TEL AVIV UNIVERSITY אוניברסיטת תל-אביב

A review: basic fractional nonlinear-wave models and solitons

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Срећан рођендан Бранку!

עד מאה ועשרים!

(Up to one hundred and twenty!)

1. Introduction. The concept **of derivatives of fractional orders** was introduced, as a mathematical curiosity, by **Niels Henrik Abel** in **1823**:

N. H. Abel, *Oplösning af et par opgaver ved hjelp af bestemte integraler*. Magazin for Naturvidenskaberne, Aargang I, Bind 2, Christiania, 1823.

IV.

Oplösning af et Par Opgaver ved Hjelp af bestemte Integraler.

Af

N. H. Abel.

I.

Det er som bekiendt ofte Tilfældet, at man ved Hjelp af bestemte Integraler (intégrales difinies) kan oplöse mange Opgaver, som man paa anden Maade enten aldeles ikke eller dog meget vanskelig kan oplöse, og især har man anvendt dem med Held paa Oplösningen af flere vanskelige Opgaver i Mechaniken, f. Ex. om Bevægelsen af en elastisk Flade, i Bolgetheorien &c. En anden Anvendelse af disse Integraler vil jeg vise i Oplösningen af fölgende Opgave:

"Lad *CB* Tavle 1, Fig. 4, være en horizontal Linie, *A* et "givet Punkt; *AB* lodret paa *BC*, *AM* en krum Linie, "hvis retvinklede Koordinater ere $AP = x$. $PM = y$. "Endvidere være $AB = a$ og $KM = s$. Tænker man "sig at et Legeme gjennemlober Buen *CA* med en

In that work, Abel had introduced a fractional-order derivative of function ***f*(*t*)**, which, in the modern literature, is usually called the ***Caputo derivative***, that was reintroduced in **1967**:

M. Caputo, Linear models of dissipation whose *Q* is almost frequency independent – II. *Geophysical Journal of the Royal Astronomical Society*, **13**, Issue 5 (1967), cited ca. **2,600** times.

Book: **Michele Caputo**, *Elasticitá e Dissipazione*. Zanichelli, Bologna, 1969.

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-1-\alpha} f^{(n)}(\tau)\, d\tau,$$

where the **integer part** of α, $n \equiv [\alpha]$, is an integer closest to α, such that $n-1 < \alpha < n$.

In physics, the concept of fractal derivatives was introduced by **Nikolai Laskin** (University of Toronto, Canada) in **2000**, in the context of ***fractional quantum mechanics***:

N. Laskin, Fractional quantum mechanics and Lévy path integrals, Phys. Lett. A **268**, 298-305 (2000) (cited about **1,500 times**).

ELSEVIER

17 April 2000

PHYSICS LETTERS A

Physics Letters A 268 (2000) 298–305

www.elsevier.nl/locate/physleta

Fractional quantum mechanics and Lévy path integrals

Nikolai Laskin

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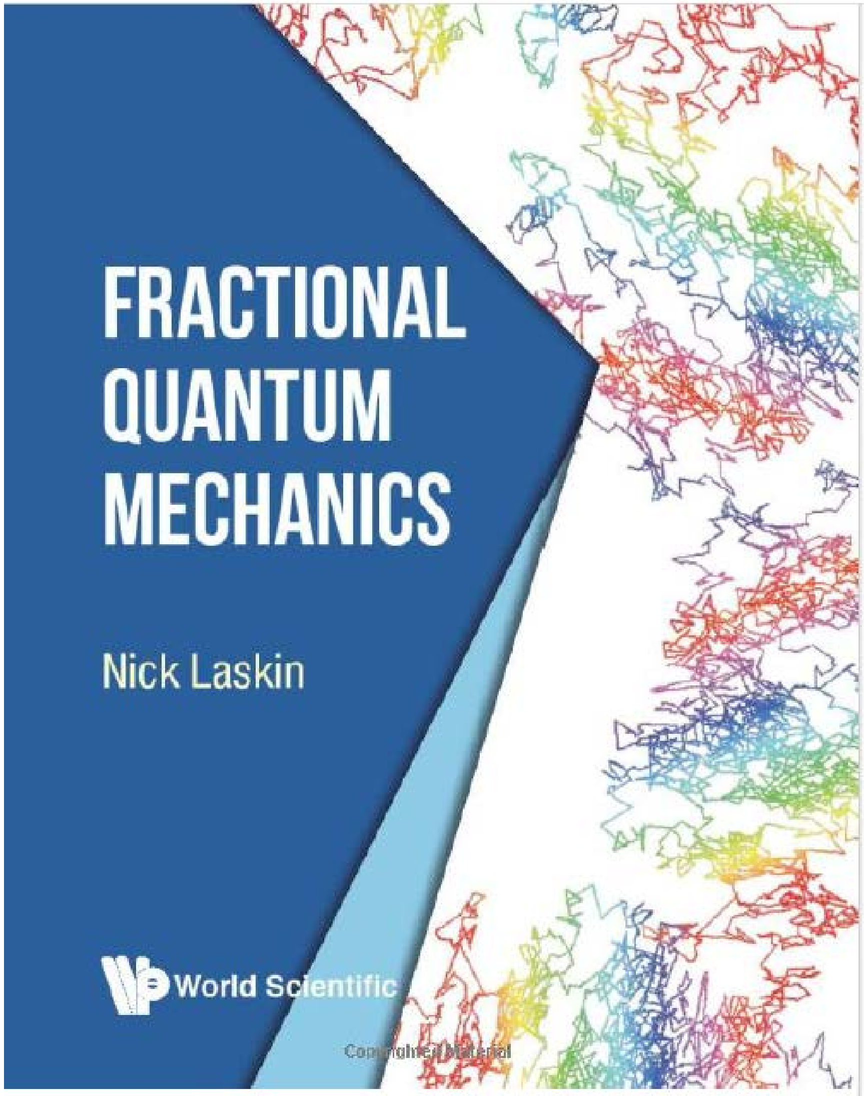
Communicated by A.R. Bishop

A book by the same author:

N. Laskin, *Fractional Quantum Mechanics*

(World Scientific, Singapore, 2018)



In these works, the **fractal Schrödinger equation** was derived, by means of the **Feynman's integrals**, alias **path integrals** (~ $\int$**exp(i*S*)d(path)**), for a quantum particle whose classical stochastic motion, with **action *S***, does not follow the usual **Brownian law**, but proceeds through **random jumps** (***Lévy flights***).

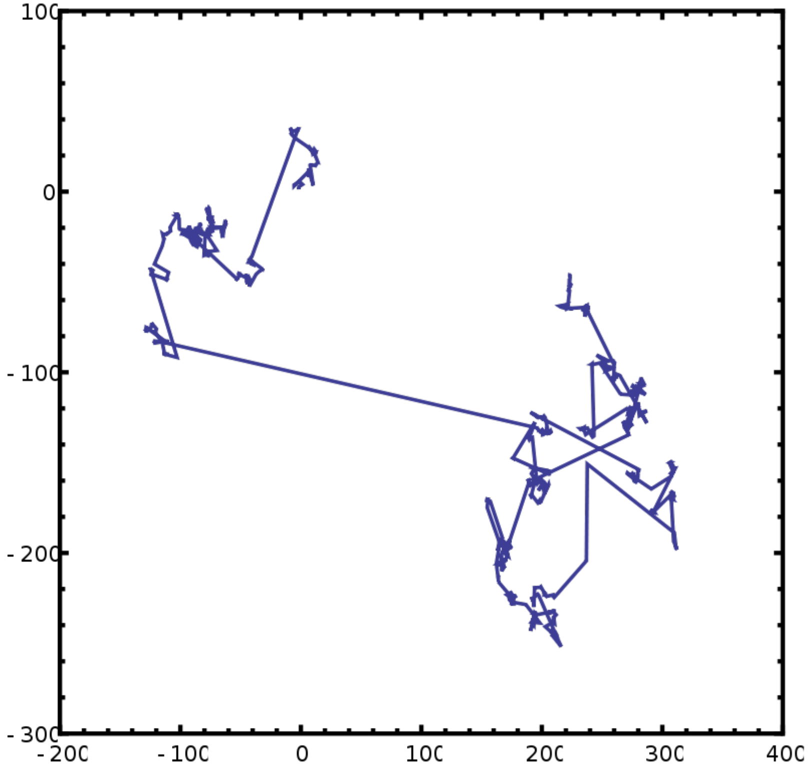
The average distance from the initial position of a classical particle moving by Lévy flights (along axis ***x***) grows with time as

$$\langle|x|\rangle \sim t^{1/\alpha}, \text{where } \alpha \leq 2 \text{ is called the } \textbf{Levy index}.$$

