicial complex with maximal subcomplexes called apartments
- **Apartments** are tessellations of $\mathbb{X}^n$ by copies of chambers (i.e. they are Coxeter complexes).
- **Chambers** are simplices (convex polytopes) in $\mathbb{X}^n = \mathbb{H}^n, \mathbb{E}^n, \text{or } \mathbb{S}^n$.
- *Any* two chambers belong to a common apartment
- Between any two apartments, there exists an isometry that fixes their intersection
- Two geodesics starting from the same point are equivalent if they remain within a bounded distance of each other
- Visual boundary: Equivalence class of geodesics starting at ‘origin’

Example of Euclidean Building

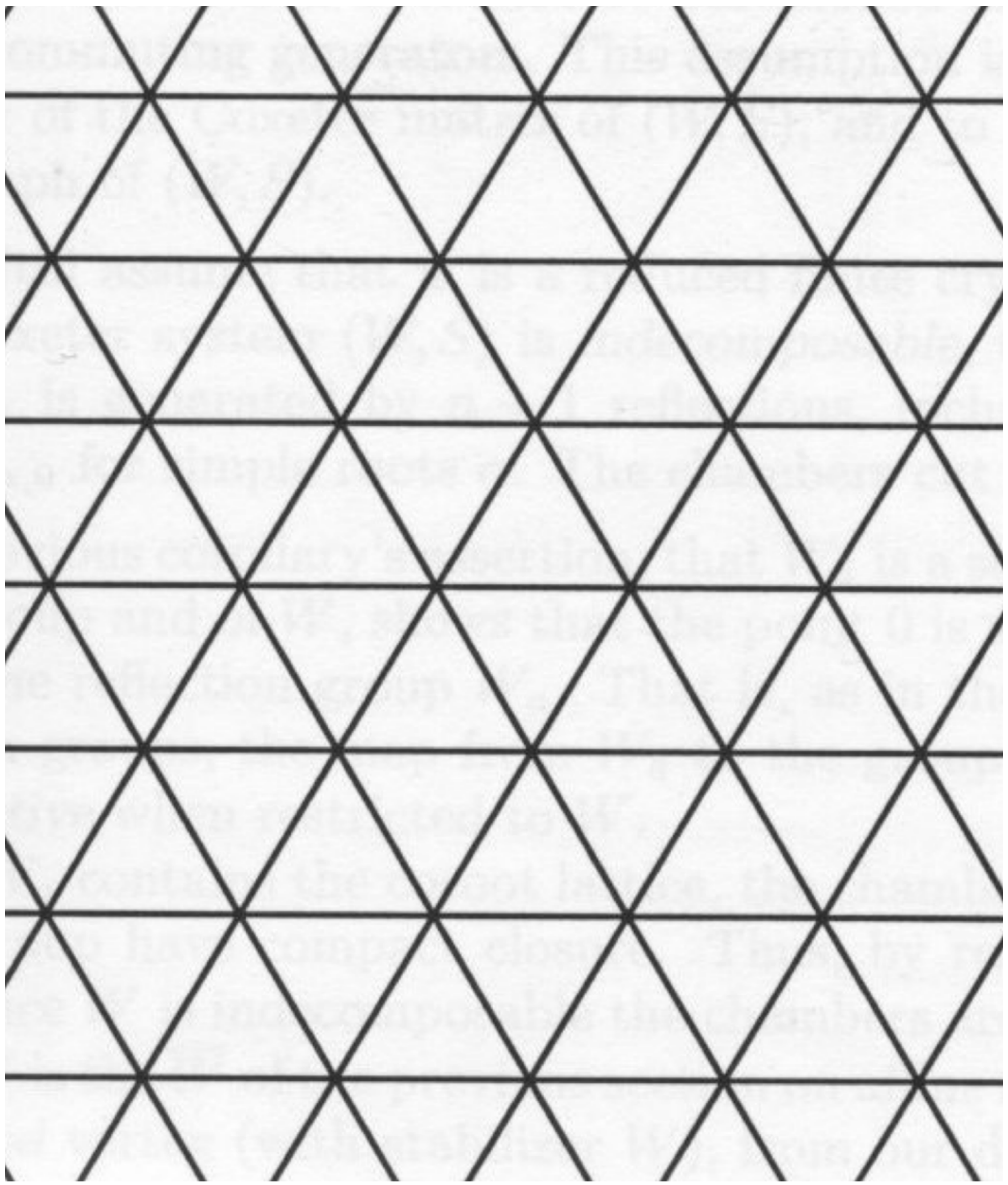


