Fixed points of transformation of a renormalization group in a generalized fermionic hierarchical model

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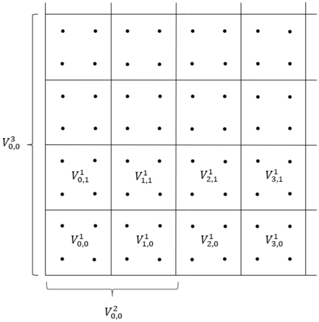
Slide 1 Renormalization group in a generalized hierarchical model

$$\begin{array}{ccc} (Z^d, R) & \longrightarrow & (Z^d, \Gamma) \\ \downarrow & & \downarrow \\ (T^d, R) & \longrightarrow & (T^d, \Gamma) \end{array}$$



$$\phi'(i) \equiv (r(\alpha)\phi)(i) = 2^{-\alpha/2} \sum_{j \in V_i^1} \phi(j)$$

Slide 2 Renormalization group in a generalized hierarchical model

Let $T = \{0, 1, 2, \ldots\}$ and $V_k^s = \{j \in T : \ k \cdot 2^s \leq j < (k+1) \cdot 2^s\}$, where $k \in T$, $s \in N = \{1, 2, 3, \ldots\}$.
The hierarchical distance $d_2(i, j)$, $i, j \in T$, $i \neq j$ is defined as $d_2(i, j) = 2^{s(i,j)}$, where

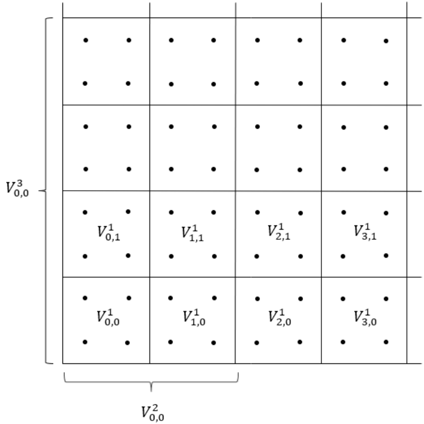
$$s(i, j) = \min\{s : \text{ is } k \in T \text{ such that } i, j \in V_k^s\}$$

Let $T^2 = T \times T$, $k = (k_1, k_2) \in T^2$,

$$V_k^s = \{(j_1, j_2) \in T^2 : \ k_1 \cdot 2^s \leq j_1 < (k_1+1) \cdot 2^s, \ k_2 \cdot 2^s \leq j_2 < (k_2+1) \cdot 2^s\}.$$

For any $k = (k_1, k_2) \in T^2$, $l = (l_1, l_2) \in T^2$, $k \neq l$ we define $s(k, l) = \max(s(k_1, l_1), s(k_2, l_2))$. The hierarchical distance on T^2 is defined as $d_2(k, l) = 2^{s(k,l)}$.

Slide 3 Renormalization group in a generalized hierarchical model



Slide 4 Renormalization group in a generalized hierarchical model

Let us consider a 4-component fermionic field

$$\psi^*(i) = (\bar{\psi}_1(i), \psi_1(i), \bar{\psi}_2(i), \psi_2(i)), \quad i \in T^2,$$

all components of which are generators of the Grassmann algebra. Let us redenote V_0^N by Λ_N. Let Γ_N be the Grassmann subalgebra generated by all generators

$$\bar{\psi}_1(i), \psi_1(i), \bar{\psi}_2(i), \psi_2(i), i \in \Lambda_N.$$

The block-spin renormalization group transformation is defined by the formula

$$\psi^{*\prime}(i) \equiv (r(\alpha)\psi^*)(i) = 2^{-\alpha/2} \sum_{j \in V_i^1} \psi^*(j), \tag{1}$$

where $\alpha \in R^1$ is the renormalization group parameter.