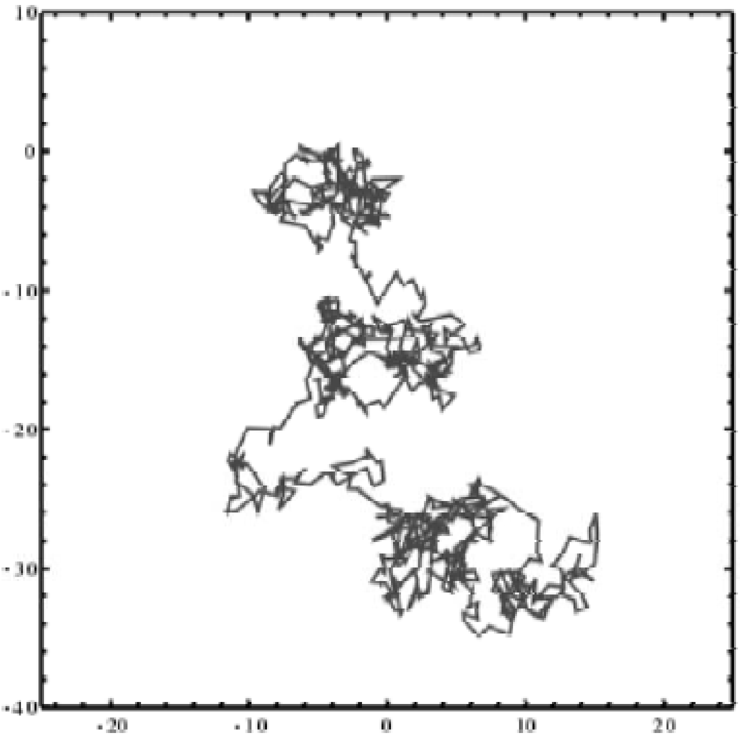
That is, in the case of **α < 2**, the stochastic motion of the Lévy particle (at ***t*** $\rightarrow$**∞**) is ***faster*** than the classical **random (Brownian) walk**, which corresponds to **α = 2**, i.e., **$\langle x^2 \rangle \sim t$** .

A typical example of the trajectory built of **1000** **stochastic Lévy flights** of a particle, corresponding to ***α* = 1** in two dimensions [**e.g., a shark in the search of food in the ocean** (even if a shark can scarcely be considered as a prototype of a quantum particle); the picture is borrowed from Wikipedia]. For comparison, a trajectory built of **1000** random steps of the usual Brownian particle (***α* = 2**) is shown too (right) (note the **difference in the spatial scales**):

***α* = 1 (Lévy flights)**



***α* = 2 (Brownian motion)**



The Schrödinger equation derived by Laskin for the quantum particle moving by the Lévy flights, written in a scaled form, is

$$i\frac{\partial\psi}{\partial t}=\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\psi+U(x)\psi,$$

where $U(x)$ is an external potential, and the kinetic-energy operator, $\left(-\partial^2/\partial x^2\right)^{\alpha/2}$, is represented by the *Riesz derivative* (named after Marcel Riesz), which is defined as follows: take the Fourier transform of ψ, with wavenumber k; in the Fourier space, the action of operator $\left(-\partial^2/\partial x^2\right)^{\alpha/2}$ amounts to the multiplication by $|k|^{\alpha}$; after that, return from the Fourier space back to the coordinate space, applying the inverse Fourier transform.

Thus, the fractional differential operator, which represents the kinetic energy in the one-dimensional version of **fractional quantum mechanics**, is actually an integral operator, generated by the juxtaposition of the direct and inverse Fourier transforms:

$$\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\psi=\int_{-\infty}^{+\infty}dk\,|k|^{\alpha}\int_{-\infty}^{+\infty}d\xi e^{ik(x-\xi)}\psi(\xi).$$

Similarly, the kinetic-energy operator appearing in the Schroedinger equation for the two-dimensional quantum Levy particle takes the following integral form:

$$\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)^{\alpha/2}\psi=\iint dkdq\left(p^2+q^2\right)^{\alpha/2}\iint d\xi d\eta e^{ik(x-\xi)+iq(y-\eta)}\psi\left(\xi,\eta\right).$$

The structure of the talk

2. A proposal to **emulate** the fractional Schrödinger equation **in optics**.

3. **Experimental realization** of the temporal fractional group-velocity dispersion in fiber optics.

4. Adding **nonlinearity** to fractional systems.

5. **Experimental realization** of bright solitons in fiber optics.

6. **Domain walls** in a two-component fractional system with self-defocusing nonlinearity

7. Two-component solitons in the fractional system with the **second-harmonic generation.**

8. Conclusion.

2. A proposal to *emulate* the fractional Schrödinger equation in optics

The fractional quantum mechanics has not been, as yet, realized experimentally. Making use of the commonly known fact that the quantum-mechanical Schrödinger equation is tantamount to the classical equation for the paraxial propagation of light, it was proposed to ***emulate*** the fractional Schrödinger equation in optical cavities (this paper was cited ca. **300 times**):

March 15, 2015 / Vol. 40, No. 6 / OPTICS LETTERS 1117

Fractional Schrödinger equation in optics

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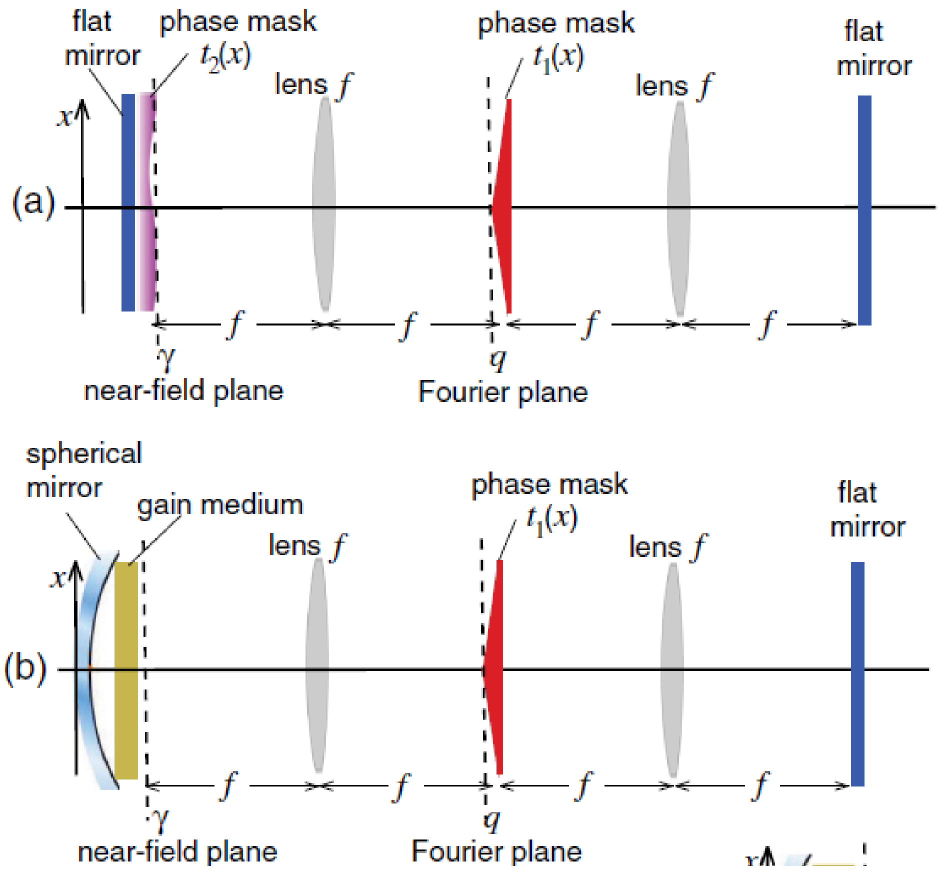
posted February 10, 2015 (Doc. ID 231827); published March 13, 2015