An apartment of the Euclidean Building

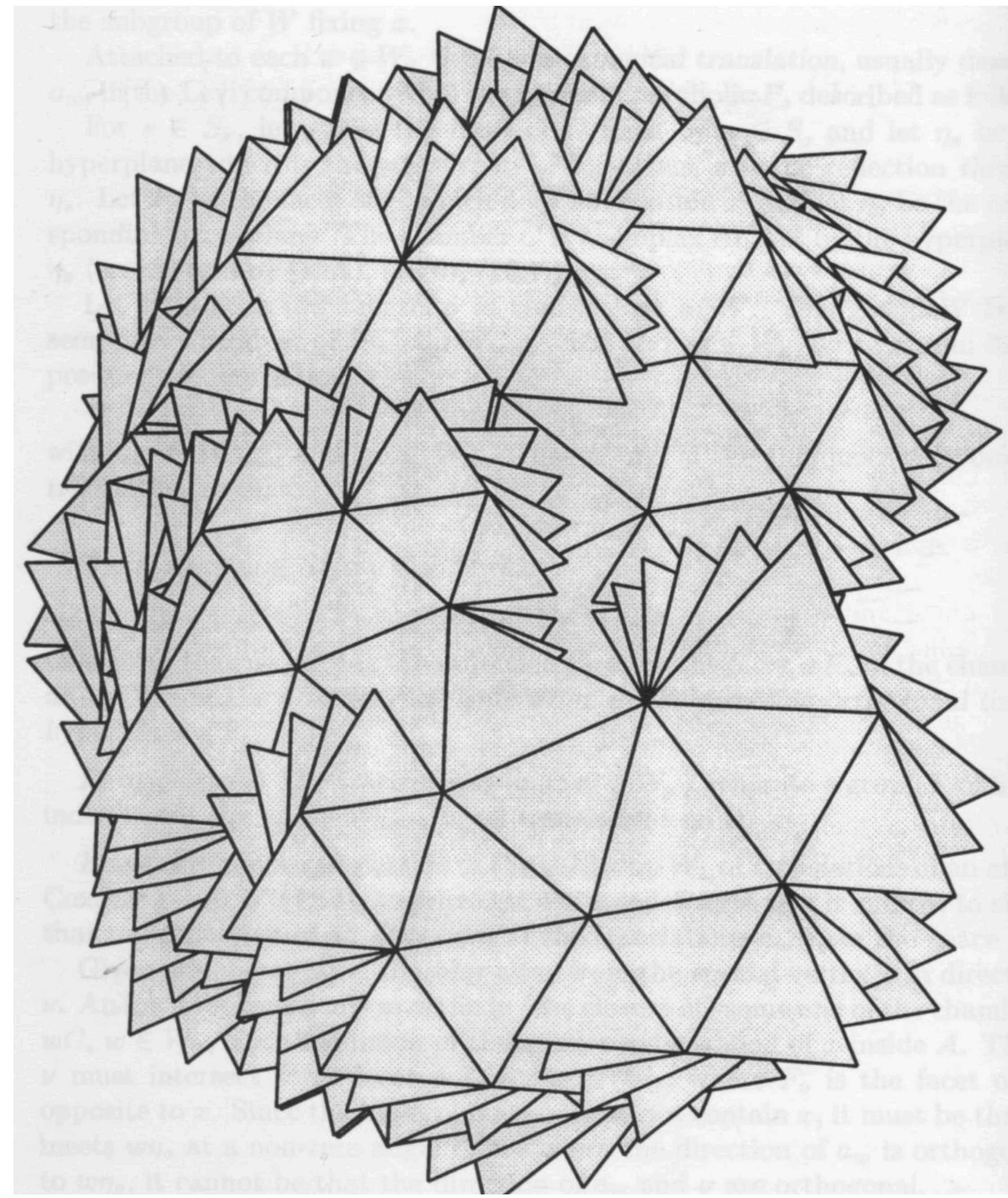


Subset of the Euclidean building

Example of Euclidean Building

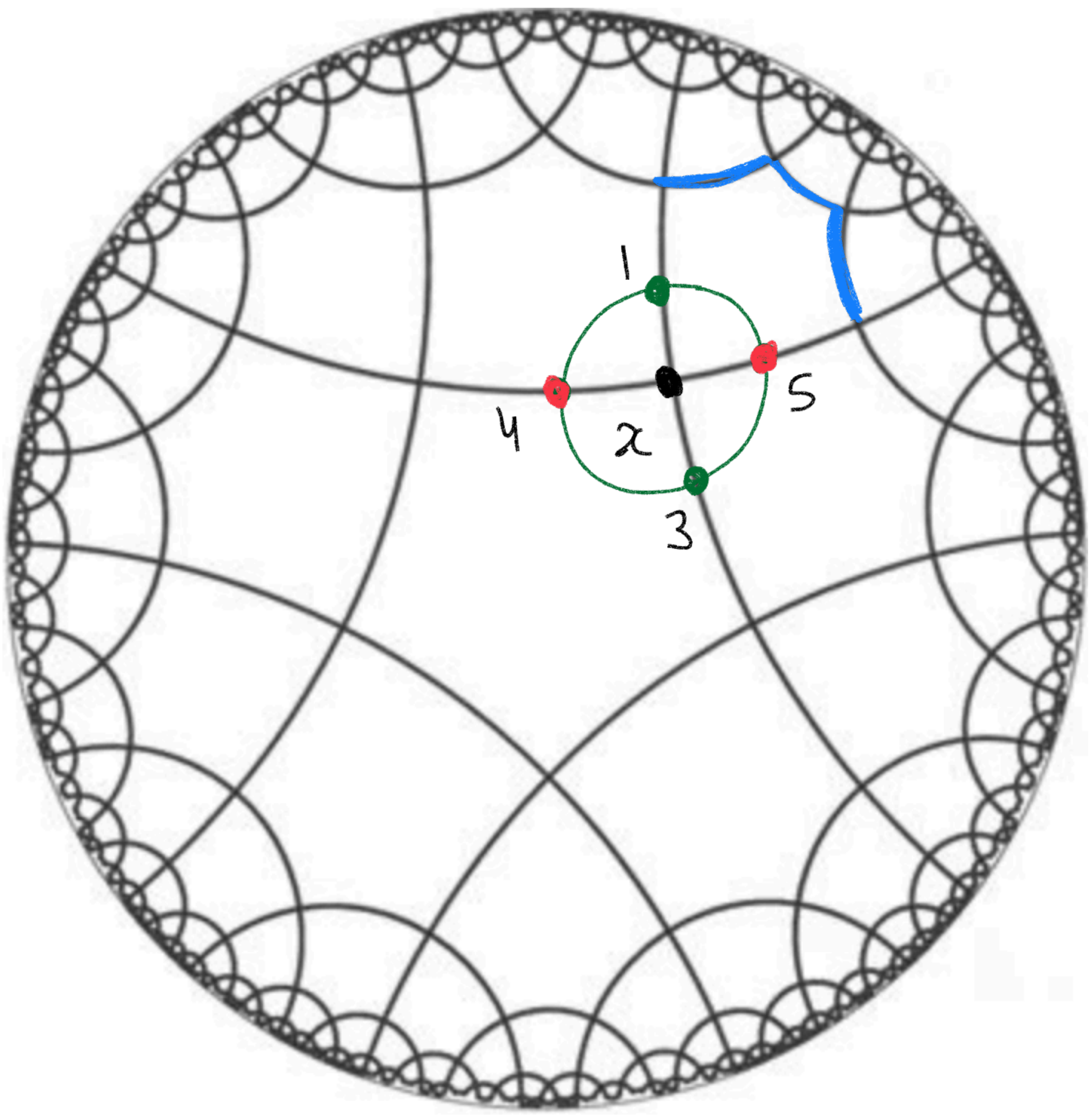


An apartment of the Euclidean Building

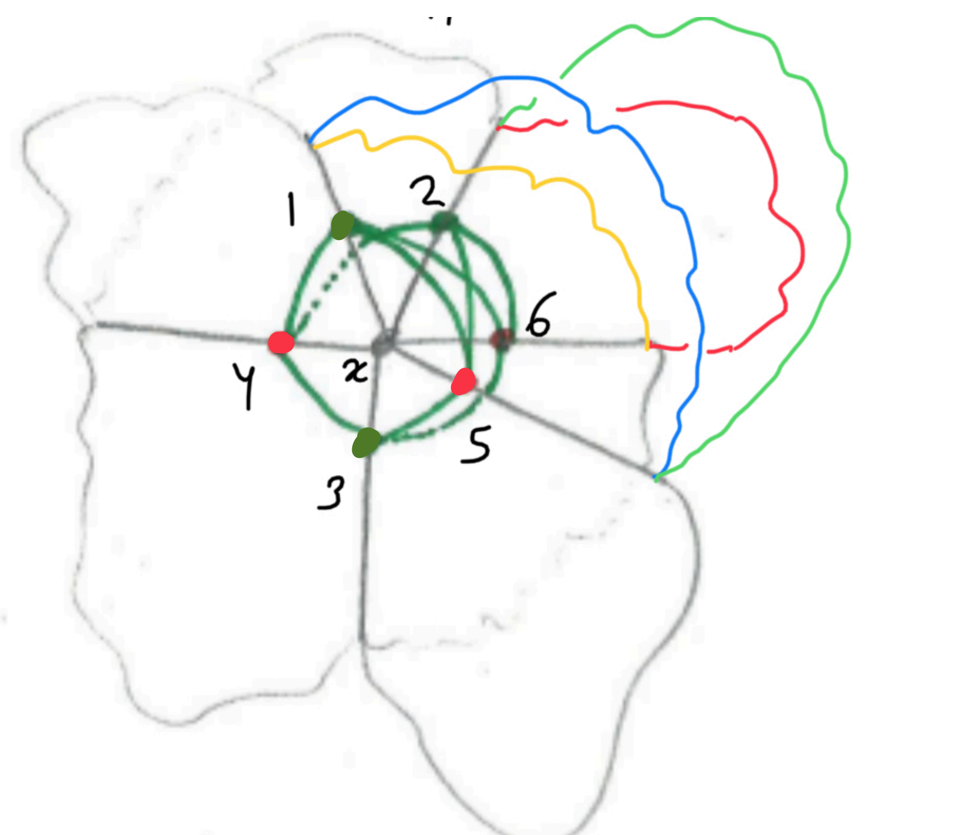


Subset of a Euclidean building [P Garrett]

Example of Hyperbolic Building



An apartment of the Hyperbolic Building



Subset of a Hyperbolic building

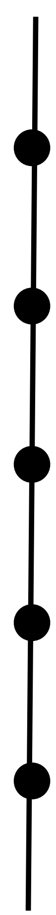
Outline

1. Buildings
2. Euclidean/Bruhat-Tits buildings
   - Effective field theory
   - Conformal correlators
3. Hyperbolic buildings
   - Tensor network
   - Ryu-Takayanagi formula

