Wavefunction of the Universe and the Effective Action

Aleksandar Miković [1]
Lusófona University
and
Mathematical Physics Group, IST, Lisbon

Branko Dragovich's 80-th birthday meeting, Belgrade, 26 - 30 May, 2025

Abstract

We review the Hartle-Hawking construction for the full quantum gravity and show how it can be realized in the PFQG theory. We then review the QFT effective action and show how to relate it to the wavefunction of the Universe.

[1] Work supported by FCT through the project GFM-IST/2025.

1 Wavefunction of the Universe

- The WFU can be determined by the Hartle-Hawking construction, which is a path integral for M with $\partial M = \Sigma$

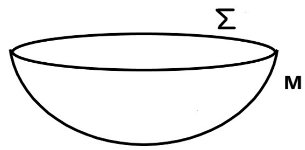

Figure 1: Topology of the Hartle-Hawking manifold

- The main problem is the definition of the path integral

$$\Psi_0(q) = \int \mathcal{D}Q \, e^{iS(Q)/\hbar} \,,$$

where

$$Q = (g, \varphi) \,, \quad q = Q|_\Sigma = (h, \phi) \,, \quad S(Q) = S_{AHC}(g) + S_m(g, \varphi) \,,$$

$$S_{AHC} = \int_M \sqrt{g} \, (R(g) + \Lambda) \, d^4x \,.$$

- Discretization approach

$$M \to T(M) \,, \quad Q \to (L, \varphi) \,, \quad S(L, \varphi) = \frac{1}{G_N} S_{RC}(L) + S_m(L, \varphi) \,,$$

$$S_{RC}(L) = \sum_{\Delta \in T(M)} A_\Delta(L) \, \delta_\Delta(L) + \Lambda V_4(L) \,,$$

$$V_4(L) = \sum_{\sigma \in T(M)} V_\sigma(L) \,,$$

$$\mathcal{D}Q = \mu(L) \prod_{\epsilon=1}^{N} dL_\epsilon \prod_{v=1}^{n} d\varphi_v \,,$$

and find the limit when $N \to \infty$ and $L_\epsilon \to 0$.

- HH construction
$$\mu(L) = \text{const.}\,, \quad S(L,\varphi) = iS_E(L,\varphi)\,.$$

- HH is problematic because $e^{-S_E(L,\varphi)}$ is not a bounded function on the L space.

- One can avoid this problem by using the minisuperspace approximation and complex contours, but HH does not work in full QG.

- PLQG approach: Let N and n be large and
$$\mu(L) = e^{-V_4(L)/L_0^4} \prod_{\epsilon=1}^{N} \left(1 + \frac{|L_\epsilon|^2}{l_0^2}\right)^{-p}\,,$$
where L_0, l_0 and p parameters.

- When $S = S_G$, easy to show that the path integral is convergent for
$$p > \frac{1}{2}\quad.$$

- The exponential damping factor is necessary in order to have the correct SC expansion of the EA, see [1, 2].

- When $S = S_G + S_{SM}$, the PLQG path integral is convergent for
$$p > 52{,}5\quad.$$
See [5].

- The smooth-manifold approximation
$$\Psi_0(q) \approx \Phi_0[h(\vec{x}), \phi(\vec{x})]\,, \quad \vec{x} \in \Sigma\,,$$
when $N \to \infty$ and $l_\epsilon = O(l_0/N)$.