The proposal aimed to emulate ***the fractional diffraction*** in an optical **4*f* setup**. The transverse structure of a spatial light beam is converted into the ***Fourier form*** by a lens, then an appropriately designed ***phase mask*** adds ***phase shifts*** to different ***spatially separated*** Fourier components. The phase shifts are ***the same*** as would be produced by the **fractional Riesz derivative**. Finally, another lens casts the optical field back into the form of a parallel-propagating beam (the bottom scheme realizes the fractional Schrödinger equation including the harmonic-oscillator trapping potential, $\boldsymbol{U(x) = \mathrm{const}\cdot x^2}$:



Circulation of light in this optical cavity is governed by the effective averaged fractional Schrödinger equation, which ***emulates*** the corresponding equation in quantum mechanics:

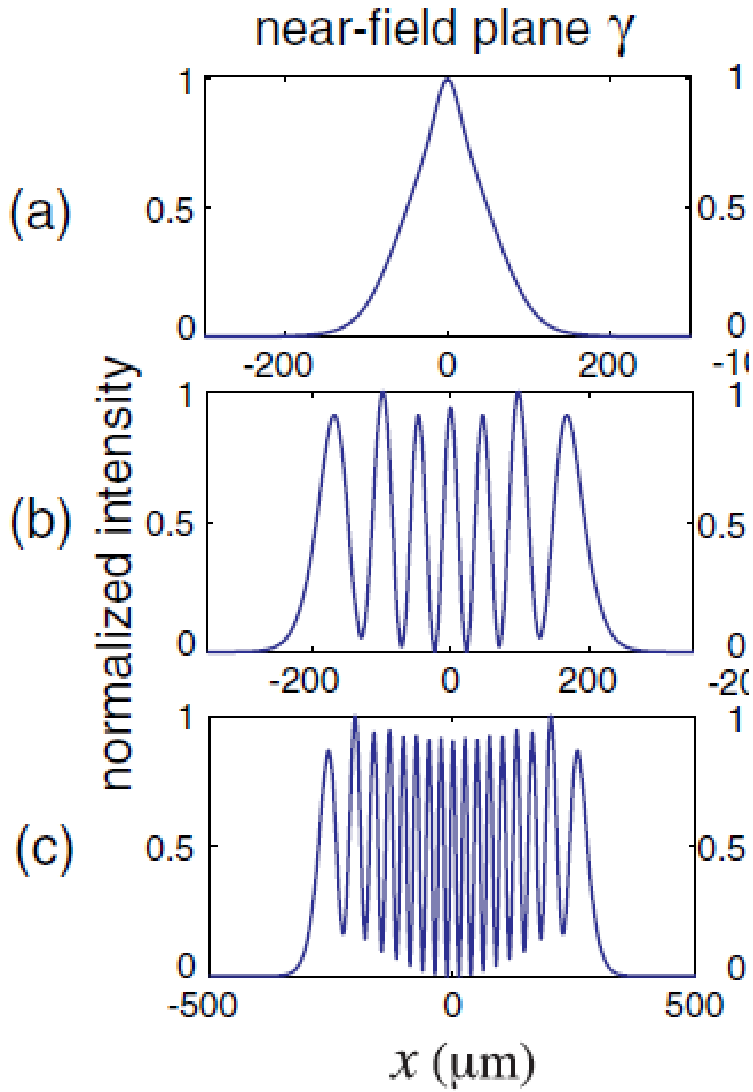
$$i\frac{\partial\psi}{\partial z}=\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\psi+U(x)\psi,$$

where $U(x)$ is an effective potential, and z is the propagation distance instead of time in quantum mechanics.

The two-dimensional fractional Schroedinger equation may be realized in this setting as well, in the form of

$$i\frac{\partial\psi}{\partial z}=\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)^{\alpha/2}\psi+U(x,y)\psi.$$

Examples: the ground and excited **eigenstates** produced by the fractional **one-dimensional** Schrödinger equation with Lévy index **α = 1,** including the harmonic-oscillator trapping potential, **$U(x)$ = const·x^2**:



3. Experimental realization of the temporal fractional group-velocity dispersion (instead of the spatial diffraction) in fiber optics

The **cardinal problem** is the absence of any previously reported experimental realization of the fractional diffraction in linear or nonlinear optics (experimental realization of the fractional Schrödinger equation in quantum mechanics was not reported either).

An experimental realization of ***fractional dispersion*** (in the ***temporal domain***, rather than fractional diffraction in the spatial domain) has been reported, using a fiber-laser cavity.

Nature Communications **14**, 222 (2023)
(for the time being, cited **62 times**)

nature communications

Article https://doi.org/10.1038/s41467-023-35892-8

Experimental realisations of the fractional Schrödinger equation in the temporal domain

