The correspondence of generalised entropic cosmology theory with $F(T)$ and $F(Q)$ modified gravity and gravitational waves

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Abstract

This talk is based on the collaboration with S Nojiri, arXiv:2502.15272 [gr-qc]. We investigate the correspondence between modified gravity theories and general entropic cosmology theory. Such a theory is proposed by an analogy with Jacobson's work, where the Einstein equation was derived from the Bekenstein-Hawking entropy. We compare FLRW equations obtained in entropic gravity with those in modified gravity theories. Some correspondence was found between $F(T)$ and $F(Q)$ gravity and general entropic gravity. We regard the $F(T)$ and $F(Q)$ gravity theories as effective local theories corresponding to the entropic gravity theories, and we investigate the gravitational waves. The obtained equation of the gravitational wave is identical to that in Einstein's concept of gravity, except that the gravitational coupling is modified by the functional forms of $F(T)$ and $F(Q)$. It is interesting that, in the case of the Tsallis entropic cosmology, gravitational coupling becomes small or large, which may suppress or enhance the emission of the gravitational wave.

S. Nojiri and S. D. Odintsov, "The correspondence of generalised entropic cosmology theory with $F(T)$ and $F(Q)$ modified gravity and gravitational waves," Phys. Dark Univ. **48** (2025), 101899 doi:10.1016/j.dark.2025.101899 [arXiv:2502.15272 [gr-qc]].

Introduction

Gravity $\Leftrightarrow$ Thermodynamics???
A black hole has a temperature and entropy.
Bekenstein-Hawking entropy

$$S = \frac{A}{4G} \, . \tag{1}$$

$A = 4\pi r_{\mathrm{h}}^2$ is the area of the horizon given by the horizon radius r_{h}.

J. D. Bekenstein, "Black holes and entropy," Phys. Rev. D **7** (1973), 2333-2346 doi:10.1103/PhysRevD.7.2333

S. W. Hawking, "Particle Creation by Black Holes," Commun. Math. Phys. **43** (1975), 199-220 [erratum: Commun. Math. Phys. **46** (1976), 206] doi:10.1007/BF02345020

Jacobson has shown that the Einstein equation can be obtained from the Bekenstein-Hawking entropy by thermodynamical considerations.

T. Jacobson, "Thermodynamics of space-time: The Einstein equation of state," Phys. Rev. Lett. **75** (1995), 1260-1263 doi:10.1103/PhysRevLett.75.1260 [arXiv:gr-qc/9504004 [gr-qc]].

Jacobson has assumed that the equation can be written in terms of the spacetime curvature.

In different physical and statistical systems, various entropies exist.

Tsallis entropy
C. Tsallis, "Possible Generalization of Boltzmann-Gibbs Statistics," J. Statist. Phys. **52** (1988), 479-487 doi:10.1007/BF01016429

Rényi entropy
A. Rényi, "On measures of information and entropy" Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability, University of California Press (1960), 547-56

Barrow entropy
J. D. Barrow, "The Area of a Rough Black Hole," Phys. Lett. B **808** (2020), 135643 doi:10.1016/j.physletb.2020.135643 [arXiv:2004.09444 [gr-qc]].

Sharma-Mittal entropy

A. Sayahian Jahromi, S. A. Moosavi, H. Moradpour, J. P. Morais Graça, I. P. Lobo, I. G. Salako and A. Jawad, "Generalized entropy formalism and a new holographic dark energy model," Phys. Lett. B **780** (2018), 21-24 doi:10.1016/j.physletb.2018.02.052 [arXiv:1802.07722 [gr-qc]].

Kaniadakis entropy

G. Kaniadakis, "Statistical mechanics in the context of special relativity. II.," Phys. Rev. E **72** (2005), 036108 doi:10.1103/PhysRevE.72.036108 [arXiv:cond-mat/0507311 [cond-mat]].

N. Drepanou, A. Lymperis, E. N. Saridakis and K. Yesmakhanova, "Kaniadakis holographic dark energy and cosmology," Eur. Phys. J. C **82** (2022) no.5, 449 doi:10.1140/epjc/s10052-022-10415-9 [arXiv:2109.09181 [gr-qc]].