- Time evolution
$$\Psi(q,t) = Z_T(M(t))\,,$$
where
$$M(t) = M_0 \sqcup (\Sigma \times [t_i, t])$$

- The time function is determined by the triangulation $T(\Sigma \times I)$ such that T is a temporal triangulation, i.e.
$$T(\mathcal{U}) = \bigcup_{k=1}^{n'} T_k(\Sigma) \cup T(\bar{\mathcal{U}})\,,$$
where
$$\mathcal{U} \equiv \Sigma \times I\,, \quad \bar{\mathcal{U}} = \mathcal{U} \setminus \bigcup_{k=1}^{n'} \Sigma_k\,,$$

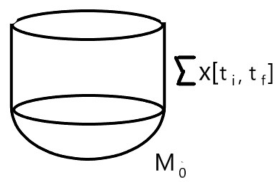

Figure 2: PFQG spacetime manifold with a time variable interval.

while all the edge lengths in $T_k(\Sigma)$ are spacelike and all the edge lengths in $T(\bar{\mathcal{U}})$ are timelike. Then

$$\Delta t_k = \min\{L_\gamma = \sum_{\epsilon\in\gamma} |L_\epsilon| : \gamma \in T(\bar{\mathcal{U}})\}\,,$$

where γ is a PL curve with an inital point in $T_1(\Sigma)$ and a final point in $T_k(\Sigma)$.

- When $n' \to \infty$ then $\Delta t_k \approx t - t_i$ and

$$\Psi(q,t) \approx \Phi[h(\vec{x}), \phi(\vec{x}), t]\,, \quad \vec{x} \in \Sigma\,.$$

- We can write

$$\Psi(q,t) = \hat{U}_T(t,t_i)\Psi_0(q)\,,$$

and we expect that the evolution operator will be unitary for a hamiltonian triangulation of $\Sigma \times I$

$$T_1(\Sigma) = T_2(\Sigma) = \cdots = T_{n'}(\Sigma)\,.$$

2 The Effective Action

- QFT EA

$$M = \Sigma \times I = \mathbf{R}^3 \times \mathbf{R}\,,$$

the metric on M is flat and

$$Z[J] = \int \mathcal{D}\varphi\, e^{\frac{i}{\hbar}\left(S[\varphi] + \int_M J(x)\varphi(x)\, d^4x\right)}\,,$$

where $S = S_m$.

- Legendre transform

$$\Gamma[\varphi] = W[J] - \int_M J(x)\varphi(x)\, d^4x \,,$$

where

$$W[J] = -i\hbar \log Z[J] \,, \quad \varphi(x) = \frac{\delta W}{\delta J(x)} \,.$$

- It is easy to show that

$$e^{i\Gamma[\varphi]/\hbar} = \int \mathcal{D}\phi \exp\left[\frac{i}{\hbar}\left(S[\varphi+\phi] - \int_M \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}\phi(x)\, d^4x\right)\right] .$$

- The EA equation has a perturbative solution

$$\Gamma[\varphi] = S[\varphi] + \hbar\Gamma_1[\varphi] + \hbar^2\Gamma_2[\varphi] + \cdots \,,$$

which is valid in the perturbative regime

$$|S| \gg \hbar|\Gamma_1| \gg \hbar^2|\Gamma_2| \gg \cdots \quad .$$

- From the relationship

$$\varphi(\vec{x},t) = \langle\Psi_0|\hat{U}(t_2,t)\,\hat{\varphi}(\vec{x})\,\hat{U}(t,t_1)|\Psi_0\rangle = \frac{\delta W}{\delta J(x)}\Big|_{J=0} \,,$$

where $|\Psi_0\rangle$ is the vacuum state, $\hat{U}$ is the evolution operator and $t_1 \to -\infty$, $t_2 \to +\infty$, we see that the QFT EA is defined with respect to the QFT vacuum.

- In the QG case one can define the analog of the QFT $Z[J]$, by using the PLQG path integral for $M = \Sigma \times I$. However, then it is not clear what is the Ψ_0 state, because the QG Hamiltonian is not bounded from bellow.