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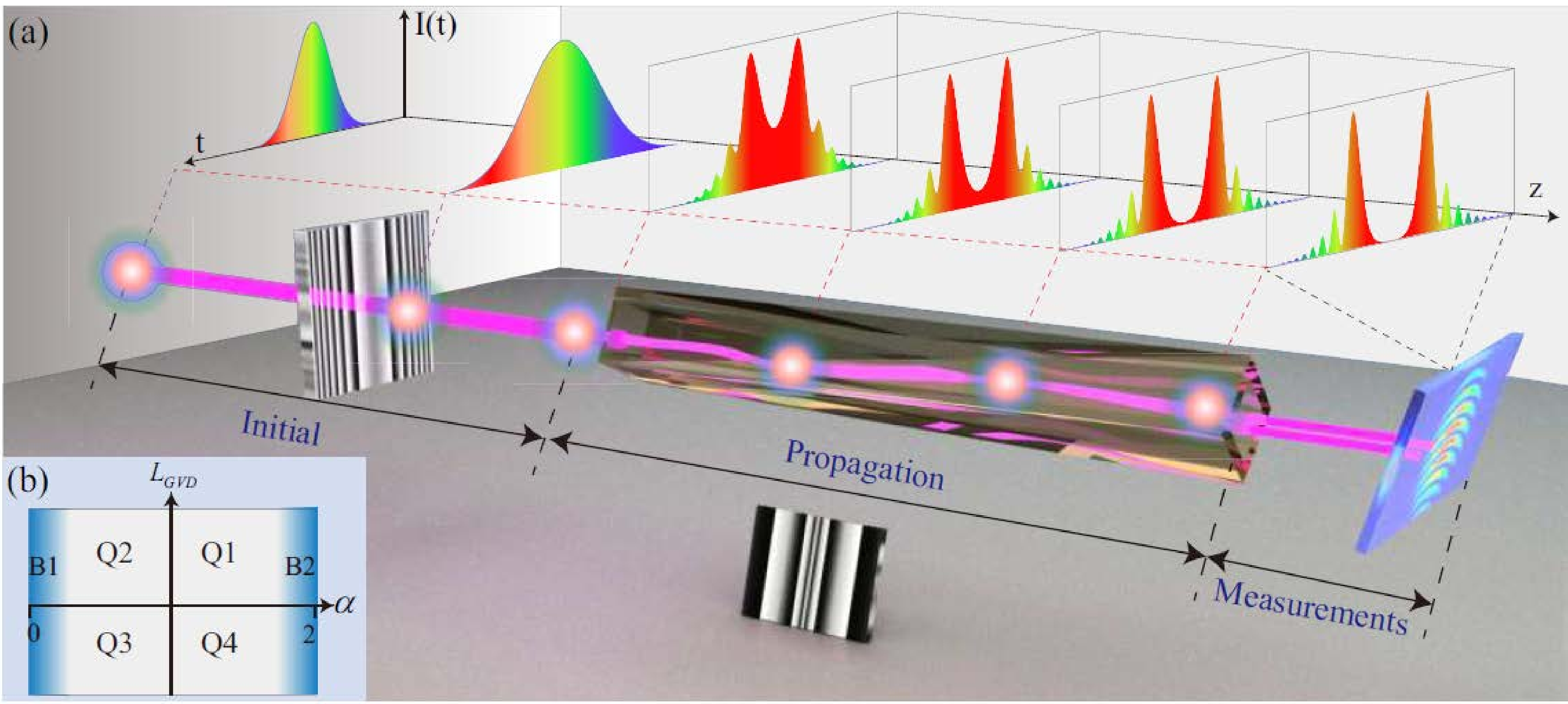
Accepted: 6 January 2023

Published online: 14 January 2023

Shilong Liu[1,2], Yingwen Zhang[1,3], Boris A. Malomed[4,5] & Ebrahim Karimi[1,3]

The **main principle** is to split the temporal wave packet into its spectral components, and pass the light signal with the **spatially separated** spectral components through a **phase mask, realized as a hologram,** which imparts a **particular phase shift** to each component, so as to **emulate** the action of the **fractional GVD** (in the combination with the **regular GVD**) onto the original wave packet. With the Lévy index α, the phase shift emulating the action of the fractional GVD onto a spectral component with frequency ω should be **const·**$|\omega|^{\alpha}$.

The schematic setup used in the experiment (the central hologram, which plays the role of the phase mask **emulating the fractional GVD,** is shown in the rotated form, to display its intrinsic structure). The operation of the setup is controlled by the emulated **Lévy index**, $\boldsymbol{\alpha}$, and regular-**GVD** length, $\boldsymbol{L_{GVD}}$.



The theoretical model

The propagation of light in the fiber laser may be affected by the action of both the **fractional** and usual (non-fractional) **GVD**:

$$i\frac{\partial\psi}{\partial z}=\frac{D}{2}\left(-\frac{\partial^2}{\partial\tau^2}\right)^{\alpha/2}\psi-\sum_{k=2,3,\ldots}\frac{\beta_k}{k\,!}\left(i\frac{\partial}{\partial\tau}\right)^k\psi,$$

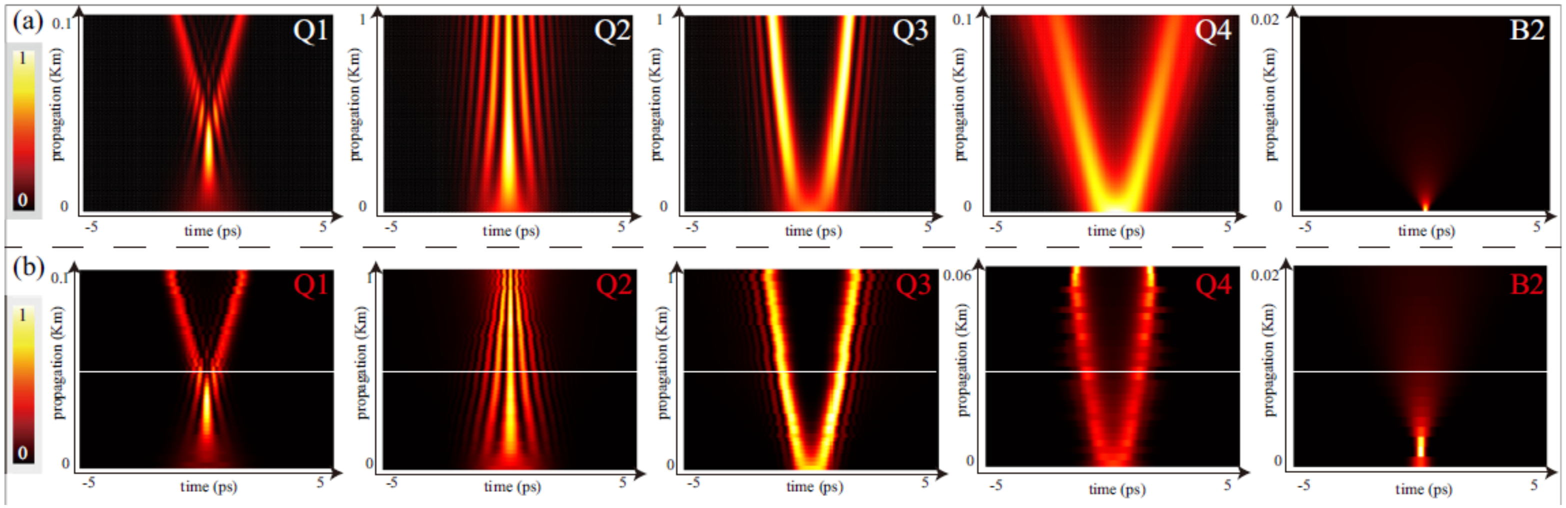
where z is again the propagation distance (along the fiber), while the *temporal coordinate* is $\tau=t\text{-}z/V_{gr}$. Further, D is an effective coefficient of the fractional dispersion with Levy index α, and β_k are coefficients of the regular (usual) dispersion of **integer** k-th orders.

Note that **no nonlinearity** is included here.

Basic experimental results and the corresponding simulations

Row (a): simulations; **row (b)**: experiment.

α = 1.25	**α = 0.25**	**α = 0.25**	**α = 1.25**	**close to**
L_{GVD} = 5	**L_{GVD} = 5**	**L_{GVD} = -5**	**L_{GVD} = -5**	**α = 2**



4. The interplay of the fractional diffraction and nonlinearity

Because the optical medium naturally includes the **Kerr nonlinearity (self-focusing),** the corresponding cubic term may be added to the fractional Schrödinger equation:

$$i\frac{\partial\psi}{\partial z}=\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\psi+U(x)\psi-\gamma|\psi|^2\psi,$$

where γ is the nonlinearity coefficient. In the case of $\gamma>0$ (self-attraction), this 1D equation gives rise to the **critical collapse** (catastrophic self-compression of the wave field) at $\alpha = 1$, and **supercritical collapse** at $\alpha < 1$. **Stable solutions** are possible at $1<\alpha \leq 2$.

The **two-dimensional version** of the nonlinear fractional Schroedinger equation:

$$i\frac{\partial\psi}{\partial z}=\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)^{\alpha/2}\psi+U(x,y)\psi-\gamma|\psi|^2\psi.$$

The two-dimensional equation gives rise to the **supercritical collapse** at $\alpha < 2$.