Loop Quantum Gravity entropy

A. Majhi, "Non-extensive Statistical Mechanics and Black Hole Entropy From Quantum Geometry," Phys. Lett. B **775** (2017), 32-36 doi:10.1016/j.physletb.2017.10.043 [arXiv:1703.09355 [gr-qc]].

Y. Liu, "Non-extensive statistical mechanics and the thermodynamic stability of FRW universe," EPL **138** (2022) no.3, 39001 doi:10.1209/0295-5075/ac3f52 [arXiv:2112.15077 [gr-qc]].

Shinichi, Valerio, Tanmoy, and/or I proposed more generalised entropies,

S. Nojiri, S. D. Odintsov and V. Faraoni, “From nonextensive statistics and black hole entropy to the holographic dark universe,” Phys. Rev. D **105** (2022) no.4, 044042 doi:10.1103/PhysRevD.105.044042 [arXiv:2201.02424 [gr-qc]].

S. Nojiri, S. D. Odintsov and T. Paul, “Early and late universe holographic cosmology from a new generalized entropy,” Phys. Lett. B **831** (2022), 137189 doi:10.1016/j.physletb.2022.137189 [arXiv:2205.08876 [gr-qc]].

S. D. Odintsov and T. Paul, “A non-singular generalized entropy and its implications on bounce cosmology,” Phys. Dark Univ. **39** (2023), 101159 doi:10.1016/j.dark.2022.101159 [arXiv:2212.05531 [gr-qc]].

What could be the gravity theories corresponding to generalised entropy?

Tsallis (like) entropy

$$S_{\mathrm{T}} = \frac{A_0}{4G} \left(\frac{A}{A_0} \right)^{\delta} . \quad (2)$$

A_0: a constant with the dimension of area, δ: an exponent.

Rényi (like) entropy

$$S_{\mathrm{R}} = \frac{1}{\alpha} \ln \left(1 + \alpha S \right) . \quad (3)$$

α: a parameter, S: Bekenstein-Hawking entropy.

Barrow entropy

$$S_{\mathrm{B}} = \left(\frac{A}{A_{\mathrm{Pl}}} \right)^{1+\Delta/2} . \quad (4)$$

A: the area of the black hole horizon, $A_{\mathrm{Pl}} = 4G$: the Planck area.

Sharma-Mittal entropy

$$S_{\mathrm{SM}} = \frac{1}{R}\left[(1+\delta\ S)^{R/\delta} - 1\right]. \tag{5}$$

R and δ; parameters.

Kaniadakis entropy

$$S_{\mathrm{K}} = \frac{1}{K}\sinh(KS). \tag{6}$$

K: a phenomenological parameter

Loop Quantum Gravity entropy

$$S_q = \frac{1}{(1-q)}\left[\mathrm{e}^{(1-q)\Lambda(\gamma_0)S} - 1\right]. \tag{7}$$

q: the exponent, $\Lambda(\gamma_0) = \ln 2/\left(\sqrt{3}\pi\gamma_0\right)$, γ_0: Barbero-Immirzi parameter.

Four- and six-parameter generalised entropies were proposed as,

$$S_4(\alpha_\pm,\delta,\gamma) = \frac{1}{\gamma}\left[\left(1+\frac{\alpha_+}{\delta}S\right)^{\delta} - \left(1+\frac{\alpha_-}{\delta}S\right)^{-\delta}\right], \tag{8}$$

$$S_6(\alpha_\pm,\delta_\pm,\gamma_\pm) = \frac{1}{\alpha_+ + \alpha_-}\left[\left(1+\frac{\alpha_+}{\delta_+}S^{\gamma_+}\right)^{\delta_+} - \left(1+\frac{\alpha_-}{\delta_-}S^{\gamma_-}\right)^{-\delta_-}\right], \tag{9}$$

For some parameter choices, as particular examples, the above two entropies describe all the known entropies.
Simplified version: three-parameter entropy,

$$S_3(\alpha,\delta,\gamma) = \frac{1}{\gamma}\left[\left(1+\frac{\alpha}{\delta}S\right)^{\delta} - 1\right]. \tag{10}$$