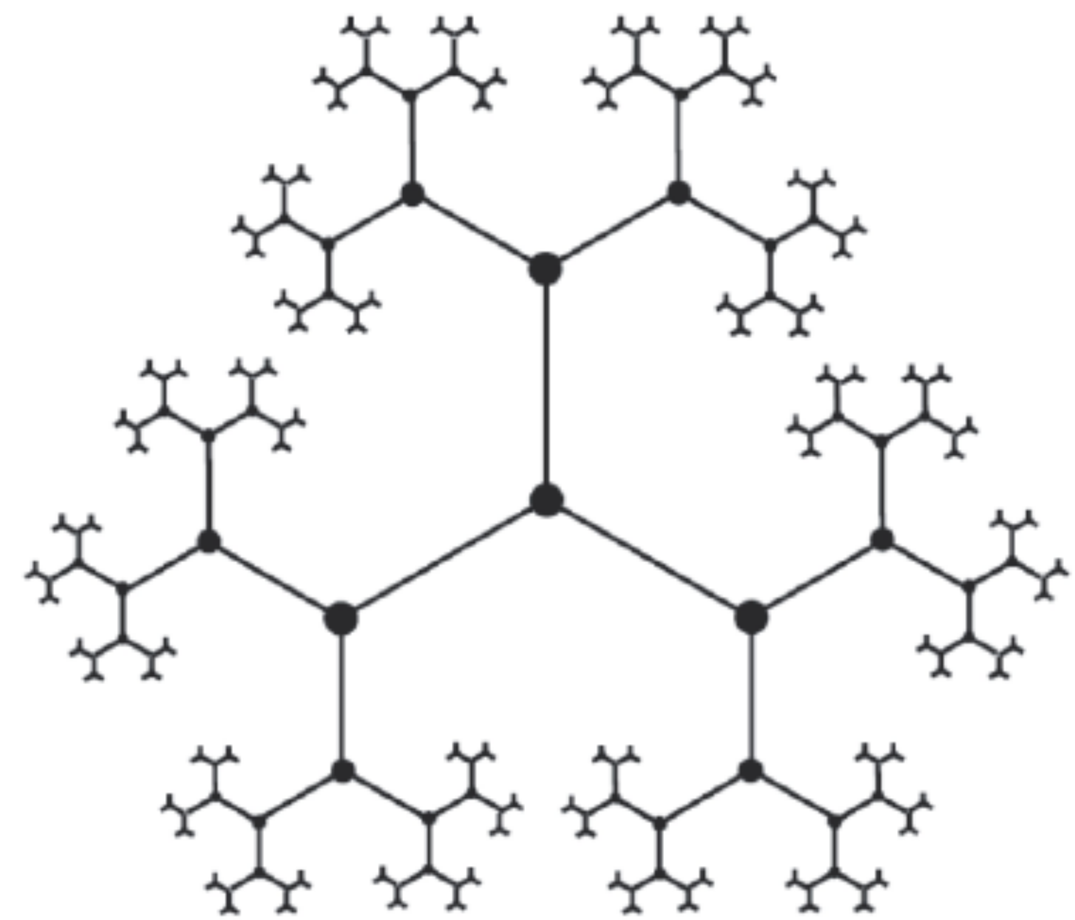
Euclidean/Bruhat-Tits Buildings

- A building is Euclidean if its apartments are Euclidean Coxeter complexes
- Isometry group: semi-simple group G over a field with discrete valuation. Set of chambers of building = G/K
- Nonpositive curvature (CAT(0) spaces)
- Boundary of BT buildings: spherical building
  - generalization of “celestial” sphere at infinity of Euclidean space
  - totally disconnected topology, ultrametric