The nonlinear equations produce various one- and two-dimensional modes supported by the self-focusing (or defocusing) of light, such as bright and dark solitons, fronts, vortices, etc. Such modes were considered in many theoretical works. **A brief review**:
Photonics **8**, 353 (2021) (for the time being, cited ca. **150 times**):

photonics MDPI

Review

Optical Solitons and Vortices in Fractional Media: A Mini-Review of Recent Results

Boris A. Malomed [1,2]

An updated recent review:

Chaos REVIEW pubs.aip.org/aip/cha

Basic fractional nonlinear-wave models and solitons

Cite as: Chaos **34**, 022102 (2024); doi: 10.1063/5.0190039
Submitted: 2 December 2023 · Accepted: 9 January 2024 ·
Published Online: 7 February 2024

View Online Export Citation CrossMark

Boris A. Malomed[a)]

5. A recent result: experimental realization of solitons in fiber optics

Laser & Photonics Reviews **2025**, 2401714 (2025).

RESEARCH ARTICLE

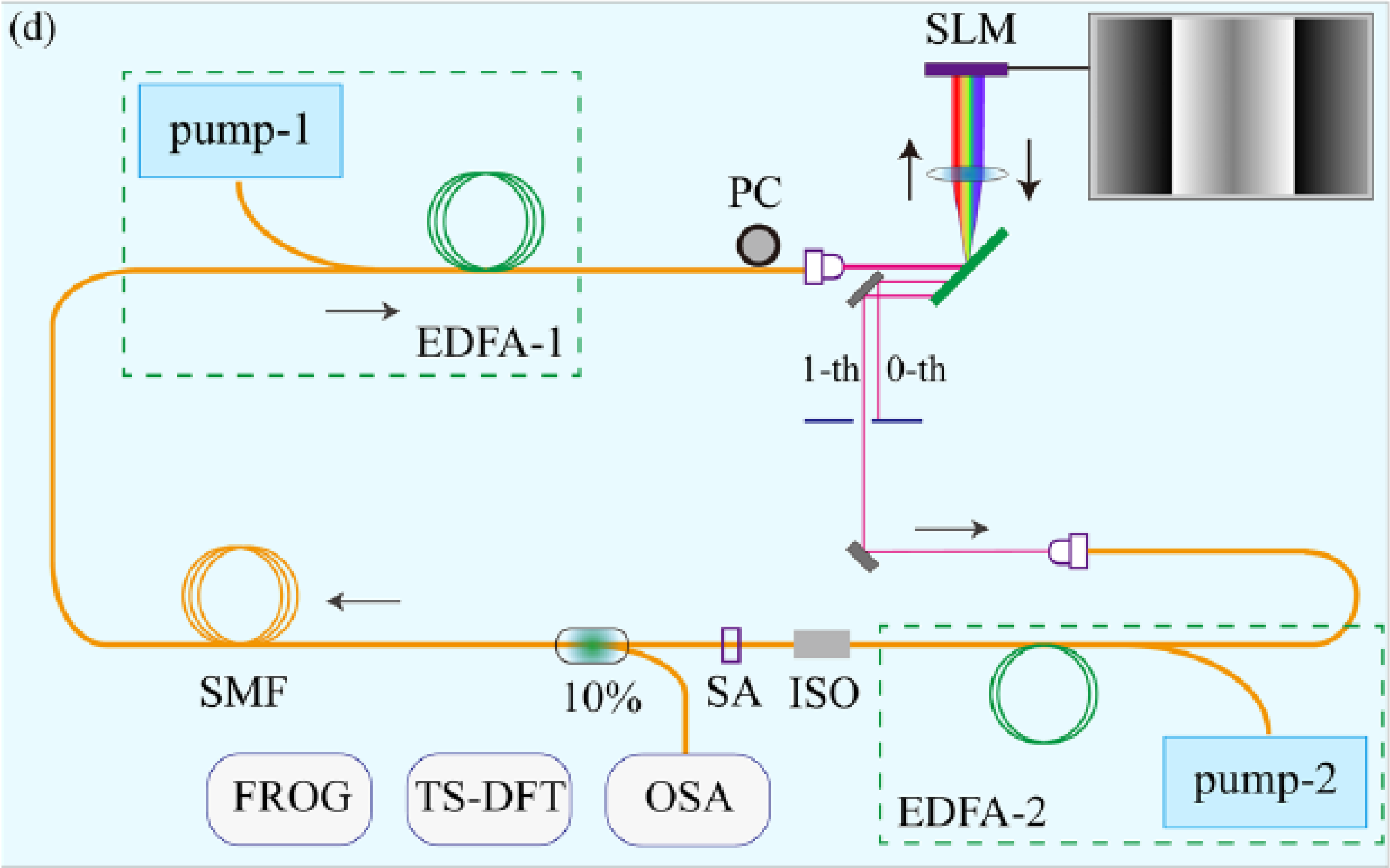
LASER & PHOTONICS REVIEWS

www.lpr-journal.org

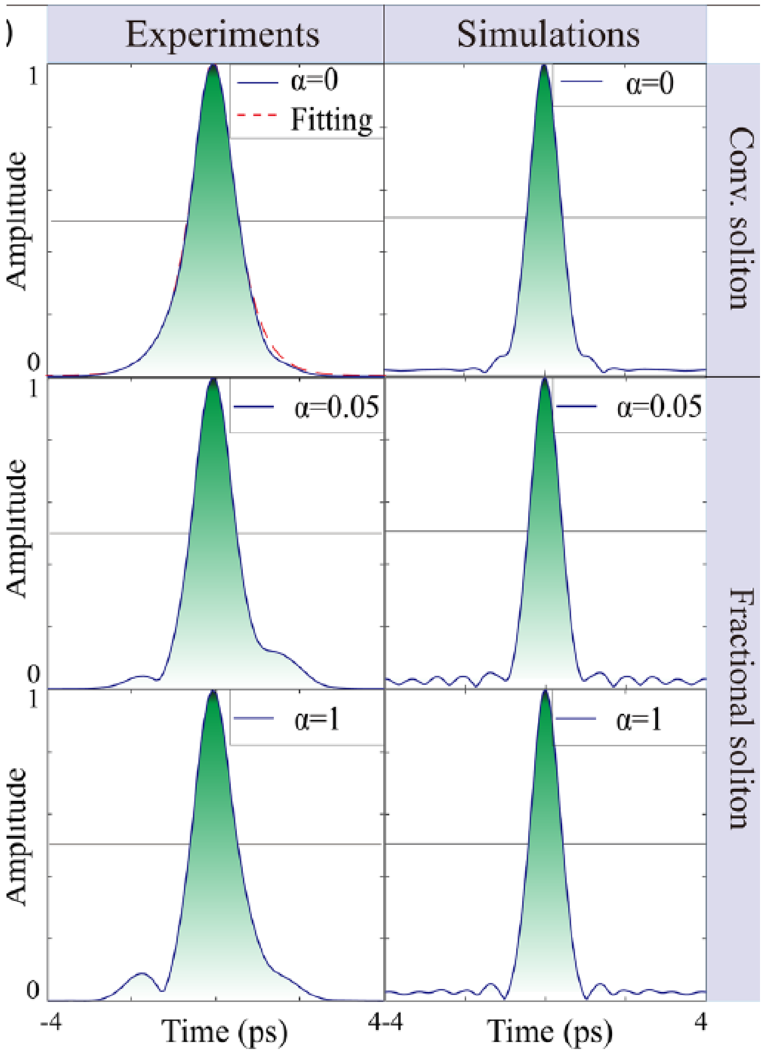
Experimental Emulator of Pulse Dynamics in Fractional Nonlinear Schrödinger Equation