- The answer comes from the QM formulas

$$\langle q_2|\hat{U}(t_2,t_1)|q_1\rangle = \int \mathcal{D}q \exp\left(\frac{i}{\hbar}\int_{t_1}^{t_2} L(q,\dot{q})\, dt\right) \,,$$

where $q(t_k) = q_k$, $k = 1,2$ and

$$Z_{1,2}[J(t)] = \int \mathcal{D}q \exp\left(\frac{i}{\hbar}\int_{t_1}^{t_2} [L(q,\dot{q}) + J(t)q(t)]\, dt\right) .$$

- Consequently

$$\langle\hat{q}(t)\rangle_{1,2} = \frac{\delta Z_{1,2}}{\delta J(t)}\Big|_{J=0} = \langle q_2|\hat{U}(t_2,t)\,\hat{q}\,\hat{U}(t,t_1)|q_1\rangle \,,$$

so that the PLQG path integral on $\Sigma \times I$ is just the Feynman formula for the evolution operator

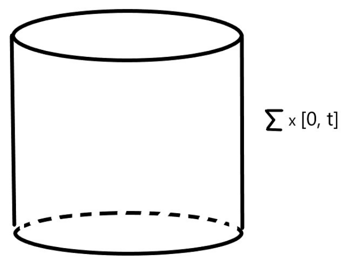

Figure 3: Topology of the QM propagator manifold.

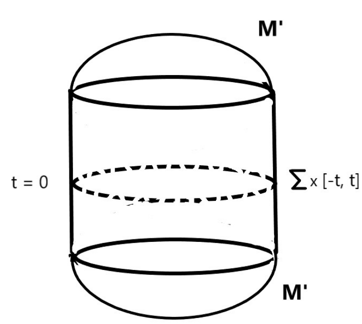

Figure 4: Topology of the QFT effective action manifold

- If we assume that the analog of the QFT vacuum is the HH state $\Psi_0(q)$, then we obtain
$$\langle \hat{q}(t)\rangle_{\Psi_0} = \int d^n q_1 \int d^n q_2 \Psi_0^*(q_2)\langle \hat{q}(t)\rangle_{1,2}\Psi_0(q_1)\,,$$
which will be generated by the path integral for the manifold $M = M_0 \sqcup \mathcal{U} \sqcup M_0$

- Taking the closed manifold $M_0 \sqcup \mathcal{U} \sqcup M_0$ is not physically justifiable, because it means that the history of our universe will end in a big crunch. Hence we take the open manifold $M_0 \sqcup \mathcal{U}$ in order to construct the EA that corresponds to the WFU $\Psi(q,t)$.

3 EA and the manifold topology

- Let
$$M = M_- \sqcup \mathcal{U}\,, \quad \bar{M} = M_- \sqcup \mathcal{U} \sqcup M_+$$
where $\mathcal{U} \equiv \Sigma \times [t_i, t_f]$, while $M_\pm$ indicates the manifold M_0 with the boundary Σ at the time t_f and at the time t_i, respectively.

- We can then write
$$Z_{\bar{M}} = \int d^n q_- \int d^n q_+ \, Z_0(q_-)\, Z_U(q_-, q_+)\, Z_0^*(q_+)\,,$$
$$Z_{\bar{M}}(\bar{J}) = \int d^n q_- \int d^n q_+ \, Z_0(J_-, j_-, q_-)\, Z_U(q_-, \tilde{J}, q_+)\, Z_0^*(q_+, j_+, J_+)\,,$$
where
$$\bar{J} = (J_-, j_-, \tilde{J}, j_+, J_+)\,, \quad Q = (Q_-, q_-, \tilde{Q}, q_+, Q_+)\,,$$
$$Q_\alpha = (L_\alpha, \Phi_\alpha)\,, \quad q_\pm = (l_\pm, \varphi_\pm)\,, \quad n = n_l + n_\varphi\,,$$
where n_l is the number of edges and n_φ is the number of vertices of $T_i(\Sigma)$ and $T_f(\Sigma)$.

- The simbol Z_0^* means that we take $e^{-iS/\hbar}$ instead of $e^{iS/\hbar}$ in the integrand, where S is the classical action on $T(M_+)$.