Slide 5 Renormalization group in a generalized hierarchical model

The Gaussian fermionic field with zero mean and binary correlation function

$$\langle \psi_n(k)\bar{\psi}_m(l)\rangle = \delta_{n,m} b(k,l), \quad n,m = 1,2, \quad k,l \in T^2$$

is defined on the entire lattice as a quasi-state (expectation) $\langle \cdot \rangle$ on the algebra of all monomials such that $\langle F \rangle$ for a monomial of even degree F is calculated according to Wick's rules and $\langle F \rangle = 0$ for any monomial of odd degree, $\delta_{n,m}$ is the Kronecker symbol. Here we use the following notations:

$$\psi(i) = (\psi_1(i), \psi_2(i)), \quad \bar{\psi}(i) = (\bar{\psi}_1(i), \bar{\psi}_2(i)),$$

$$\bar{\psi}(i)\eta(i) = \bar{\psi}_1(i)\eta_1(i) + \bar{\psi}_2(i)\eta_2(i), \quad i \in T^2.$$

Slide 6 Renormalization group in a generalized hierarchical model

We define the following functions on T^2:

$$d(k,l;\lambda) = d_2(k,l),\ \text{if } s(k_1,l_1) \neq s(k_2,l_2),$$

$$d(k,l;\lambda) = \lambda d_2(k,l),\ \text{if } s(k_1,l_1) = s(k_2,l_2),$$

$$f(k,l;\lambda;\alpha) = d^\alpha(k,l;\lambda),\ \text{if } k \neq l, \quad f(k,k;\lambda;\alpha) = \frac{2+\lambda^\alpha}{4(1-2^{-(2+\alpha)})},$$

λ is a real parameter, $\lambda > 0$.

Slide 7 Renormalization group in a generalized hierarchical model

Let $b(k,l;\lambda;\alpha) = f(k,l;\lambda;\alpha-4)$. It was shown in [1] that a zero-mean Gaussian fermionic field with a binary correlation function

$$\langle \psi_n(k)\bar{\psi}_m(l)\rangle = \delta_{n,m} b(k,l;\lambda;\alpha), \quad n,m = 1,2, \quad k,l \in T^2$$

is invariant under the transformation of the renormalization group with the parameter α (1). Let us denote the corresponding Gaussian quasi-state as $\rho_0(\lambda;\alpha)$.

Slide 8 Renormalization group in a generalized hierarchical model

Consider the restriction of the Gaussian state $\rho_0(\lambda;\alpha)$ to the volume Γ_N. It follows from [1, Theorem 1] that for any $F(\psi^*) \in \Gamma_N$:

$$\rho_0(\lambda;\alpha)(F(\psi^*)) = Z_N^{-1}(\lambda;\alpha)\int F(\psi^*)\exp\{-H_{0,N}(\psi^*;\lambda;\alpha)\}\,d\psi^*,$$

$$d\psi^* = \prod_{i\in\Lambda_N} d\psi_1(i)\,d\bar{\psi}_1(i)\,d\psi_2(i)\,d\bar{\psi}_2(i),$$

integration is carried out according to the rules of superanalysis,

$$Z_N(\lambda;\alpha) = \int \exp\{-H_{0,N}(\psi^*;\lambda;\alpha)\}\,d\psi^*.$$

Slide 9 Renormalization group in a generalized hierarchical model

Consider the local potential (self-action) for a 4-component fermionic field:

$$L(\psi^*; r, g) = r(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) + g\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2.$$

Denote $u(\psi^*(i)) = \exp\{-L(\psi^*(i); r, g)\}$. We will use the notation $\rho_N(\lambda; \alpha; u)$ for the state in the algebra Γ_N defined by the formula:

$$\rho_N(\lambda; \alpha; u)(F) = Z_N^{-1}(\lambda; \alpha; u) \int F \exp\{-H_{0,N}(\psi^*; \lambda; \alpha)\} \prod_{i\in\Lambda_N} u(\psi^*(i))\, \mathrm{d}\psi^*,$$

$$Z_N(\lambda; \alpha; u) = \int \exp\{-H_{0,N}(\psi^*; \lambda; \alpha)\} \prod_{i\in\Lambda_N} u(\psi^*(i))\, \mathrm{d}\psi^*.$$