Euclidean Building for $PGL_2(\mathbb{Q}_p)$

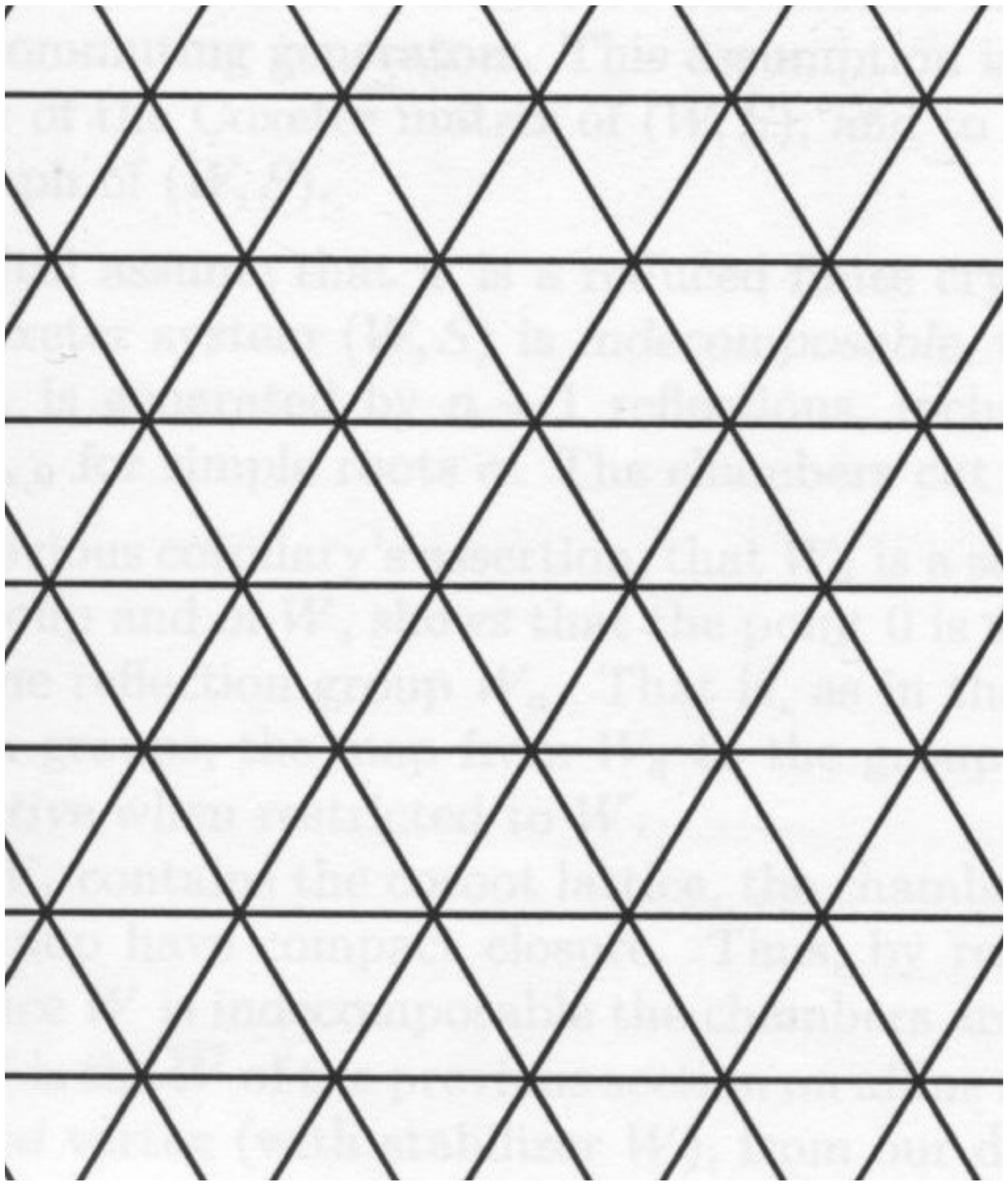


An apartment of a Euclidean Building

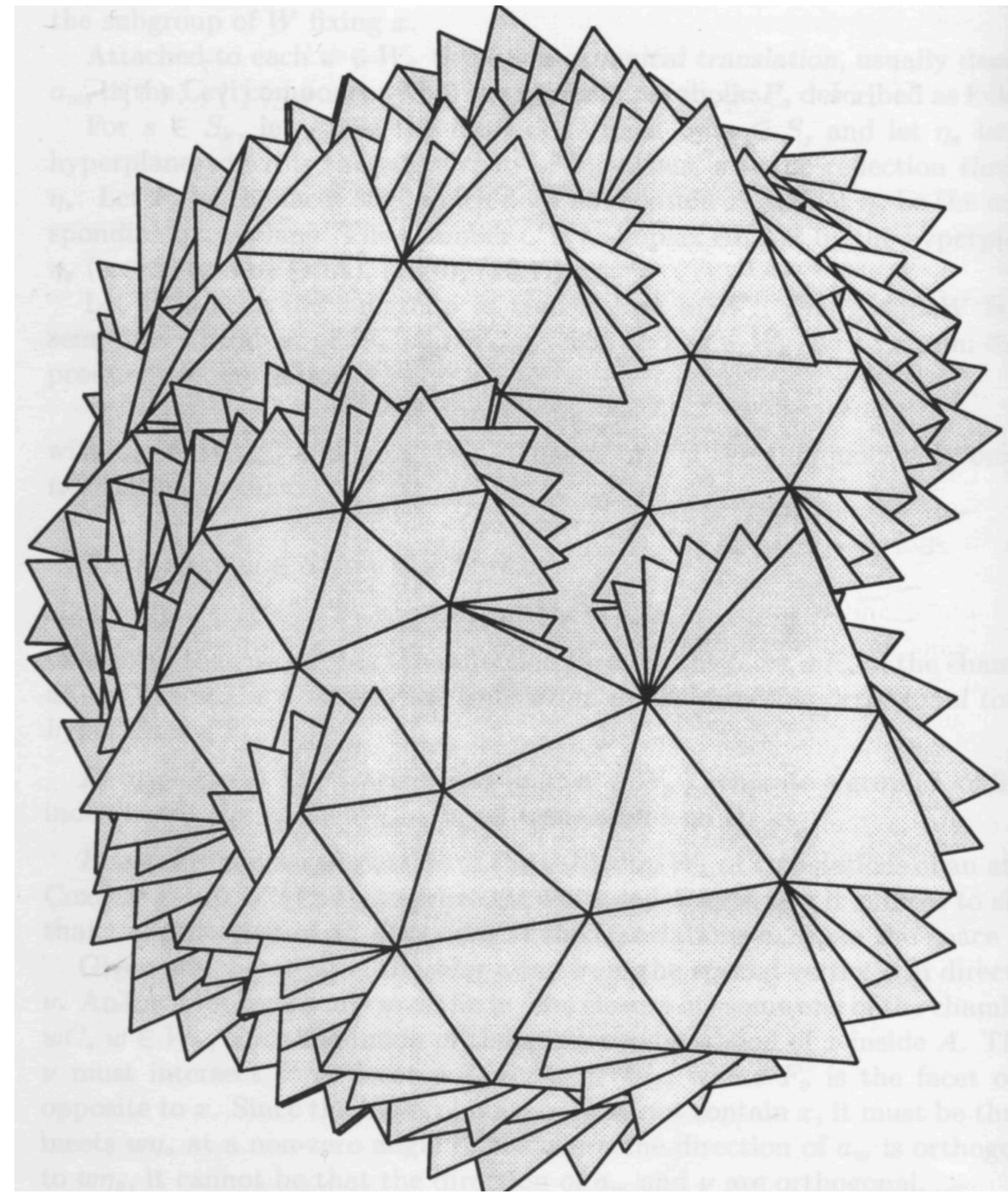


Subset of a Euclidean building

Euclidean Building for $PGL_3(\mathbb{Q}_p)$



An apartment of a Euclidean Building

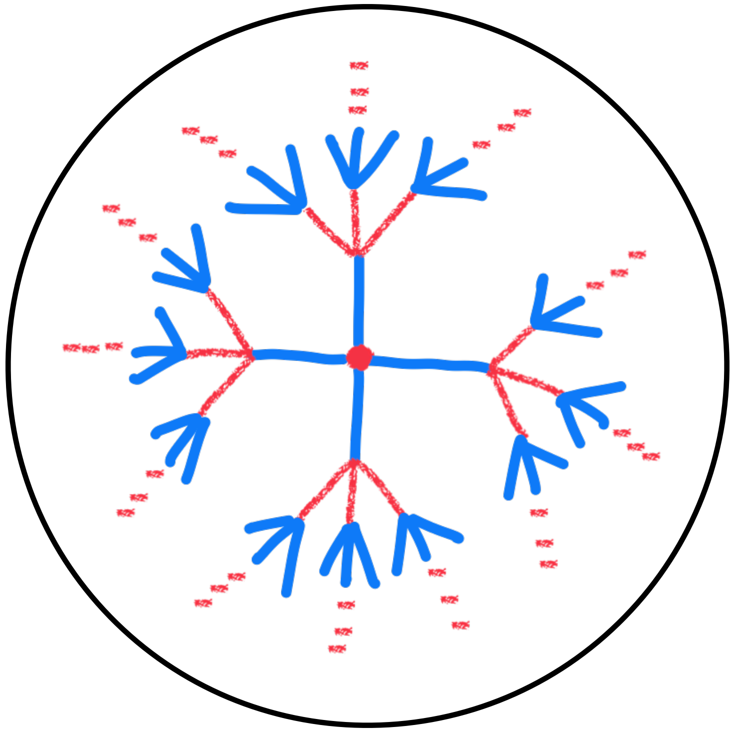


Subset of a Euclidean building [P Garrett]

Statistical model on regular trees

$$G = \mathrm{PGL}_2(\mathbb{Q}_{p^n})$$

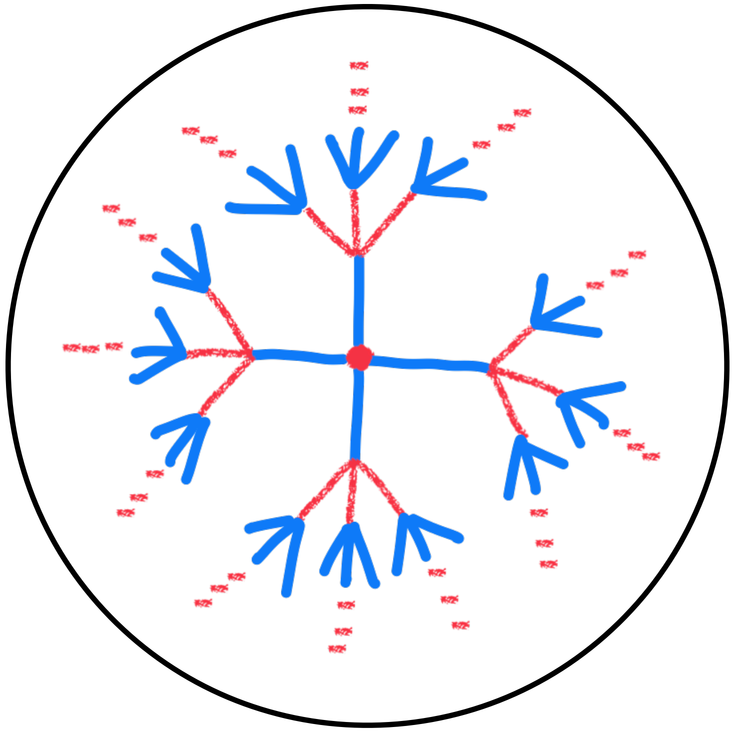
Gubser, Knaute, SP, Samberg, Witaszczyk. CMP ‘17
Heydeman, Marcolli, Saberi, Stoica. ATMP ’18
Gubser, SP. PRD ‘17



$$S_{\text{lattice}} = \sum_{\text{edges } \langle ab \rangle} \frac{1}{2}(\phi_a - \phi_b)^2 + \sum_{\text{vertices } a} \left(\frac{1}{2} m^2 \phi_a^2 + \frac{1}{3!} g_3 \phi_a^3 + \cdots \right)$$

Conformal field theory on *p*-adic boundary