- In the standard QFT we are not interested in the dynamics of $Q_\pm$, so that we use
$$\tilde{Z}_{\mathcal{U}}(\tilde{J}) = Z_{\bar{M}}(0,0,\tilde{J},0,0) = \int d^n q_- \int d^n q_+ \, Z_0(q_-)\, Z_U(q_-, \tilde{J}, q_+)\, Z_0^*(q_+)\,,$$
$$= \int d^n q_- \int d^n q_+ \, \Psi_0(q_-)\, Z_U(q_-, \tilde{J}, q_+)\, \Psi_0^*(q_+)\,.$$

- Consequently
$$\tilde{\Gamma}_{\mathcal{U}}(\tilde{Q}) = \tilde{W}_{\mathcal{U}}(\tilde{J}) - \tilde{J}\tilde{Q}\,,$$
where
$$\tilde{Q} = \partial \tilde{W}_{\mathcal{U}}/\partial \tilde{J}\,, \quad \tilde{W}_{\mathcal{U}} = -i\hbar \log \tilde{Z}_{\mathcal{U}}\,.$$

- Note that one can also use

$$\Gamma_{\mathcal{U}}(q_-, \tilde{Q}, q_+) = W_{\mathcal{U}}(j_-, \tilde{J}, j_+) - j_- q_- - \tilde{J}\tilde{Q} - j_+ q_+ \,,$$

where

$$W_{\mathcal{U}} = -i\hbar \log Z_{\mathcal{U}} \,, \quad \tilde{Q} = \partial W_{\mathcal{U}}/\partial \tilde{J} \,, \quad q_\pm = \partial W_{\mathcal{U}}/\partial j_\pm \,,$$
$$\tilde{J}\tilde{Q} = J_L L + J_\Phi \Phi \,, \quad jq = j_l l + j_\varphi \varphi \,.$$

- When

$$\Psi_0(q) = \delta(q - q_0) = \delta(l - l_0)\delta(\varphi - \varphi_0) \,,$$

where l_0 gives a flat metric on $T(\Sigma)$ and $\varphi_0 = 0$ we have

$$\tilde{\Gamma}_{\mathcal{U}}(L, \Phi) = \Gamma_{\mathcal{U}}(q_0, L, \Phi, q_0) \,.$$

- If $N_{\mathcal{U}} \to \infty$ such that $L_\epsilon = O(1/N_{\mathcal{U}})$ for all $\epsilon \in T(\mathcal{U})$ and for the trivial WFU then

$$\tilde{\Gamma}_{\mathcal{U}}(L, \Phi) \approx \tilde{\Gamma}_{\mathcal{U}}[g, \varphi] \equiv \Gamma_{\mathcal{U},K}[g, \varphi] \,,$$

where $\Gamma_{\mathcal{U},K}$ is the usual QFT EA for the momentum cutoff $\hbar K$, which is proportional to $\hbar/\bar{L}_{\mathcal{U}}$, where $\bar{L}_{\mathcal{U}}$ the average edge length in $T(\mathcal{U})$.

4 The WFU correction

- When the WFU is nontrivial, instead of using the effective action $\tilde{\Gamma}_{\mathcal{U}}(\tilde{Q})$, it is easier and more resonable to use the effective action for the manifold $M = M_0 \sqcup \mathcal{U}$, which we denote as $\Gamma_M(Q_-, q_-, \tilde{Q}, q_+)$.

- We can write

$$\Gamma_M(Q_-, q_-, \tilde{Q}, q_+) = \Gamma_{\mathcal{U}}(q_-, \tilde{Q}, q_+) + \Delta\Gamma_M(Q_-, q_-, \tilde{Q}, q_+) \,,$$

and the correction $\Delta\Gamma_M$ can be calculated perturbatively by using the perturbative expansions

$$\Gamma_{\mathcal{U}}(q_-, \tilde{Q}, q_+) = \sum_{k\geq 0} \hbar^k \, \Gamma_{\mathcal{U},k}(q_-, \tilde{Q}, q_+) \,,$$

and

$$\Gamma_M(Q_-, q_-, \tilde{Q}, q_+) = \sum_{k\geq 0} \hbar^k \, \Gamma_{M,k}(Q_-, q_-, \tilde{Q}, q_+) \,,$$

when $|L_\epsilon| \gg l_P$ and $|\tilde{\varphi}_v| < 1$ for $\epsilon, v \in T(M)$.