Slide 10 Renormalization group in a generalized hierarchical model

If ρ is a state on Γ_N, then the renormalized state of ρ' is defined on Γ_{N-1} according to the formula: $\rho'\left(F(\psi^{*\prime})\right) = \rho\left(F(r(\alpha)\psi^*)\right)$. Let's define a quadratic form:

$$\sum_{i,j\in V_k^1} f(i,j;\mu(\lambda);-\alpha)\bar{\eta}^*(i)\eta^*(j) = Q_k(\eta^*), \quad k \in \Lambda_{N-1}.$$

$$f(i,j;\mu;-\alpha) = \frac{2+\mu^{-\alpha}}{4(1-2^{\alpha-2})}, \quad \text{if } i=j,$$

$$\mu(\lambda) = \left(\frac{4(1-2^{2-\alpha})-3\lambda^{\alpha-4}}{\lambda^{\alpha-4}(1-2^{2-\alpha})-3\cdot 2^{2-\alpha}}\right)^{-\frac{1}{\alpha}},$$

$$f(i,j;\mu;-\alpha) = 2^{-\alpha}, \quad \text{if } d_2(i,j)=1 \text{ and } s(i_1,j_1)\neq s(i_2,j_2), \quad (2)$$

$$f(i,j;\mu;-\alpha) = (2\mu)^{-\alpha}, \quad \text{if } d_2(i,j)=1 \text{ and } s(i_1,j_1)=s(i_2,j_2).$$

Slide 11 Renormalization group in a generalized hierarchical model

In [1, Theorem 2], it was proved that

$$\rho'_N(\lambda;\alpha;u) = \rho_{N-1}(\lambda;\alpha;u')$$

where

$$u'(\psi^{*\prime}) = \int \delta\left(\sum_{i\in V_0^1} \eta^*(i)\right) \exp\{-Q_0(\eta^*)\} \quad * \\ * \prod_{i\in V_0^1} \left(u(2^{\alpha/2-2}\psi^{*\prime} + \eta^*(i)\right) d\eta^*(i). \quad (3)$$

Let us denote the transformation in the density space given by the right side (3) as $R(\alpha)u$. From this formula we see that the transformation of the renormalization group $R(\alpha)$ is local and does not depend on N.

Slide 12 Renormalization group in a generalized hierarchical model

In this section, instead of regular densities of the form

$$u(\psi^*) = \exp\{-L(\psi^*; r, g)\}$$

we will use Grassmann-valued densities of the general form

$$u(\psi^*) = c_0 + c_1(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) + c_2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2.$$

Slide 13 Renormalization group in a generalized hierarchical model

The triple $c = (c_0, c_1, c_2)$ can be naturally treated as a point in the two-dimensional real projective space because two sets that differ by a nonzero factor represent the same Gibbs state. Let

$$\delta = 2\frac{(2^{2-\alpha} - 1)(\lambda^{\alpha-4} - 1)}{\lambda^{\alpha-4} + 2^{4-\alpha} - 2} + 2, \gamma = 2^{\alpha-2}.$$

Slide 14 Renormalization group in a generalized hierarchical model

Theorem

Let $u^{'}(\psi^{*'}) = R(\alpha)u$. *Then*

$$u^{'}(\psi^{*'}) = c_0^{'} + c_1^{'}(\bar{\psi}_1^{'}\psi_1^{'} + \bar{\psi}_2^{'}\psi_2^{'}) + c_2^{'}\bar{\psi}_1^{'}\psi_1^{'}\bar{\psi}_2^{'}\psi_2^{'} \quad (4)$$

$$c_0^{'} = f_0(c;\delta) = c_0^4(\delta^2 - 2\delta + 1) + c_0^3 c_1(-4\delta^2 + 6\delta - 2) +$$
$$+ c_0^3 c_2(\delta^2 - \delta + \frac{1}{4}) + c_0^2 c_1^2(5\delta^2 - 3\delta - \frac{5}{4}) + c_0^2 c_1 c_2(-2\delta^2 - \delta + 1) +$$
$$+ c_0^2 c_2^2(\frac{\delta^2}{4} + \frac{1}{2}) + c_0 c_1^3(-2\delta^2 - 3\delta + 3) + c_0 c_1^2 c_2(\frac{\delta^2}{2} + 5\delta - \frac{3}{2}) +$$
$$+ c_0 c_1 c_2^2(-\delta - 1) + \frac{c_0 c_2^3}{4} + c_1^4(\frac{\delta^2}{4} + \delta) + c_1^3 c_2(-\delta - 1) + \frac{3c_1^2 c_2^2}{4}, \quad (5)$$