Lerner, Missarov, Khajrullin,…, Abdesselam, Chandra, Guadagni
Gubser, Jepsen, SP, Trundy. JHEP '17
Gubser, SP. PRD '17



$$S_{\text{bdy}} = \int_{\mathbb{Q}_{p^n}} dx \frac{1}{2} \Phi^i(x) D^s \Phi^i(x) + \frac{\lambda}{4!} \int_{\mathbb{Q}_{p^n}} dx \left(\Phi^i(x)\Phi^i(x)\right)^2$$

$$D^s\Phi(x) = \frac{1}{\Gamma(-s)} \int_{\mathbb{Q}_{p^n}} dy \frac{1}{2} \frac{\left(\Phi^i(y) - \Phi^i(x)\right)}{|y-x|^{d+s}}$$

Bulk dual of generalized free field theory

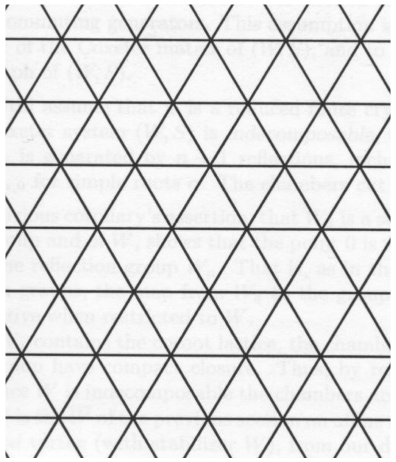
Gubser, SP. PRD '17

$$S = \frac{1}{2}\sum_{\langle ab \rangle}(\phi_a - \phi_b)^2 + \sum_a \left(\frac{1}{2}m^2\phi_a^2 + \frac{g_3^*}{3\,!}\phi_a^3 + \sum_{k=0}^{\infty}\lambda_k^*\phi_a^2\,\Box^k\,\phi_a^2\right)$$

- Discrete/non-Archimedean version of holographic duality between O(N) model and Vasiliev higher spin theory