Shilong Liu, Yingwen Zhang, Stéphane Virally, Ebrahim Karimi, Boris A. Malomed, and Denis V. Seletskiy**

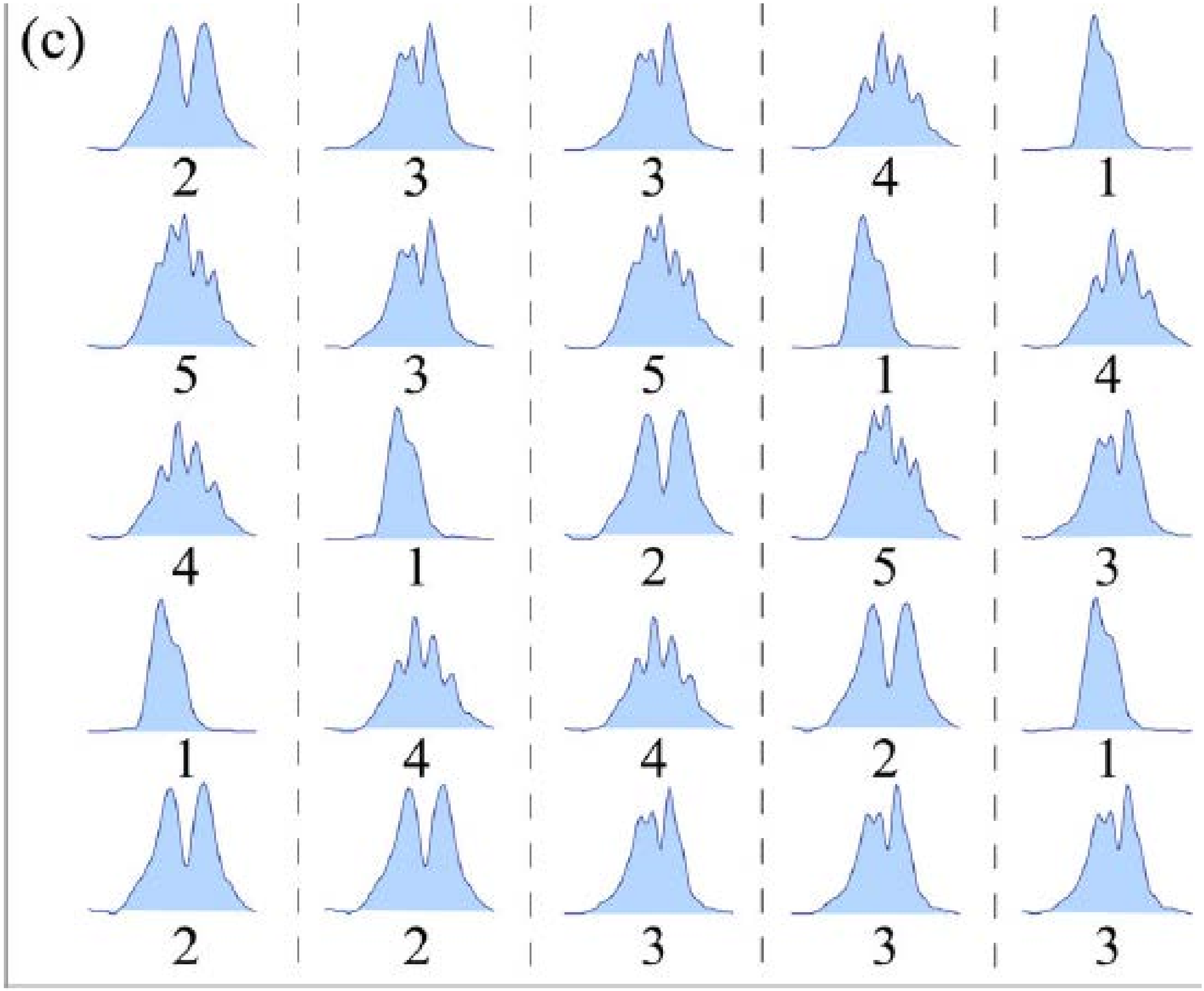
The setup used in the experimental realization of the nonlinear fiber-laser cavity:



Examples of the temporal profiles of the experimentally created and numerically simulated solitons in the fractional fiber-laser cavity:



Examples of multi-soliton complexes that may be used to encode datasets in the system:



6. A relatively simple example of theoretically elaborated nonlinear states in the fractional medium:

a one-dimensional **domain wall** separating two **immiscible** (mutually repelling) wave fields, produced by a system of coupled **fractional nonlinear Schrödinger** (**FNLS**) equations with the self- and cross-**defocusing** nonlinearities.

PHYSICAL REVIEW E **106**, 054207 (2022)

Domain walls in fractional media

Shatrughna Kumar,[1] Pengfei Li,[2,3] and Boris A. Malomed[1,4]
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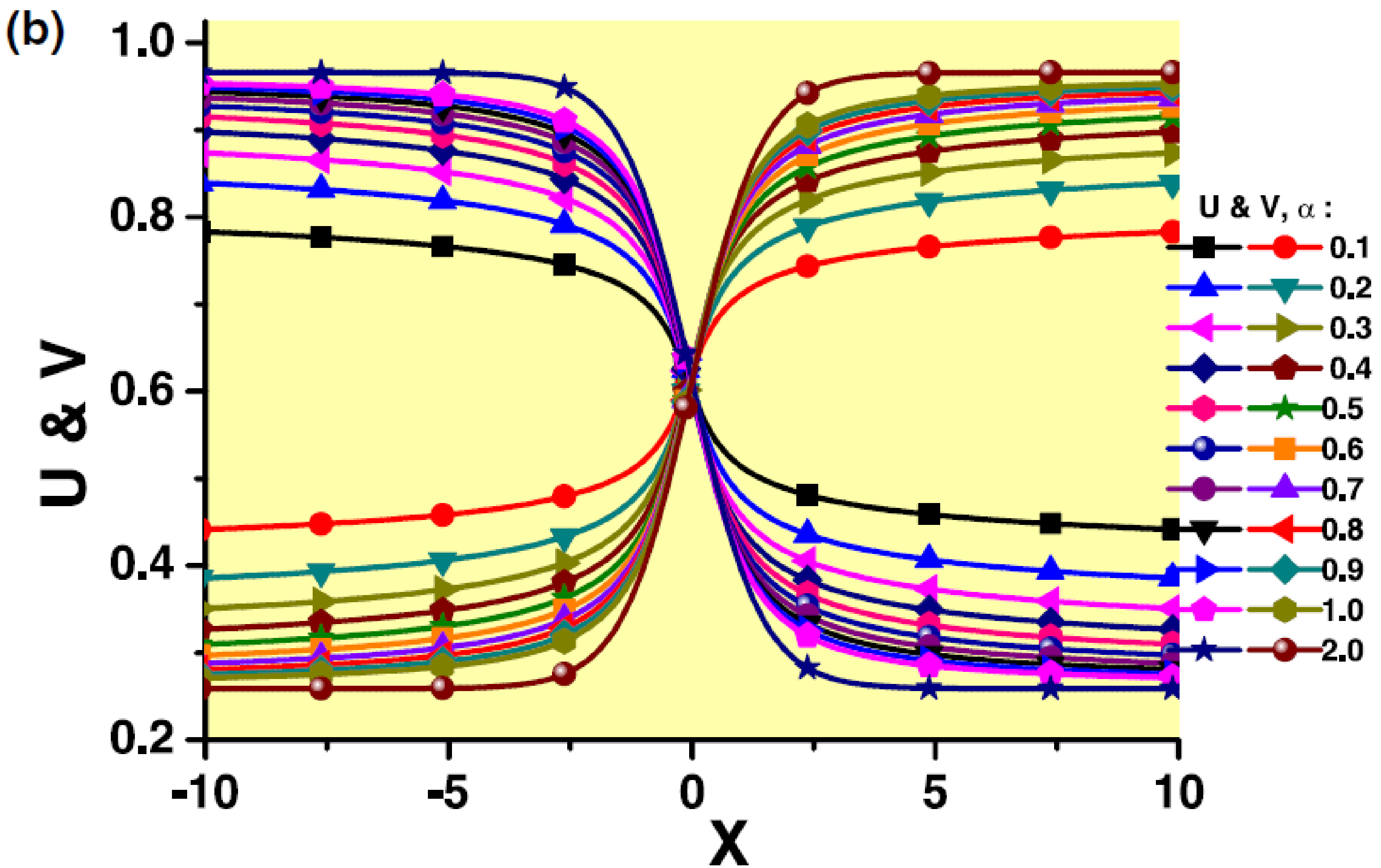
[2]*Department of Physics, Taiyuan Normal University, Jinzhong 030619, China*
[3]*Institute of Computational and Applied Physics, Taiyuan Normal University, Jinzhong 030619, China*
[4]*Instituto de Alta Investigación, Universidad de Tarapacá, Casilla 7D, Arica, Chile*