Slide 15 Renormalization group in a generalized hierarchical model

Theorem

$$c_1' = \gamma f_1(c;\delta) = \gamma(c_0^3 c_1(\delta^2 - 2\delta + 1) + c_0^3 c_2(-\frac{\delta^2}{2} + \frac{3\delta}{4} - \frac{1}{4}) +$$
$$+ c_0^2 c_1^2(-\frac{7\delta^2}{2} + \frac{21\delta}{4} - \frac{7}{4}) + c_0^2 c_1 c_2(\frac{5\delta^2}{2} - 2\delta) + c_0^2 c_2^2(-\frac{\delta^2}{4} - \frac{\delta}{2} + \frac{1}{2}) +$$
$$+ c_0 c_1^3(\frac{7\delta^2}{2} - 2\delta - 1) + c_0 c_1^2 c_2(-3\delta^2 - \frac{3\delta}{2} + \frac{3}{2}) + c_0 c_1 c_2^2(\frac{\delta^2}{2} + 2\delta) +$$
$$+ c_0 c_2^3(-\frac{\delta}{4} - \frac{1}{4}) + c_1^4(-\frac{3\delta^2}{4} - 2\delta + 2) + c_1^3 c_2(\frac{\delta^2}{2} + 4\delta - 1) +$$
$$+ c_1^2 c_2^2(-\frac{7\delta}{4} - \frac{7}{4}) + c_1 c_2^3) \quad (6)$$

Slide 16 Renormalization group in a generalized hierarchical model

Theorem

$$c_2' = \gamma^2 f_2(c;\delta) = \gamma^2 (c_0^3 c_2 (\frac{\delta^2}{4} - \frac{\delta}{2} + \frac{1}{4}) + c_0^2 c_1^2 (\frac{3\delta^2}{4} - \frac{3\delta}{2} + \frac{3}{4}) +$$
$$+ c_0^2 c_1 c_2 (-2\delta^2 + 3\delta - 1) + c_0^2 c_2^2 (\frac{3\delta^2}{4} - \delta + \frac{1}{2}) + c_0 c_1^3 (-2\delta^2 + 3\delta - 1) +$$
$$+ c_0 c_1^2 c_2 (4\delta^2 - 2\delta - \frac{3}{2}) + c_0 c_1 c_2^2 (-2\delta^2 - \delta + 1) + c_0 c_2^3 (\frac{\delta^2}{4} + \frac{\delta}{2} + \frac{1}{4}) +$$
$$+ c_1^4 (\frac{5\delta^2}{4} - \delta) + c_1^3 c_2 (-2\delta^2 - 3\delta + 3) + c_1^2 c_2^2 (\frac{3\delta^2}{4} + \frac{11\delta}{2} - \frac{5}{4}) +$$
$$+ c_1 c_2^3 (-2\delta - 2) + c_2^4) \quad (7)$$

Slide 1 Fixed points

Note that the mapping given by the formulas (5), (6), (7) depends not only on α, but also on the parameter δ. Therefore, let us denote the mapping of the renomalization group $R(\alpha)$ in c-space as $R(\alpha;\delta)$:

$$R(\alpha;\delta)(c_0, c_1, c_2) = (f_0(c;\delta), f_1(c;\delta), f_2(c;\delta)).$$

Slide 2 Fixed points

Next, consider the representation of our model in (r, g)-coordinates. Let:

$$u(\psi^*) = exp\{-r(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) + g\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2\}$$

In the projective representation, such a density is given by the triple of coefficients

$$c = (1, -r, r^2 - g).$$

Slide 3 Fixed points

The transformation of the renormalization group in the plane of coupling constants (r, g) will also be denoted by $R(\alpha; \delta)$:

$$(r^{'}, g^{'}) = R(\alpha; \delta)(r, g).$$

Lemma

The transformation of the renormalization group $R(\alpha; \delta)$ in the plane (r, g) has the form:

$$r^{'} = 2^{\alpha-2} \frac{r_1(r, g)}{r_2(r, g)}, \tag{8}$$

$$g^{'} = \frac{2^{2(\alpha-2)}}{4} g \left(\frac{r_3(r, g)}{r_2(r, g)} \right)^2, \tag{9}$$