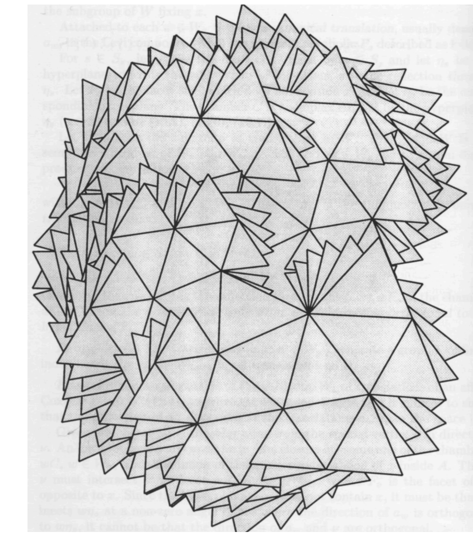
Klebanov, Polyakov '02

- Reproduce Wick contractions of boundary free field theory



biregular trees

$$G = PU_3(\mathbb{Q}_p)$$



- PGL3($\mathbb{Q}_p$): each edge contained in $p+1$ triangles; each vertex contained in $2(p^2+p+1)$ edges
- PU3($\mathbb{Q}_p$) is a subgroup of PGL3(E), where E is the unramified quadratic extension of $\mathbb{Q}_p$: elements fixed under a certain involution
- BT building of PU3($\mathbb{Q}_p$) is a subcomplex of BT building of PGL3(E)
  - infinite $(p^3+1, p+1)$-biregular tree
  - boundary: totally disconnected topology and ultrametric
- Automorphism group of biregular trees: not transitive. Only even translations allowed ⇒ semihomogeneous space

$(q_+ + 1, q_- + 1)$-biregular tree bulk dynamics

Mondal, Pradhan, SP, Sengar. '25 (to appear)

$$S_{\text{lattice}} = \sum_{\text{edges } \langle ab \rangle} \frac{1}{2}(\phi_a - \phi_b)^2 + \sum_{\text{vertices } a} \left(\frac{1}{2} m^2 \phi_a^2 + \frac{1}{3!} g_3 \phi_a^3 + \cdots \right)$$

$$(\Box + m^2) G(a,b) = \delta_{a,b}$$ Greens function

$$G(a,b) = \psi_\Delta \left((-1)^{d(a,b)} \right) \mathfrak{q}^{-\Delta d(a,b)}$$ $$\mathfrak{q} = \sqrt{q_+ q_-}$$

$$(q_+ + 1 + m^2)(q_- + 1 + m^2) = (q_+^\Delta + q_-^{1-\Delta})(q_-^\Delta + q_+^{1-\Delta})$$ mass-dim. relation

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle_{\mathfrak{q}} = \frac{c_{123}(q_o)}{|x_{12}|_{\mathfrak{q}}^{\Delta_{12,3}} |x_{23}|_{\mathfrak{q}}^{\Delta_{23,1}} |x_{31}|_{\mathfrak{q}}^{\Delta_{31,2}}}$$ $$o = \text{join}(x_1, x_2, x_3)$$

$(q_+ + 1, q_- + 1)$-biregular tree bulk dynamics

- (Quasi)conformal non-Archimedean boundary dual?
  - Higher-point correlators
- Archimedean analogue of biregular holography a la O(N) model/Vasiliev higher spin correspondence?
- Bulk dynamics on the full BT building of PGL_3?

Outline

1. Buildings
2. Euclidean/Bruhat-Tits buildings
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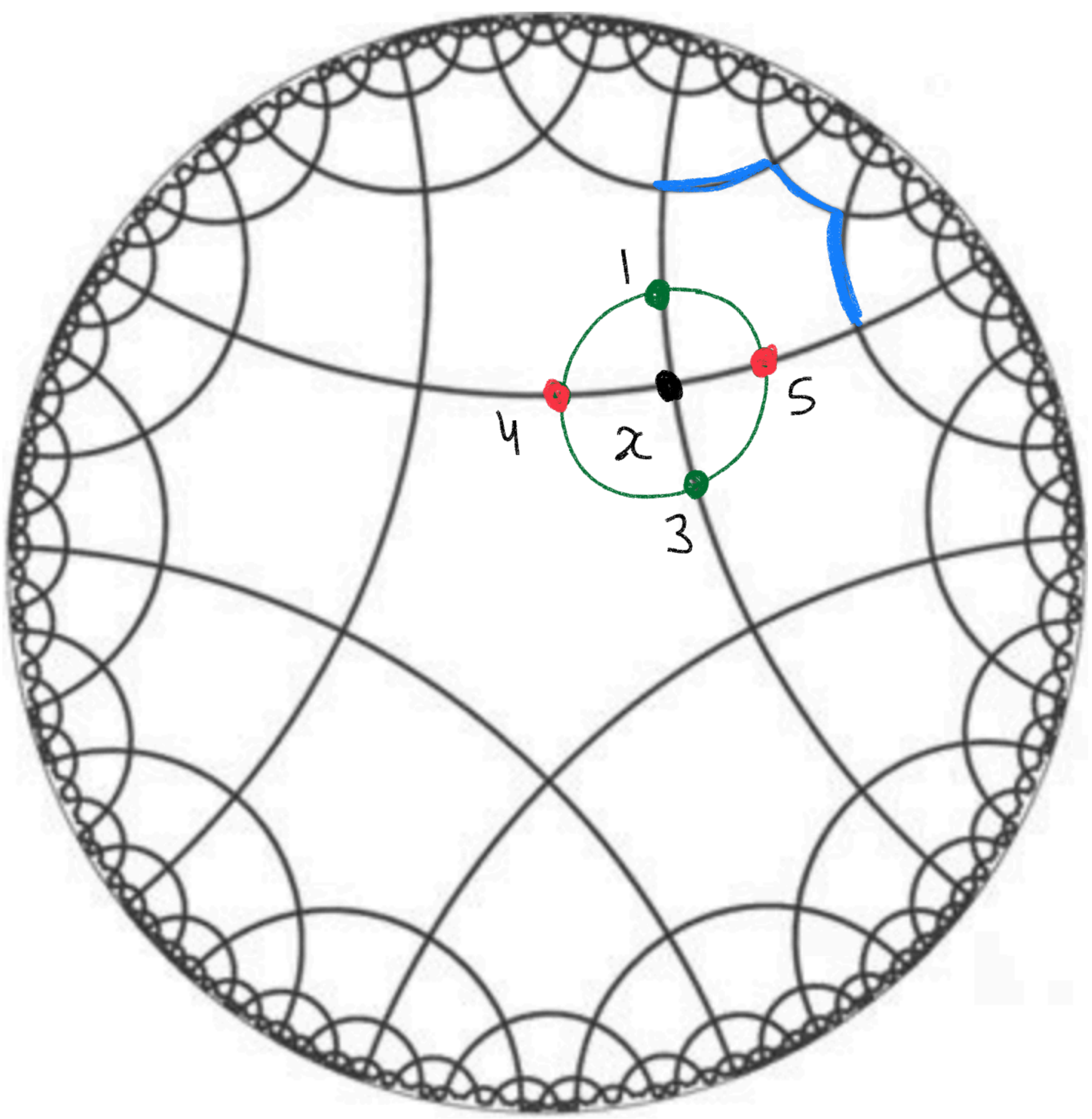
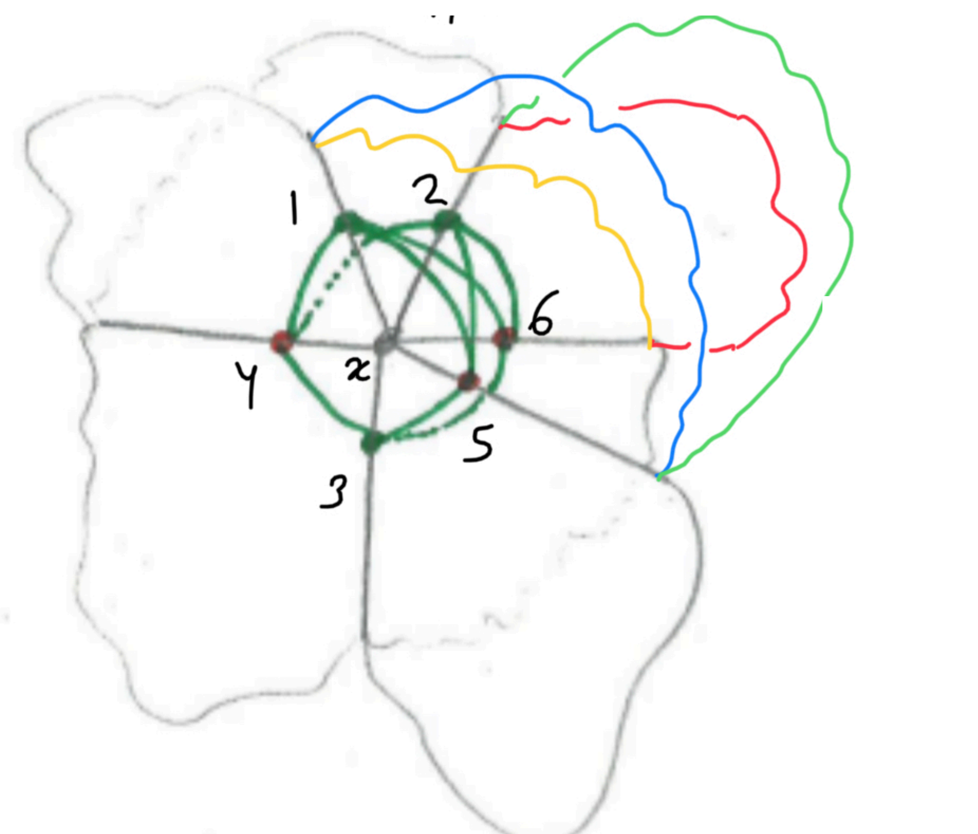
   - Tensor network
   - Ryu-Takayanagi formula