S_3 gives all the above entropies except the Kaniadakis entropy.
Five-parameter entropy (to solve the problem of singularity)

$$S_5(\alpha_\pm,\delta,\gamma,\epsilon) = \frac{1}{\gamma}\left[\left\{1+\frac{1}{\epsilon}\tanh\left(\frac{\epsilon\alpha_+}{\delta}S\right)\right\}^{\delta} - \left\{1+\frac{1}{\epsilon}\tanh\left(\frac{\epsilon\alpha_-}{\delta}S\right)\right\}^{-\delta}\right]. \tag{11}$$

Generalised entropies and cosmology

Entropy $\Leftrightarrow$ Gravity,
Simplified version: Entropy $\Leftrightarrow$ Cosmology $\Rightarrow$ Further simplified version

For the detailed version,

R. G. Cai and S. P. Kim, "First law of thermodynamics and Friedmann equations of Friedmann-Robertson-Walker universe," JHEP **02** (2005), 050 doi:10.1088/1126-6708/2005/02/050 [arXiv:hep-th/0501055 [hep-th]].

M. Akbar and R. G. Cai, "Friedmann equations of FRW universe in scalar-tensor gravity, f(R) gravity and first law of thermodynamics," Phys. Lett. B **635** (2006), 7-10 doi:10.1016/j.physletb.2006.02.035 [arXiv:hep-th/0602156 [hep-th]].

We consider FLRW spacetime with a flat spatial part,

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} \left(dx^i\right)^2 . \tag{12}$$

$a(t)$: a scale factor.

The radius r_{H} of the cosmological horizon,

$$r_{\mathrm{H}} = \frac{1}{H} . \tag{13}$$

$H = \dot{a}/a$: Hubble rate.

Bekenstein-Hawking entropy: entropy inside the cosmological horizon.

$$S = \frac{A}{4G} = \frac{4\pi {r_{\mathrm{H}}}^2}{4G} . \tag{14}$$

The flux of the energy E = the increase of the heat Q in the region,

$$dQ = -dE = -\frac{4\pi}{3} r_{\rm H}^3 \dot{\rho} dt = -\frac{4\pi}{3H^3} \dot{\rho} dt = \frac{4\pi}{H^2} \left(\rho + p \right) dt \,. \tag{15}$$

ρ: energy density, p: pressure.
Conservation law: $0 = \dot{\rho} + 3H \left(\rho + p \right)$.

Hawking temperature

$$T = \frac{1}{2\pi r_{\rm H}} = \frac{H}{2\pi} \,, \tag{16}$$

First law of thermodynamics $TdS = dQ \Rightarrow \dot{H} = -4\pi G \left(\rho + p \right)$
$\Rightarrow$ Integration,

$$H^2 = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} \quad \text{First Friedmann equation} \ \Leftarrow \ \text{Einstein equation} \tag{17}$$

Λ: constant of the integration, regarded as a cosmological constant.

Thermodynamics $\Leftrightarrow$ Einstein Gravity (Cosmology)

Simplified version

In the case of generalised entropy S_{g}, the first law of thermodynamics $\Rightarrow$

$$\dot{H}\left(\frac{\partial S_{\mathrm{g}}}{\partial S}\right)\bigg|_{S=\frac{A}{4G}=\frac{\pi}{GH^2}} = -4\pi G\left(\rho+p\right) . \tag{18}$$

$\dot{\rho}+3H\left(\rho+p\right)=0 \Rightarrow$

$$\left(\frac{\partial S_{\mathrm{g}}}{\partial S}\right)\bigg|_{S=\frac{A}{4G}=\frac{\pi}{GH^2}} d\left(H^2\right) = \left(\frac{8\pi G}{3}\right) d\rho\,. \tag{19}$$

Integration $\Rightarrow$ a generalised first FLRW equation,

$$\mathcal{H}\left(H^2\right)^2 \equiv \int^{H^2} dx\left(\frac{\partial S_{\mathrm{g}}}{\partial S}\right)\bigg|_{S=\frac{\pi}{Gx}} = \left(\frac{8\pi G}{3}\right)\rho\,. \tag{20}$$