Slide 4 Fixed points

Lemma

$$r_1(r,g) = -g^3 r - g^3(\frac{\delta}{4}+\frac{1}{4}) + 3g^2 r^3 - g^2 r^2(-\frac{5\delta}{2}-\frac{5}{2}) - g^2 r(-\frac{\delta^2}{2}-2\delta) -$$

$$-g^2(-\frac{\delta^2}{4}-\frac{\delta}{2}+\frac{1}{2}) - 3gr^5 - gr^4(\frac{17\delta}{4}+\frac{17}{4}) - gr^3(\frac{3\delta^2}{2}+8\delta-1) -$$

$$-gr^2(\frac{7\delta^2}{2}+\frac{5\delta}{2}-\frac{5}{2}) - gr(\frac{5\delta^2}{2}-2\delta) - g(\frac{\delta^2}{2}-\frac{3\delta}{4}+\frac{1}{4}) + r^7 - r^6(-2\delta-2) -$$

$$-r^5(-\delta^2-6\delta+1) - r^4(-4\delta^2-4\delta+4) - r^3(-6\delta^2+4\delta+1) -$$

$$-r^2(-4\delta^2+6\delta-2) + r(\delta-1)^2$$

Slide 5 Fixed points

Lemma

$$r_2(r,g) = -\frac{g^3}{4} + \frac{3g^2r^2}{2} + g^2r(\delta+1) + g^2(\frac{\delta^2}{4}+\frac{1}{2}) - \frac{9gr^4}{4} + gr^3(-3\delta-3)+$$

$$+gr^2(-\delta^2-5\delta+\frac{1}{2}) + gr(-2\delta^2-\delta+1) + g(-\delta^2+\delta-\frac{1}{4}) + r^6 + r^5(2\delta+2)+$$

$$+r^4(\delta^2+6\delta-1) + r^3(4\delta^2+4\delta-4) + r^2(6\delta^2-4\delta-1)+$$

$$+r(4\delta^2-6\delta+2) + (\delta-1)^2$$

$$r_3(r,g) = (-g+(r+1)^2)(\frac{-\delta^2 g}{2} - 2\delta gr + 2\delta r^3 + g^2 - 2gr^2 + r^4+$$

$$+r^2(\delta^2+2\delta-2) + r(2\delta^2-2\delta) + (\delta-1)^2).$$

Slide 6 Fixed points

Sorollary

Consequence. The domains $\{(r,g) : g > 0\}$ *(upper half-plane),* $\{(r,g) : g < 0\}$ *(lower half-plane) and* $\{(r,g) : g = 0\}$ *are invariant domains of the mapping* $R(\alpha;\delta)$*. The action of* $R(\alpha,\delta)$ *on the line* $\{(r,g) : g = 0\}$ *is linear:*

$$R(\alpha;\delta)(r,0) = (2^{\alpha-2}r, 0).$$

Slide 7 Fixed points

Let

$$A_0 = (1, 0, 0), \quad A_1 = (0, 0, 1).$$

The points A_0 and A_1 are fixed points of $R(\alpha, \delta)$ for all α and δ. In coordinates (r, g), the point A_0 is given as $(0, 0)$, so we call A_0 the Gaussian fixed point. The point A_1 does not belong to the regular plane $\{(r, g)\}$, and it corresponds to the density $u(\psi^*) = \bar{\psi}_1 \psi \bar{\psi}_2 \psi_2$ (Grassmann delta function).

Slide 8 Fixed points

Theorem

1. *The fixed point* A_0 *is repulsive for* $\alpha > 3$*, saddle point - for* $2 < \alpha < 3$*, and attractive for* $\alpha < 2$*.*
2. *The fixed point* A_1 *is attractive for* $\alpha > 2$*, the saddle point is in the interval* $1 < \alpha < 2$*, and repulsive for* $\alpha < 1$*.*

Slide 9 Fixed points

1. if $\alpha > 3$, then the fixed point A_0 is repulsive,
2. if $2 < \alpha < 3$, then A_0 is a saddle point,
3. if $\alpha < 2$, then A_0 is an attracting point,
4. the value $\alpha = 3$ is a bifurcation value and therefore a new (non-Gaussian) branch of fixed points must exist in a neighborhood of the point A_0.