- We then have

$$\Gamma_{M,0}(Q_M) = S(Q_-, q_-) + S(q_-, \tilde{Q}, q_+) \equiv S_0 + S_{\mathcal{U}} \,,$$

$$\Gamma_{M,1}(Q_M) = \frac{i}{2} Tr(\log(S_0 + S_{\mathcal{U}})'') - i \log \mu(L_M) \,,$$

$$\Gamma_{\mathcal{U},0} = S_{\mathcal{U}} \,, \quad \Gamma_{\mathcal{U},1}(Q_{\mathcal{U}}) = \frac{i}{2} Tr(\log S''_{\mathcal{U}}) - i \log \mu(L_{\mathcal{U}}) \,,$$

where $Q_U = (q_-, \tilde{Q}, q_+)$ and S''_0, $S''_{\mathcal{U}}$ denote the corresponding Hessian matrices, see [4].

- The higher-order corrections Γ_k are functions of the higher-order derivatives of $S(L,\Phi)$ and of the higher-order derivatives of $\log \mu(L)$, and a Γ_k function is determined by summing the evaluations of the connected 1PI graphs with k loops, see [4].

- Consequently we can write

$$\Gamma_{M,k}(Q_M) = \Gamma_{\mathcal{U},k}(Q_{\mathcal{U}}) + \Delta\Gamma_{M,k}(Q_M)\,,$$

for $k = 0, 1, 2, \ldots$.

- Given an arbitrary manifold M, then on $T(M)$ we can rewrite the perturbative expansion as

$$\frac{\Gamma(L,\Phi)}{\hbar} = \frac{S_{RC}(L) + \tilde{S}_m(L,\Phi)}{l_P^2} + \Gamma_1(L,\Phi) + l_P^2 \frac{\Gamma_2(L,\Phi)}{G_N} + l_P^4 \frac{\Gamma_3(L,\Phi)}{G_N^2} + \cdots$$

$$\equiv \frac{\tilde{S}(L,\Phi)}{l_P^2} + \sum_{k\geq 1} l_P^{2(k-1)}\,\tilde{\Gamma}_k(L,\Phi)\,.$$

- One can show that for N large

$$\tilde{S} = O(N(\bar{L})^2)\,, \quad \tilde{\Gamma}_1 = O(N)\,, \qquad (D.4)$$

where N is the number of edges in $T(M)$ and $\bar{L}$ is the average edge length in $T(M)$.

- From the expansion (D.3) and the result (D.4) we expect to have for $k > 1$

$$\tilde{\Gamma}_k = O\left(\frac{N}{(\bar{L})^{2(k-1)}}\right)\,.$$

- Consequently

$$\tilde{S}_{\mathcal{U}} = O(N_{\mathcal{U}}\bar{L}_{\mathcal{U}}^2)\,, \quad \tilde{S}_0 = O(N_0\bar{L}_0^2)\,,$$

so that $|S_{\mathcal{U}}| \gg |S_0|$ for $N_{\mathcal{U}} \gg N_0$ and $\bar{L}_{\mathcal{U}} \approx \bar{L}_0$.