The system of coupled **FNLS** equations [***α*** is again the Lévy index, the **immiscibility** condition is ***β* > 1** (***β*** is the relative **cross-phase-modulation** (**XPM**) coefficient), and ***λ*** represents possible linear mixing between the fields]:

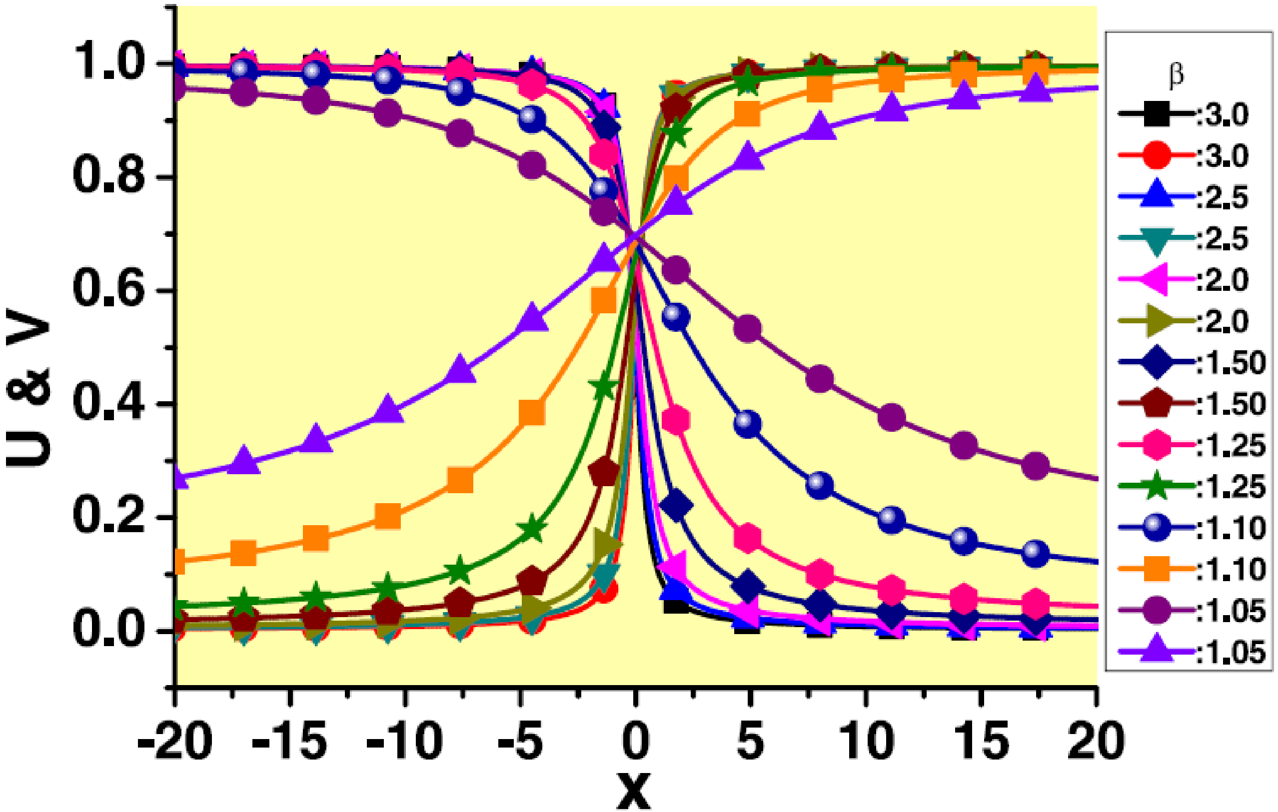
$$i\frac{\partial u}{\partial z} = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} u + (|u|^2 + \beta|v|^2)u - \lambda v,$$

$$i\frac{\partial v}{\partial z} = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} v + (|v|^2 + \beta|u|^2)v - \lambda u,$$

Stable **domain-wall patterns** produced by the coupled **FNLS** equations for ***β* = 3**, ***λ* = 0.5**, and Lévy indices between ***α* = 0.1** and ***α* = 1:**



Stable **domain-wall patterns** produced by the coupled **FNLS** equations for $\boldsymbol{\alpha = 1}$, $\boldsymbol{\lambda = 0}$, and the **XPM** coefficient taking values between $\boldsymbol{\beta = 1.05}$ and $\boldsymbol{\beta = 3}$:



7. Two-component solitons produced by the fractional second-harmonic-generation system

The **1D fractional** system for the amplitudes of the **fundamental-frequency** (**FF**) and **second-harmonic** (**SH**) fields with the fractional diffraction and quadratic nonlinearity (real ***Q*** is the **mismatch parameter, *** stands for the complex conjugation):

$$i\frac{\partial \Psi_1}{\partial z} - D_1\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\Psi_1 + \Psi_1^*\Psi_2 = 0,$$

$$2i\frac{\partial \Psi_2}{\partial z} - D_2\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\Psi_2 + Q\Psi_2 + \frac{1}{2}\Psi_1^2 = 0,$$

Stationary solutions to Eqs. (1) and (2) with FF and SH propagation constants β_1 and $\beta_2 \equiv 2\beta_1$ are looked for as

$$\Psi_1(x,z) = e^{i\beta_1 z}\psi_1(x), \Psi_2(x,z) = e^{2i\beta_1 z}\psi_2(x), \tag{5}$$

The system was introduced and analyzed in

Chaos, Solitons and Fractals 173 (2023) 113701

Contents lists available at ScienceDirect

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Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

Second-harmonic generation in the system with fractional diffraction

Pengfei Li [a,*], Hidetsugu Sakaguchi [b], Liangwei Zeng [c], Xing Zhu [c], Dumitru Mihalache [d], Boris A. Malomed [e,f]