Slide 10 Fixed points

1. if $\alpha > 2$, then A_1 is an attracting point,
2. if $1 < \alpha < 2$, then the point A_1 is a saddle fixed point,
3. if $\alpha < 1$, then the point A_1 is a repulsive point,
4. the value $\alpha = 1$ is bifurcational, therefore, in the vicinity of the point A_1, there must be a new branch of fixed points.

Slide 11 Fixed points

We consider the realization of the projective c space in the form of the hemisphere

$$S = \left\{(c_0, c_1, c_2) : c_0^2 + c_1^2 + c_2^2 = 1, c_0 \geq 0\right\},$$

where the opposite points of the boundary circle $c_1^2 + c_2^2 = 1$ are identified. To obtain the flat (two-dimensional) picture, we use the orthogonal projection S on the disk $D = \left\{(c_1, c_2) : c_1^2 + c_2^2 \leq 1\right\}$.

Slide 12 Fixed points

The regular point (r, g) then corresponds to $(c_1(r, g), c_2(r, g))$.

$$c_1(r, g) = -\frac{r}{\sqrt{1 + r^2 + (r^2 - g)^2}}, \quad c_2(r, g) = \frac{r^2 - g}{\sqrt{1 + r^2 + (r^2 - g)^2}}.$$

We note that the points $(c_1(r, g), c_2(r, g))$ belong to the interior of the disk D. The trivial fixed point $r = 0$, $g = 0$ is also represented by the point $(0, 0)$ in the (c_1, c_2) coordinates. The fixed point δ in (c_1, c_2) coordinates is determined by the point $(0, 1)$. The line $g = 0$ in the (c_1, c_2) space is described by the curve $l_0 = \{(c_1(r, 0), c_2(r, 0))\, ; r \in R\}$.

Slide 13 Fixed points

Let $v = r + 1, \tau = \delta - 2$, then

Theorem

$$r_2(r,g;\delta) = s_2(v,g;\tau) = -\frac{g^3}{4} + g^2\left(\frac{\tau^2}{4} + \tau v + \frac{3v^2}{2}\right) - \\ - \frac{gv^2(2\tau+3v)^2}{4} + v^4(\tau+v)^2;$$

$$r_3(r,g;\delta) = s_3(v,g;\tau) = (-g+v^2)\left(g^2 - \frac{g(\tau+2u)^2}{2} + v^2(\tau+v)^2\right);$$

$$r_1(r,g;\delta) = s_1(v,g;\tau) = \frac{1}{4}g\tau(g-v^2)^2 - s_2(v,g;\tau) + s_3(v,g;\tau)v. \tag{10}$$

Slide 14 Fixed points

Note that the parameters τ, γ depend on α, but we will not explicitly designate this dependence.

Theorem

Let $\alpha \neq 2$, the renormalization group transformation on a generalized hierarchical model in the space of coupling constants (v,g) have 4 branches of fixed points, where

$$v_{\pm}^{1} = \gamma - 1 + \tau(\gamma - 2\beta_{\pm}^{1}), g_{\pm}^{1} = \frac{\beta_{\pm}^{1} - 1}{\beta_{\pm}^{1} - \frac{1}{2}} {v_{\pm}^{1}}^{2},$$

$$\tau = \delta(\alpha, \lambda) - 2 = 2\frac{(2^{2-\alpha} - 1)(\lambda^{\alpha-4} - 1)}{\lambda^{\alpha-4} + 2^{4-\alpha} - 2},$$

$$\beta_{\pm}^{1} = \frac{3(\gamma(\tau+1)^2 - 1) \pm \sqrt{(\gamma(\tau+1)^2 - 1)^2 + 8(\tau+1)^2(\gamma-1)^2}}{4\tau(\tau+2)},$$