In the case of four-parameter generalised entropies,

$$\frac{GH^4\beta}{\pi\gamma}\left[\frac{1}{(2+\beta)}\left(\frac{GH^2\beta}{\pi\alpha_-}\right)^{\beta}{}_2F_1\left(1+\beta,2+\beta,3+\beta,-\frac{GH^2\beta}{\pi\alpha_-}\right)\right.$$
$$\left.+\frac{1}{(2-\beta)}\left(\frac{GH^2\beta}{\pi\alpha_+}\right)^{-\beta}{}_2F_1\left(1-\beta,2-\beta,3-\beta,-\frac{GH^2\beta}{\pi\alpha_+}\right)\right]$$
$$=\frac{8\pi G\rho}{3}+\frac{\Lambda}{3}\,. \tag{21}$$

Λ: an integration constant (cosmological constant),
${}_2F_1$: Hypergeometric function.

Tsallis entropy $\beta = \delta$ or Barrow entropy $\beta = 1 + \Delta$

$\alpha_- \to 0 \Rightarrow \gamma = (\alpha_+/\beta)^\beta$
$\Rightarrow$ Keeping β finite, $\alpha_+ \to \infty$. $\Rightarrow$ Tsallis entropy or Barrow entropy,

$$\begin{aligned} {}_2F_1(\alpha,\beta,\gamma,z) =& \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-z)^{-\alpha} F\left(\alpha,\alpha+1-\gamma,\alpha+1-\beta,\frac{1}{z}\right) \\ &+ \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z)^{-\beta} F\left(\beta,\beta+1-\gamma,\beta+1-\alpha,\frac{1}{z}\right), \\ {}_2F_1(\alpha,\beta,\gamma,0) =& 1, \end{aligned} \tag{22}$$

$\Rightarrow$

$$\mathcal{H}\left(H^2\right)^2 = \frac{\beta}{2-\beta}\left(\frac{GH^2}{\pi}\right)^{1-\beta} H^2, \tag{23}$$

Rényi entropy

$\alpha_- = 0$, $\beta \to 0$, $\alpha \equiv \frac{\alpha_+}{\beta} \to \text{finite}$, $\gamma = \alpha_+$.

$$\mathcal{H}\left(H^2\right)^2 = \frac{GH^4}{2\pi\alpha} {}_2F_1\left(1, 2, 3, -\frac{GH^2}{\pi\alpha}\right) . \tag{24}$$

- Sharma-Mittal entropy, $\alpha_- \to 0$ ($\gamma = R$, $\alpha_+ = R$, and $\beta = R/\delta$).
- Kaniadakis entropy, $\beta \to \infty$ ($\alpha_+ = \alpha_- = \frac{\gamma}{2} = K$).
- Loop Quantum Gravity entropy with $\Lambda(\gamma_0) = 1$ or $\gamma_0 = \frac{\ln 2}{\pi\sqrt{3}}$, $\alpha_- = 0$ ($\beta \to \infty$, $\gamma = \alpha_+ = (1-q)$).

What could be the gravity theories corresponding to generalised entropy?

In the case of $F(R)$ gravity,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} F(R) + S_{\mathrm{matter}} \tag{25}$$

We have,

$$\frac{1}{\kappa^2} \left(\frac{1}{2} F(R) - 3 \left(H^2 + \dot{H} \right) F'(R) + 18 \left(4H^2 \dot{H} + H\ddot{H} \right) F''(R) \right) = \rho \,, \tag{26}$$

The l.h.s. includes $\dot{H}$, $\ddot{H}$ $\Rightarrow$ cannot correspond to generalized entropies.

See also,
S. Nojiri, S. D. Odintsov, T. Paul and S. SenGupta, “Modified gravity as entropic cosmology,” [arXiv:2503.19056 [gr-qc]].

Correspondence between $F(T)$ gravity and generalised entropies

Teleparallelism with the Weitzenböck connection:
an alternative to Einstein's general relativity.
A scalar quantity T, torsion, is a fundamental ingredient instead of the curvature R.