Bourdon Building $I_{p,q}$



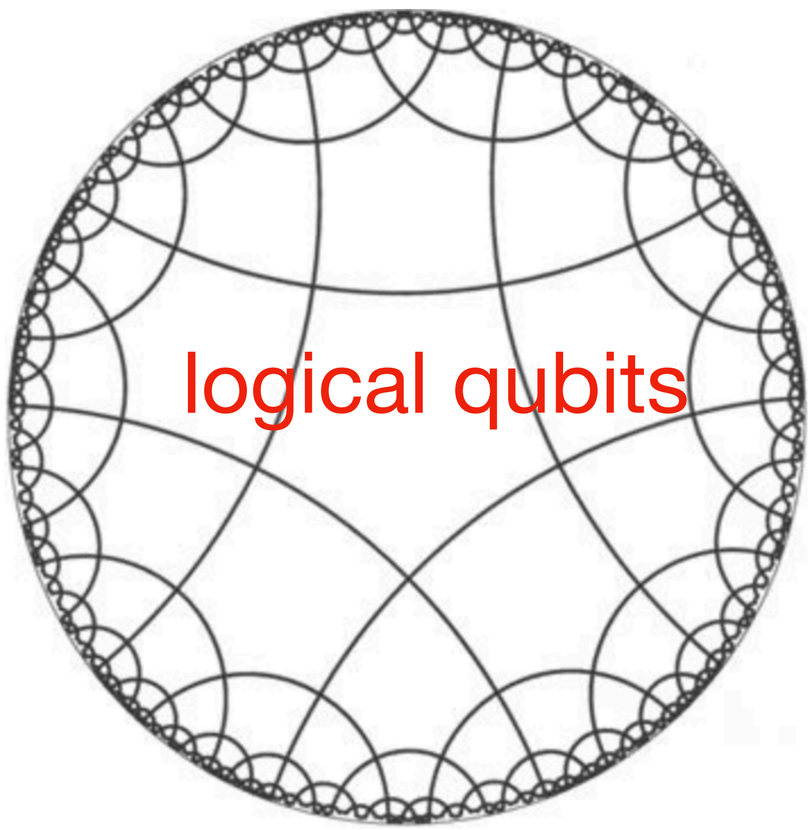
$I_{5,2}$

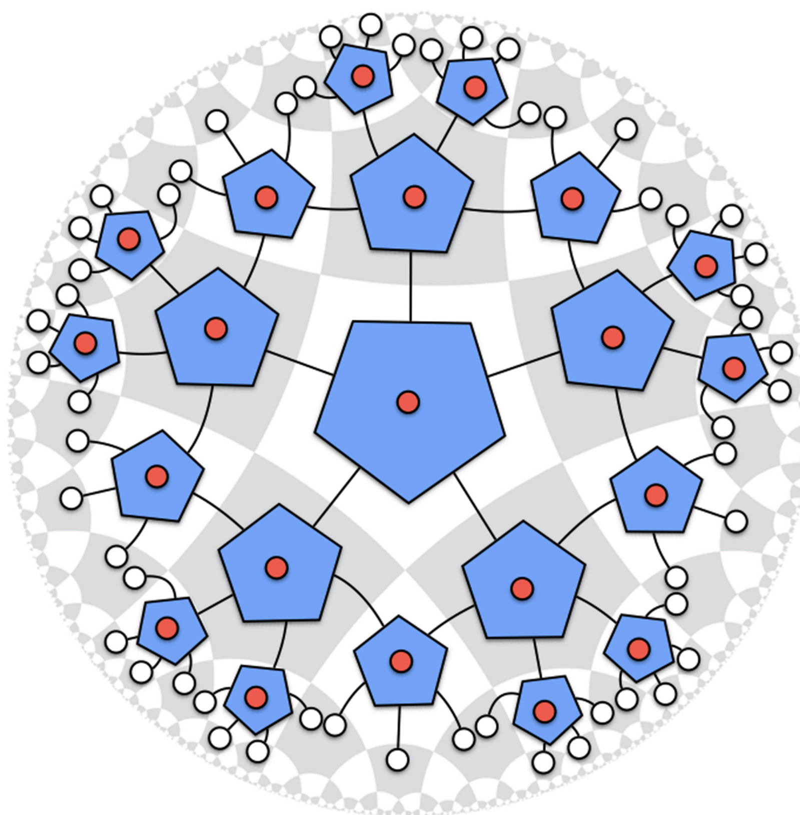


Subset of $I_{p,3}$

Quantum error-correcting codes and holography

physical qubits



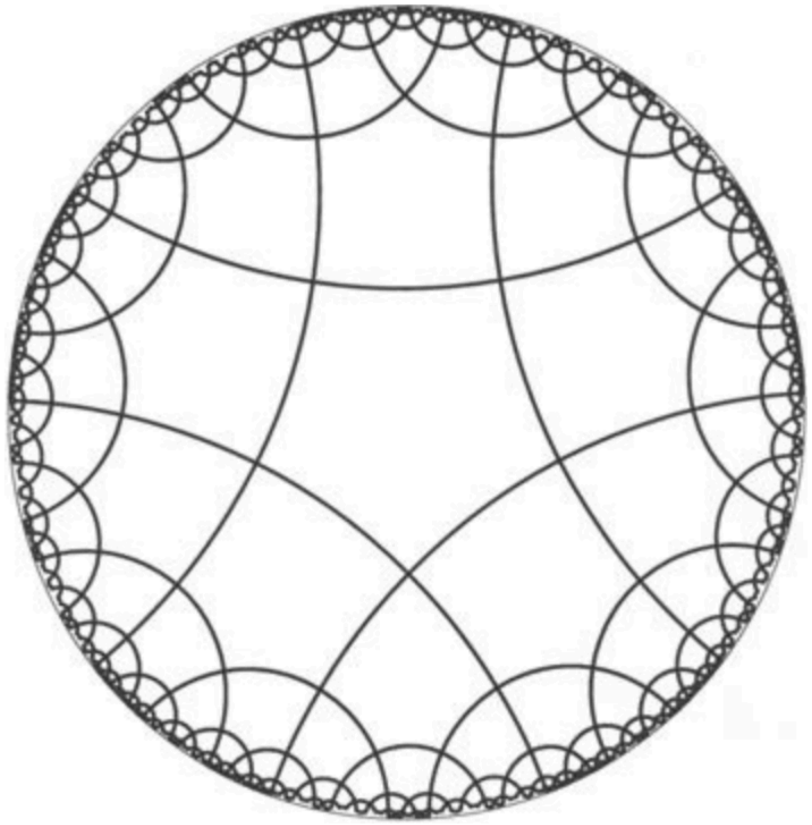


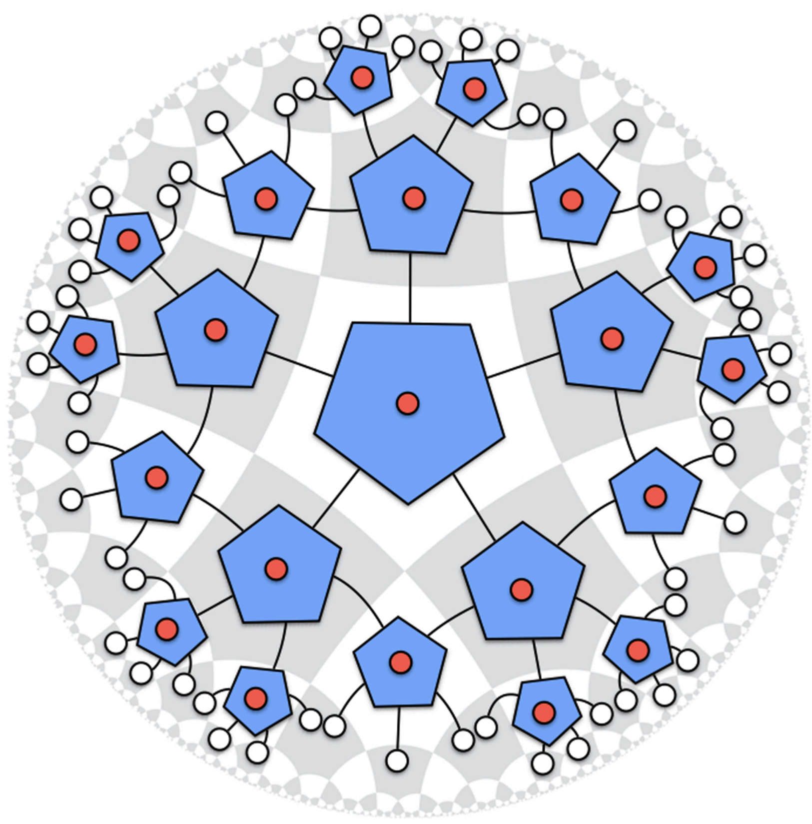
[1503.06237]

HaPPY tensor network

Pastawski, Yoshida, Harlow, Preskill ‘15

Ryu-Takayanagi formula, entanglement wedge reconstruction, complimentary recovery

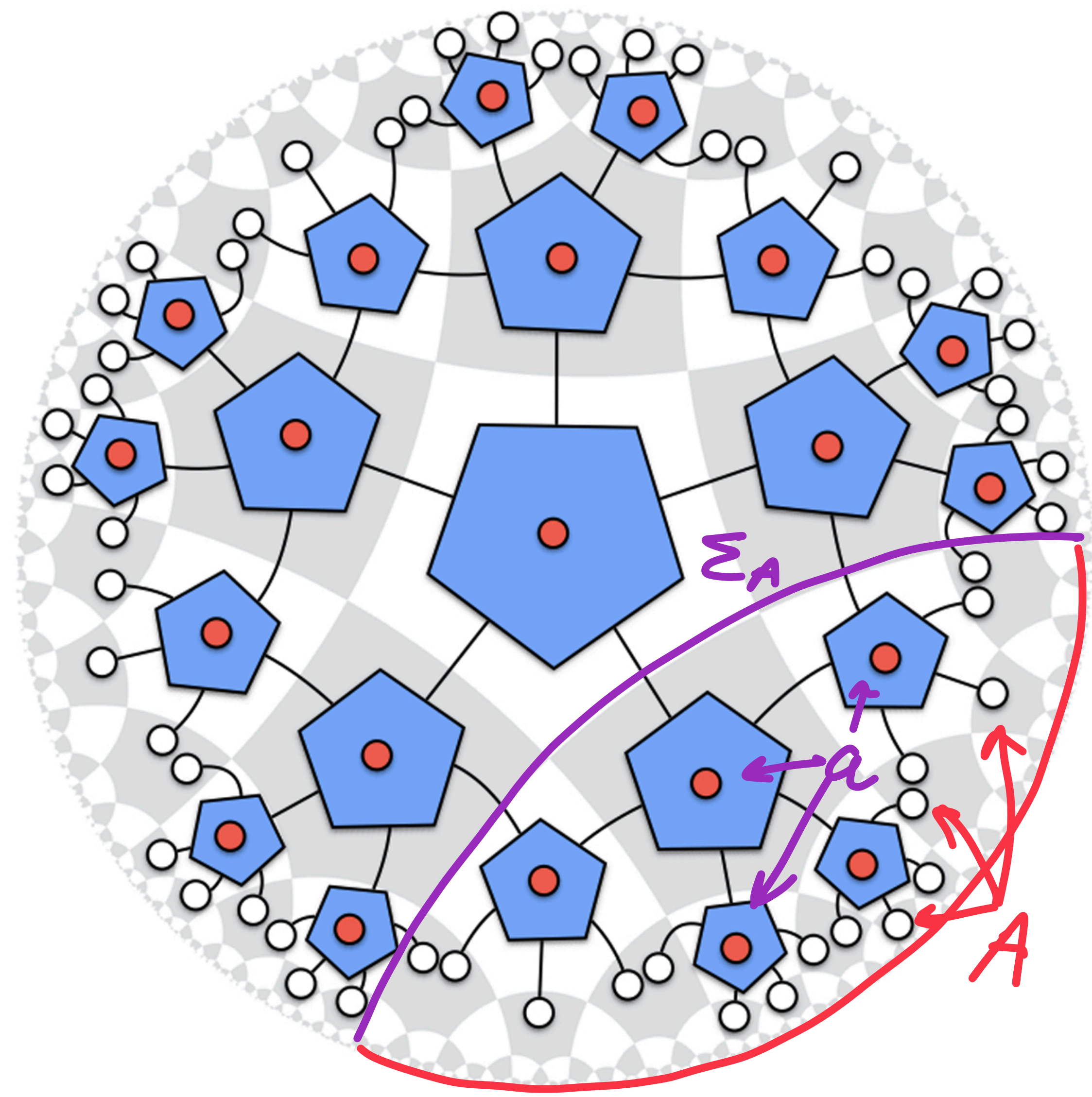




[1503.06237]