Slide 15 Fixed points

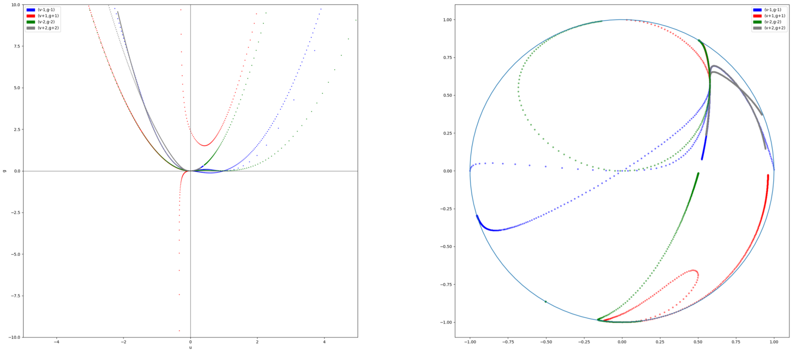
Theorem

$$v_{\pm}^2 = \frac{1-\gamma}{3} - \frac{\tau}{3}(\gamma + 2\beta_{\pm}^2), g_{\pm}^2 = \frac{\beta_{\pm}^2 - 1}{\beta_{\pm}^2 - \frac{1}{2}} {v_{\pm}^2}^2,$$

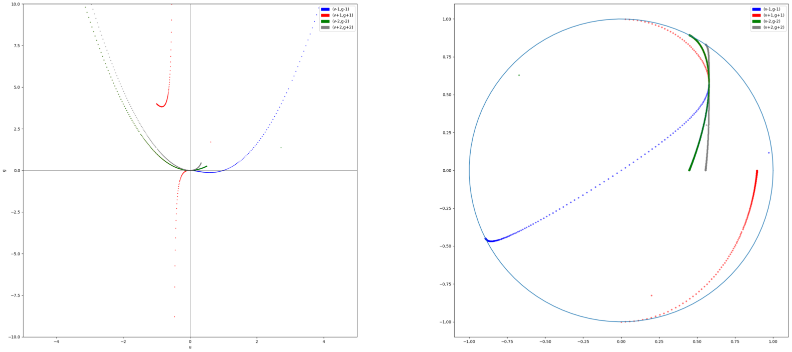
$$\beta_{\pm}^2 = -\frac{5\gamma^2\tau^2 + 2\gamma^2\tau - 11\gamma\tau^2 - 10\gamma\tau + 8\tau - 3(\gamma-1)^2 \pm 3\sqrt{\Delta(\tau,\gamma)}}{4\tau(-7\gamma\tau - 6\gamma + \tau + 6)},$$

$$\Delta(\tau,\gamma) = \gamma^4(9\tau^4 + 20\tau^3 + 14\tau^2 + 4\tau + 1) + \gamma^3(18\tau^4 + 16\tau^3 - 10\tau^2 - 12\tau - 4) + \gamma^2(9\tau^4 - 20\tau^3 - 14\tau^2 + 12\tau + 6) + \gamma(-16\tau^3 + 2\tau^2 - 4\tau - 4) + 8\tau^2 + 1.$$

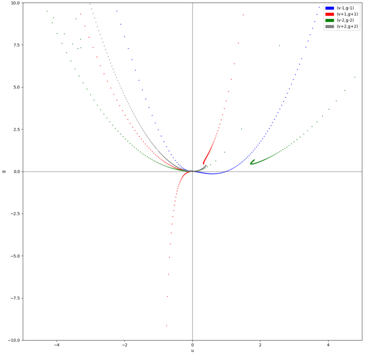
Slide 16 Fig.1 $\lambda = 1/\sqrt{2}$ Fiixed points

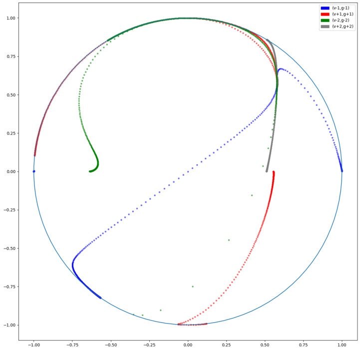


Slide 17 Fig.2 $\lambda = 1$ Fixed points



Slide 18 Fig.3 $\lambda = \sqrt{2}$ Fixed points





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[2] M. D. Missarov, D. A. Khajrullin, "The renormalization group transformation in the generalized fermionic hierarchical model Izv. Math., 2023, Vol. 87, No. 5, pp. 1011-1012.