A generalization: $F(T)$ gravity,
Many works, for example,
Y. F. Cai, S. H. Chen, J. B. Dent, S. Dutta and E. N. Saridakis, "Matter Bounce Cosmology with the f(T) Gravity," Class. Quant. Grav. **28** (2011), 215011 doi:10.1088/0264-9381/28/21/215011 [arXiv:1104.4349 [astro-ph.CO]].

Problem
The model includes superluminal propagating modes, which appear non-perturbatively.

K. Izumi and Y. C. Ong, "Cosmological Perturbation in f(T) Gravity Revisited," JCAP **06** (2013), 029 doi:10.1088/1475-7516/2013/06/029 [arXiv:1212.5774 [gr-qc]].

Y. C. Ong, K. Izumi, J. M. Nester and P. Chen, "Problems with Propagation and Time Evolution in f(T) Gravity," Phys. Rev. D **88** (2013), 024019 doi:10.1103/PhysRevD.88.024019 [arXiv:1303.0993 [gr-qc]].

$F(T)$ gravity cannot be a physically consistent theory.
$\sim$ an effective theory?

$e_A(x^\mu)$: tetrad or vierbein $\Rightarrow$ metric $g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu$, η_{AB}: flat metric

$T^\rho{}_{\mu\nu}$: torsion tensor, $K^{\mu\nu}{}_\rho$: contorsion tensor,

$$T^\rho{}_{\mu\nu} \equiv e^\rho_A \left(\partial_\mu e^A_\nu - \partial_\nu e^A_\mu \right) , \quad K^{\mu\nu}{}_\rho \equiv -\frac{1}{2} \left(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu} \right) . \quad (27)$$

T: torsion scalar

$$T \equiv S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu} , \quad S_\rho{}^{\mu\nu} \equiv \frac{1}{2} \left(K^{\mu\nu}{}_\rho + \delta^\mu{}_\rho T^{\alpha\nu}{}_\alpha - \delta^\nu{}_\rho T^{\alpha\mu}{}_\alpha \right) . \quad (28)$$

The action of the $F(T)$ gravity,

$$S = \int d^4x |e| \left[\frac{F(T)}{2\kappa^2} + \mathcal{L}_{\mathrm{matter}} \right] . \quad (29)$$

$|e| = \det \left(e^A_\mu \right) = \sqrt{-g}$, $\mathcal{L}_{\mathrm{matter}}$: the Lagrangian density of matter.
E. V. Linder, "Einstein's Other Gravity and the Acceleration of the Universe," Phys. Rev. D **81** (2010), 127301 [erratum: Phys. Rev. D **82** (2010), 109902] doi:10.1103/PhysRevD.81.127301 [arXiv:1005.3039 [astro-ph.CO]].

In the FLRW spacetime, T as $T = -6H^2$,

$$\frac{1}{6}\left(F\left(T\right) - 2TF'\left(T\right)\right)\Big|_{T=-6H^2} = \left(\frac{8\pi G}{3}\right)\rho\,. \tag{30}$$

The l.h.s. of Eq. (30) is a function of H^2 and does not depend on $\dot{H}$, $\ddot{H}$, etc.
Compared with (20),

$$\mathcal{H}\left(H^2\right)^2 \equiv \int^{H^2} dx \left(\frac{\partial S_{\mathrm{g}}}{\partial S}\right)\Bigg|_{S=\frac{\pi}{Gx}} = \left(\frac{8\pi G}{3}\right)\rho\,. \tag{20}$$

we find

$$\frac{1}{6}\left(F\left(T\right) - 2TF'\left(T\right)\right)\Big|_{T=-6H^2} = \mathcal{H}\left(H^2\right)^2 \tag{31}$$