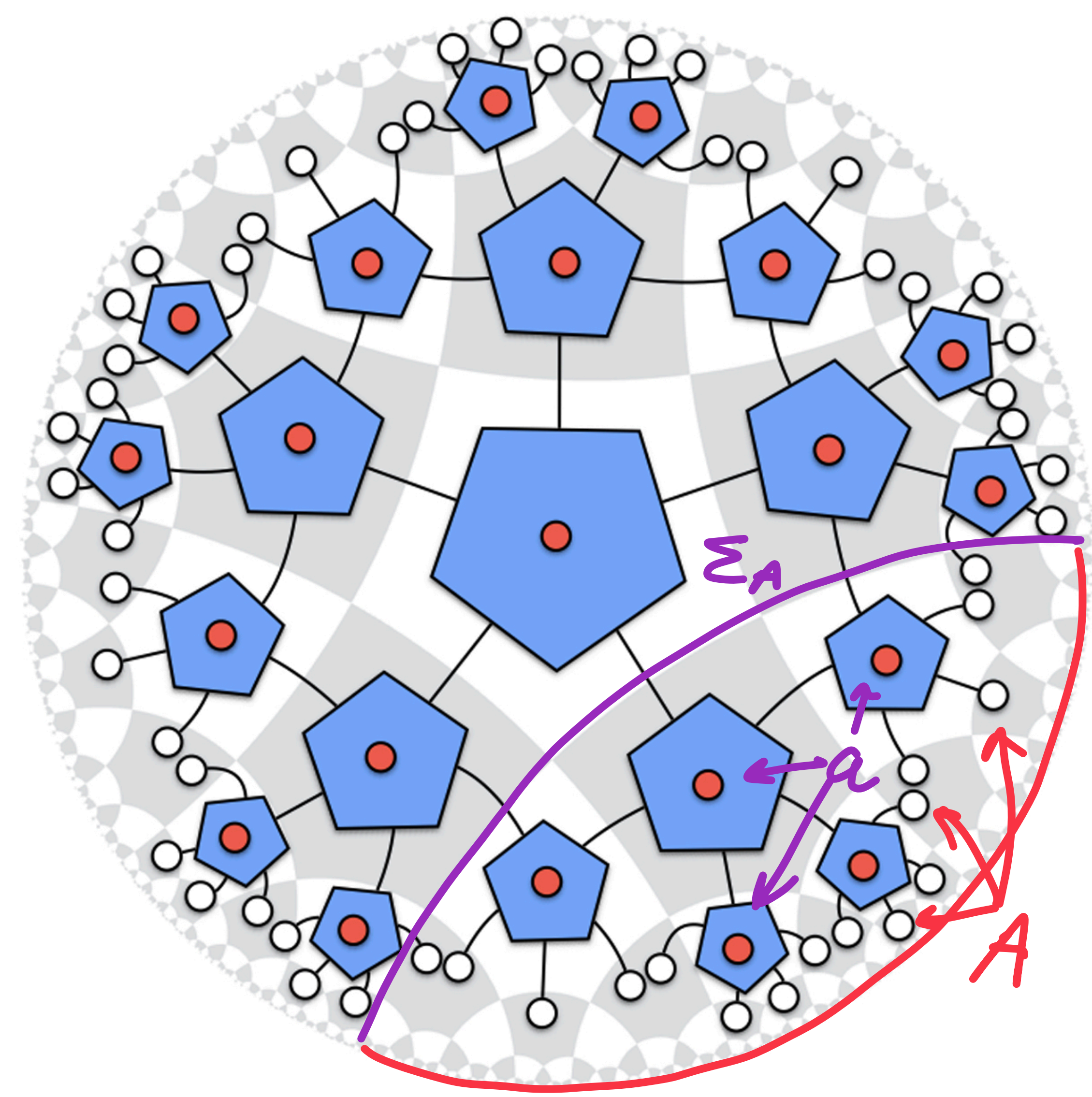
Ryu-Takayanagi formula

- For semi-classical bulk, entanglement entropy $S_{\text{bdy}}(A) = \text{Area}(\Sigma_A) + S_{\text{code}}(a)$



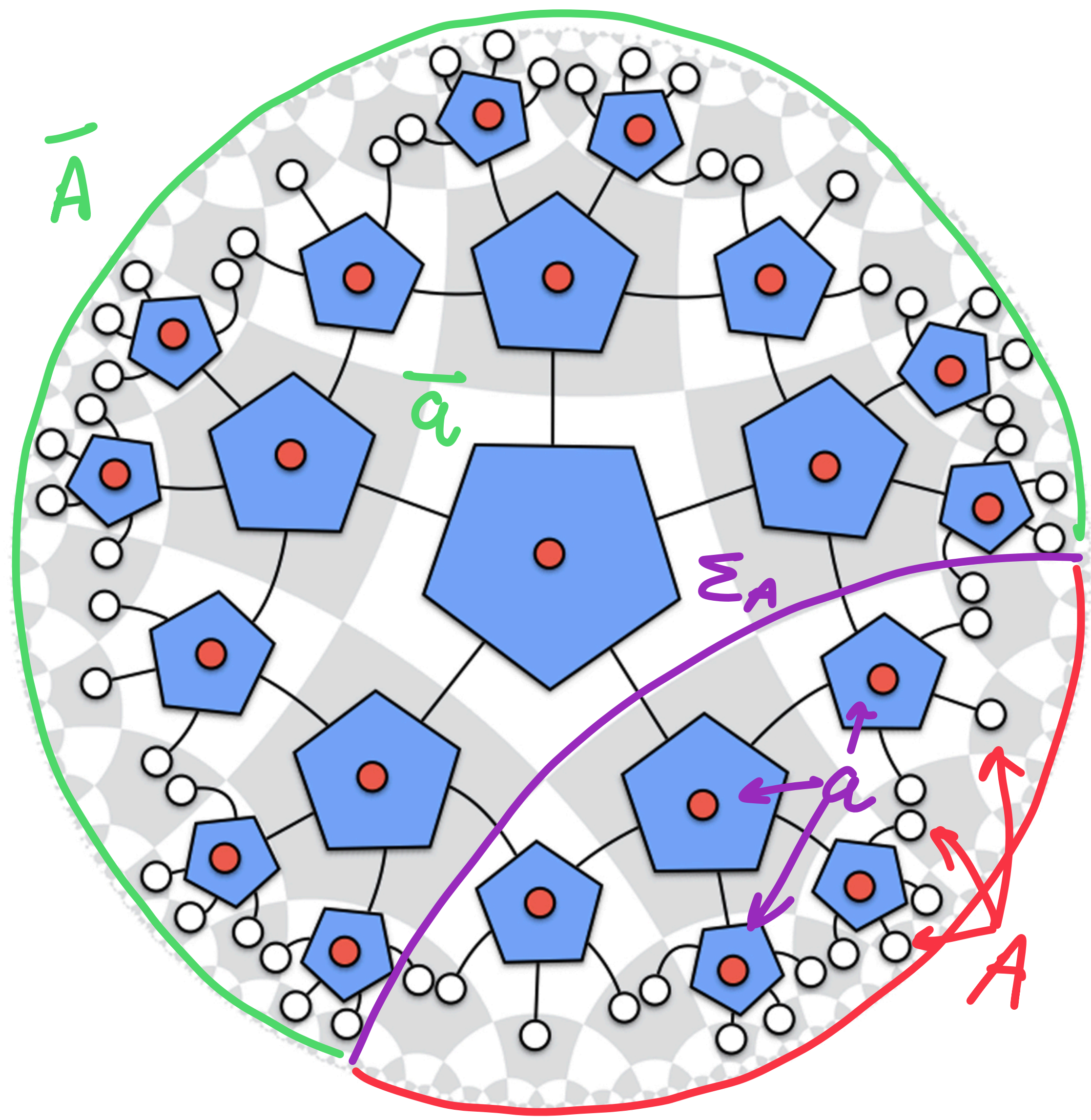
Entanglement wedge reconstruction

- Any effective local bulk operator on a (entanglement wedge of A) can be reconstructed on the boundary A



Complementary recovery

- Any effective local bulk operator on a (entanglement wedge of A) can be reconstructed on the boundary A, <u>and</u> any effective local bulk operator on $\bar{a}$ can be reconstructed on the boundary $\bar{A}$



Bourdon tensor network

Gesteau, Marcolli, SP. JHEP '22

- Insert a perfect tensor in each chamber of the building
- Perform an index contraction between every pair of chambers sharing an edge
- Add $q-1$ (bulk) dangling legs for each tensor
- Choosing a central tile, cut off the tensor at a finite distance from the central tile

Ryu-Takayanagi (RT) formula

Gesteau, Marcolli, SP. JHEP '22

$$S_{\text{bdy}}(A) = \text{Area}(\Sigma_A) + S_{\text{code}}(a)$$

$$\text{Area}(\Sigma_A) \sim \log(r/\epsilon) \qquad\qquad \text{Area}(\Sigma_A) \sim (r/\epsilon)^{D-1}$$

- On tensor networks, the area is calculated in terms of number of links of the tensor network cut by the RT surface.
  - Logarithmic in area for HaPPY tensor network
  - On building tensor networks, the scaling of area is $(r/\epsilon)^{D-1}$ where D is the Hausdorff/scaling dimension of the visual boundary
- RT surface sees the fractal nature of the boundary!

Summary

- Bruhat-Tits (BT) tree is an example of a more general object
- BT buildings provide examples of discrete spacetimes that have holographic properties. Ultrametric spaces appear naturally on the boundary
- Example: (bi)regular trees
- Biregular trees suggest an interesting analog for dual CFT
- (Bourdon) hyperbolic buildings provide a way to extend Ryu-Takayanagi formula & quantum error-correction to various (non)integer dimensions.

Outlook

- Classification of “holographic buildings”: large class of discrete models of spacetime
- Duality and string theory on higher rank buildings
- Quotient geometries: black holes
- Novel Archimedean insights
  - Euclidean buildings: flat space/celestial CFT holography
- Euclidean → Lorentzian. Time evolution… branching of apartments?

Outlook

- Classification of "holographic buildings": large class of discrete models of spacetime
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- Euclidean → Lorentzian. Time evolution… branching of apartments?

THANK YOU!

sarthak@iitd.ac.in

Extras

Bourdon Building $I_{p,q}$

- Apartments are tessellations of $\mathbb{H}^2$
- Its chambers are right-angled regular p-gons in $\mathbb{H}^2$
- Negative curvature (CAT(-1) space)
- At each edge there are exactly q chambers attached.
- Coxeter group: $W = \{s_1, \ldots, s_p \,|\, s_i^2 = 1, [s_i, s_{i+1}] = 1\}$
- Isometry group: $\Gamma_{p,q} = \{s_1, \ldots, s_p \,|\, s_i^q = 1, [s_i, s_{i+1}] = 1\}$

Hyperbolic tessellations as Coxeter group

- HaPPY tiling generated by five reflections along edges, $s_{i=1,\ldots,5}$
- Each chamber/tile is a word constructed out of reflections s_i
- Coxeter system associated to the apartment
- Distance between two chambers: minimum number of reflections needed to go from one to the other