- Similarly,

$$\tilde{\Gamma}_{\mathcal{U},k} = O\left(\frac{N_{\mathcal{U}}}{(\bar{L}_{\mathcal{U}})^{2(k-1)}}\right)\,, \quad \tilde{\Gamma}_{M,k} = O\left(\frac{N_{\mathcal{U}} + N_0}{(\bar{L}_M)^{2(k-1)}}\right)\,,$$

so that for $k \geq 1$ we obtain

$$|\Gamma_{\mathcal{U},k}| \approx |\Gamma_{M,k}|\,,$$

for $N_{\mathcal{U}} \gg N_0$ and $\bar{L}_{\mathcal{U}} \approx \bar{L}_M$, which is a consequence of $\bar{L}_{\mathcal{U}} \approx \bar{L}_0$.

- This then implies

$$\Gamma_M(Q_M) \approx \Gamma_{\mathcal{U}}(Q_{\mathcal{U}})\,,$$

and

$$|\Gamma_{\mathcal{U}}(Q_{\mathcal{U}})| \gg |\Delta\Gamma_M(Q_M)|\,.$$

5 Conclusions

- One can construct the Hartle-Hawking state $\Psi_0(q)$ by using the PFQG path integral for a PL manifold $T(M_0)$.
- The WFU time evolution $\Psi(q,t)$ is determined by the PFQG path integral for the PL manifold $T(M_0 \sqcup \Sigma \times [t_i, t])$.
- The EA can be associated to the quantum state $\Psi(q,t)$ by using the generating function for the manifold $T(M) = T(M_0 \sqcup \Sigma \times [t_i, t])$.
- The QFT effective action corresponds to the generating functional for $\mathcal{U} \equiv \Sigma \times [t_i, t_f]$, where $t_i \to -\infty$ and $t_f \to +\infty$.
- The WFU corrections to the QFT EA can be taken into account by using a non-trivial HH wavefunction, and instead of using the EA for $T(M_0 \sqcup \mathcal{U} \sqcup M_0)$, it is easier to use the EA for $T(M_0 \sqcup \mathcal{U})$.
- Note that the QFT EA coefficients $\Gamma_{M,k}[g,\varphi]$ will not be the same as the usual perturbative QFT coefficients $\Gamma_{K,k}[g,\varphi]$, where $\hbar K$ is the momentum cutoff determined by the average edge length in $T(\mathcal{U})$. This is because the coefficents $\Gamma_{K,k}$ are defined on the manifold $\mathcal{U}$ where the boundary metrics are flat and the boundary fields are vanishing.
- One can write
$$\Gamma_{M,k}[g,\varphi] = \Gamma_{K,k}[g,\varphi] + \Delta\Gamma_{M,k}[g,\varphi]\,,$$
and the corrections can be calculated by using the perturbative expansions of $\Gamma_M(L,\Phi)$ and $\Gamma_{\mathcal{U}}(L,\Phi)$.
- We expect that the corrections $\Delta\Gamma_k$ will be small compared to $\Gamma_{K,k}$ when
$$N_{\mathcal{U}} \gg N_0\,, \quad \bar{L}_0 \approx \bar{L}_{\mathcal{U}}\,.$$

References

[1] A. Miković, *Effective actions for Regge state-sum models of quantum gravity*, Adv. Theor. Math. Phys. 21 (2017) 631

[2] A. Miković and M. Vojinović, *Solution to the cosmological constant problem in a Regge quantum gravity model*, Europhys. Lett. 110 (2015) 40008

[3] A. Miković, *Effective actions for Regge piecewise flat quantum gravity*, Universe 8 (2022) 268

[4] A. Miković and M. Vojinović, *State-sum Models of Piecewise Linear Quantum Gravity*, World Scientific, Singapore (2023)

[5] A. Miković, *Finiteness of quantum gravity with matter on a PL spacetime*, Class. Quant. Grav. 40 (2023) 245011, arXiv:2306.15484

[6] A. Miković, *Physical States and Transition Amplitudes in Piecewise Flat Quantum Gravity*, Int. J. Mod. Phys. D 33 (2024) 2450048

[7] A. Miković, *Generalization of Bohmian Mechanics and Quantum Gravity Effective Action*, arXiv:2505.03305