$\Rightarrow$

One-to-one correspondence between $F(T)$ gravity and general entropic cosmology (up to a constant of the integration),

$$F(T) = 3\left(-T\right)^{\frac{1}{2}} \int^{-\frac{T}{6}} dy\, (6y)^{-\frac{3}{2}}\, \mathcal{H}\left(y\right)^2$$
$$= 3\left(-T\right)^{\frac{1}{2}} \int^{-\frac{T}{6}} dy\, (6y)^{-\frac{3}{2}} \int^{y} dx \left(\frac{\partial S_{\mathrm{g}}}{\partial S}\right)\Bigg|_{S=\frac{\pi}{Gx}} . \tag{32}$$

Four-parameter generalised entropies in (8),

$$F(T) = \frac{3G\beta}{\pi\gamma} (-T)^{\frac{1}{2}} \int^{-\frac{T}{6}} dy\, (6)^{-\frac{3}{2}}\, y^{\frac{1}{2}} \times \left[\frac{1}{(2+\beta)} \left(\frac{G\beta y}{\pi\alpha_-} \right)^{\beta} {}_2F_1 \left(1+\beta, 2+\beta, 3+\beta, -\frac{G\beta y}{\pi\alpha_-} \right) + \frac{1}{(2-\beta)} \left(\frac{G\beta y}{\pi\alpha_+} \right)^{-\beta} {}_2F_1 \left(1-\beta, 2-\beta, 3-\beta, -\frac{G\beta y}{\pi\alpha_+} \right) \right] . \tag{33}$$

Several limits for the parameters $\alpha_{\pm}$, β, and γ $\Rightarrow$ the expressions of $F(T)$ for different entropies.

Tsallis entropy (2) or Barrow entropy (4),

$$F(T) = -\frac{36\beta}{(2-\beta)(3+2\beta)}\left(\frac{G}{\pi}\right)^{1-\beta}(-T)^{1-\beta} + C(-T)^{\frac{1}{2}} . \tag{34}$$

C: a constant of the integration. We should note that there appear the fractional powers of T.

Rényi entropic cosmology (3),

$$F(T) = 3(-T)^{\frac{1}{2}} \int^{-\frac{T}{6}} dy\,(6y)^{-\frac{3}{2}} \frac{Gy^2}{2\pi\alpha} {}_2F_1\left(1, 2, 3, -\frac{Gy}{\pi\alpha}\right) . \tag{35}$$

Similarly, we can obtain the expressions of $F(T)$ corresponding to the Sharma-Mittal entropy (5), the Kaniadakis entropy (6), and the Loop Quantum Gravity entropy (7) by proceeding to take some limits of the parameters.

Correspondence between $F(Q)$ gravity and generalised entropies

Non-metricity gravity: a non-metricity scalar Q instead of the curvature R or the torsion T.

J. M. Nester and H. J. Yo, Chin. J. Phys. **37** (1999), 113 [arXiv:gr-qc/9809049 [gr-qc]].

Extension $\Rightarrow$ $F(Q)$ gravity

Review:
L. Heisenberg, "Review on F(Q) gravity," Phys. Rept. **1066** (2024), 1-78 doi:10.1016/j.physrep.2024.02.001 [arXiv:2309.15958 [gr-qc]].

General affine connection,

$$\Gamma^{\sigma}{}_{\mu\nu} = \tilde{\Gamma}^{\sigma}_{\mu\nu} + K^{\sigma}_{\mu\nu} + L^{\sigma}_{\mu\nu} \,. \tag{36}$$

$\tilde{\Gamma}^{\sigma}{}_{\mu\nu}$: the Levi-Civita connection given by the metric, $K^{\sigma}{}_{\mu\nu}$: the contortion in (28), $L^{\sigma}{}_{\mu\nu}$: deformation

$$L^{\sigma}{}_{\mu\nu} = \frac{1}{2}\left(Q^{\sigma}_{\mu\nu} - Q_{\mu}{}^{\sigma}{}_{\nu} - Q_{\nu}{}^{\sigma}{}_{\mu}\right) \,. \tag{37}$$

$Q^{\sigma}{}_{\mu\nu}$ is non-metricity tensor,

$$Q_{\sigma\mu\nu} = \nabla_{\sigma} g_{\mu\nu} = \partial_{\sigma} g_{\mu\nu} - \Gamma^{\rho}{}_{\sigma\mu} g_{\nu\rho} - \Gamma^{\rho}{}_{\sigma\nu} g_{\mu\rho} \,. \tag{38}$$

Imposing the condition that both the Riemann tensor and the torsion vanish,

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{\partial x^{\rho}}{\partial \xi^{a}} \partial_{\mu}\partial_{\nu}\xi^{a} \,. \tag{39}$$

ξ^{a} ($a = 0, 1, 2, 3$): four scalar fields ($\sim$ gauge degrees of freedom)

In the spatially flat FLRW spacetime, we can solve equations for connections with respect to ξ^a.
The ξ^a does not appear in the equations corresponding to the Friedmann equations,

$$\frac{1}{3}\left(\frac{F(Q)}{2}+6H^2F'(Q)\right)=\left(\frac{8\pi G}{3}\right)\rho\,. \tag{40}$$

In the spatially flat FLRW spacetime (12), $Q=-6H^2$.

The r.h.s. of Eq. (40) is given by a function of only $H \Rightarrow$

$$F(Q)=3\left(-Q\right)^{\frac{1}{2}}\int^{-\frac{Q}{6}}dy\,\left(6y\right)^{-\frac{3}{2}}\int^{y}dx\,\left(\frac{\partial S_{\mathrm{g}}}{\partial S}\right)\Bigg|_{S=\frac{\pi}{Gx}}\,. \tag{41}$$

$F(Q)$ gravity theory corresponding to the generalised entropy $S_{\mathrm{g}}(S)$.
We should note that the expression (41) is identical to the expression (32) for the $F(T)$ gravity by replacing T with Q.

Gravitaional wave

$F(T)$ and $F(Q)$ gravity theories are local theories corresponding to the general entropic theory.
$\Rightarrow$ We can consider the local dynamics like a gravitational wave in the general entropic theory.

Gravitational wave in $F(T)$ gravity,
K. Bamba, S. Capozziello, M. De Laurentis, S. Nojiri and D. Sáez-Gómez, "No further gravitational wave modes in $F(T)$ gravity," Phys. Lett. B **727** (2013), 194-198 doi:10.1016/j.physletb.2013.10.022 [arXiv:1309.2698 [gr-qc]].

Gravitational wave in $F(Q)$ gravity,
S. Capozziello, M. Capriolo and S. Nojiri, "Gravitational waves in F(Q) non-metric gravity via geodesic deviation," Phys. Lett. B **850** (2024), 138510 doi:10.1016/j.physletb.2024.138510 [arXiv:2401.06424 [gr-qc]].

The gravitational wave of massless and spin-two mode is only propagating mode.
We assume that the flat background $R = 0$ is a solution.
Perturbation of the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \, . \tag{42}$$

$\Rightarrow$

$$\frac{1}{2}F'(0)\left(\partial_\sigma\partial_\nu\bar{h}^\rho{}_\mu + \partial^\rho\partial_\mu\bar{h}_{\mu\nu} - \Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial^\sigma\bar{h}^\rho{}_\sigma\right) = \kappa^2 T_{\mathrm{matter}\,\mu\nu} \, . \tag{43}$$

$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$.
The gravitational coupling κ is modified by the effective one κ_{eff}

$$\kappa_{\mathrm{eff}}{}^2 = \frac{\kappa^2}{F'(0)} \, . \tag{44}$$

Summary and discussion

We considered the correspondence between gravity theories and the several kinds of entropic functions.
$\sim$ Jacobson's derivation of the Einstein equation from Bekenstein-Hawking entropy in the framework of thermodynamics.

We found the correspondence between the $F(T)$ and $F(Q)$ gravities and general entropies.

The corresponding gravity as an effective theory equivalent to the general entropic one $\Rightarrow$ the local dynamics because the gravity theories are described by the local Lagrangian density.
As one example: the gravitational wave:
The obtained equation is identical to that in Einstein's gravity but the effective gravitational coupling κ_{eff}: ${\kappa_{\mathrm{eff}}}^2 = \frac{\kappa^2}{F'(0)}$.

Black hole thermodynamics? Dynamical effects?