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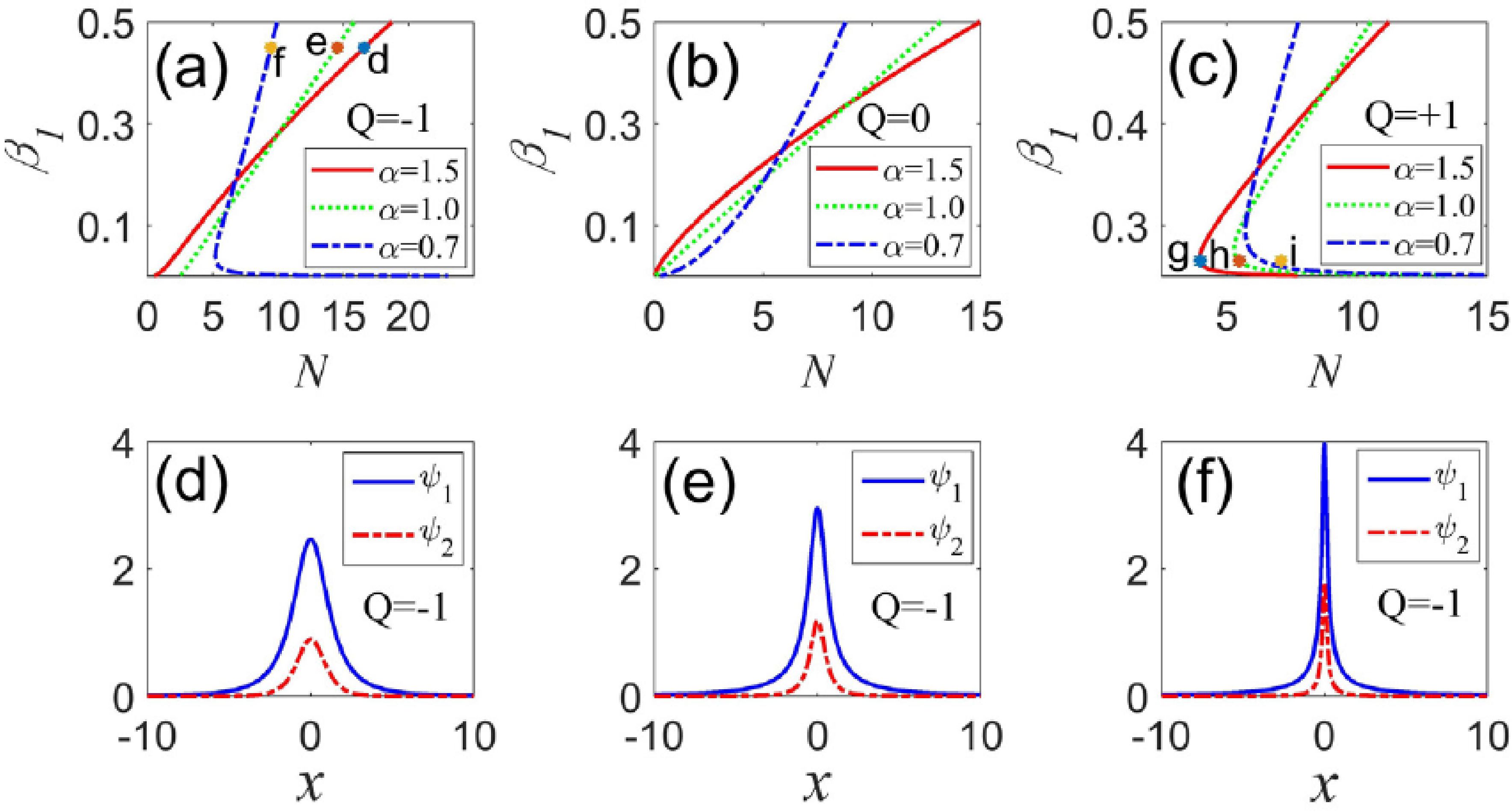
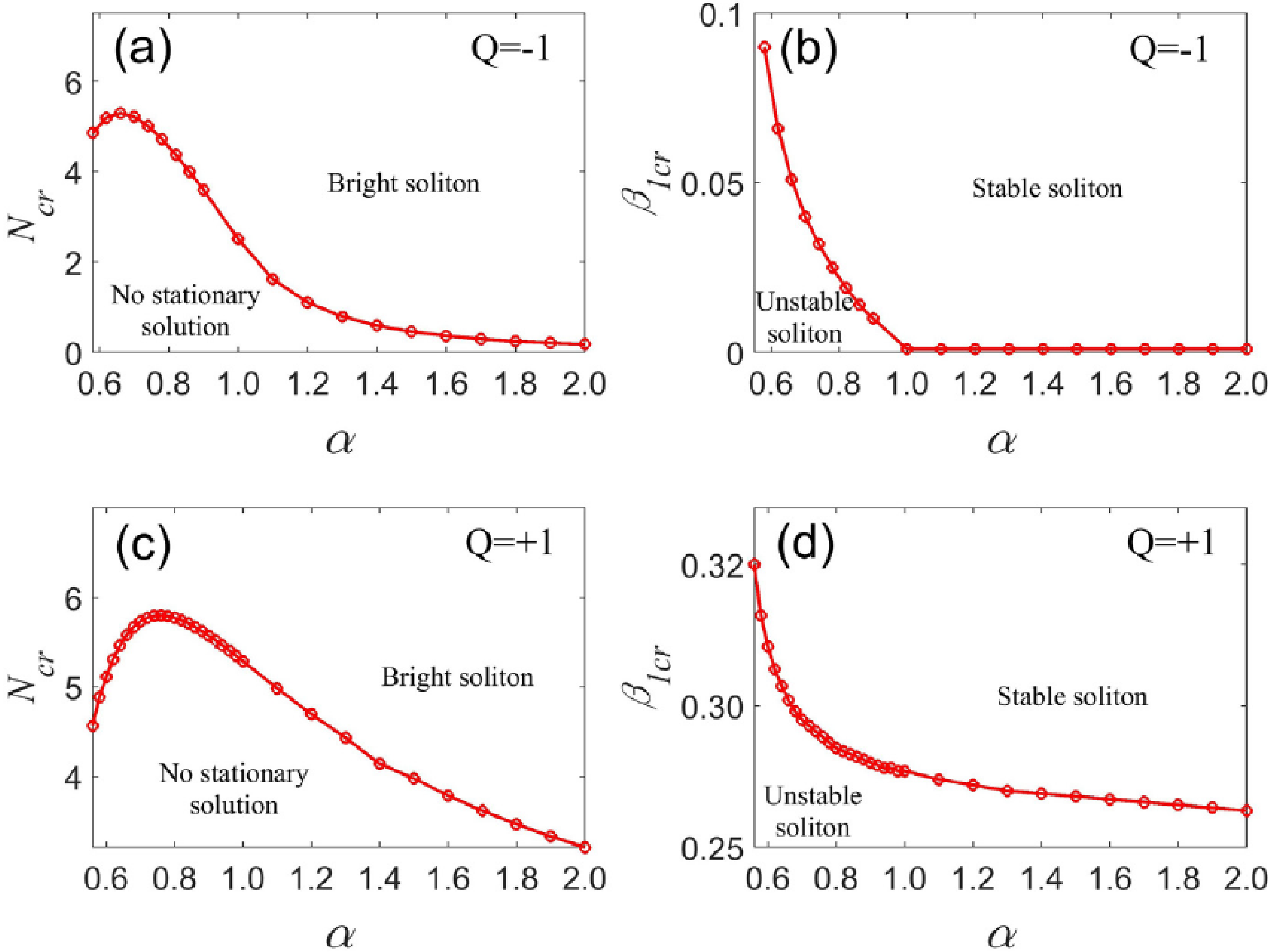
[b] *Interdisciplinary Graduate School of Engineering Sciences, Kyushu University, Kasuga, Fukuoka 816-8580, Japan*
[c] *Department of Basic Course, Guangzhou Maritime University, Guangzhou 510725, China*
[d] *Horia Hulubei National Institute of Physics and Nuclear Engineering, Magurele, Bucharest, RO-077125, Romania*
[e] *Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, and Center for Light-Matter Interaction, Tel Aviv University, Tel Aviv 69978, Israel*
[f] *Instituto de Alta Investigación, Universidad de Tarapacá, Casilla 7D, Arica, Chile*

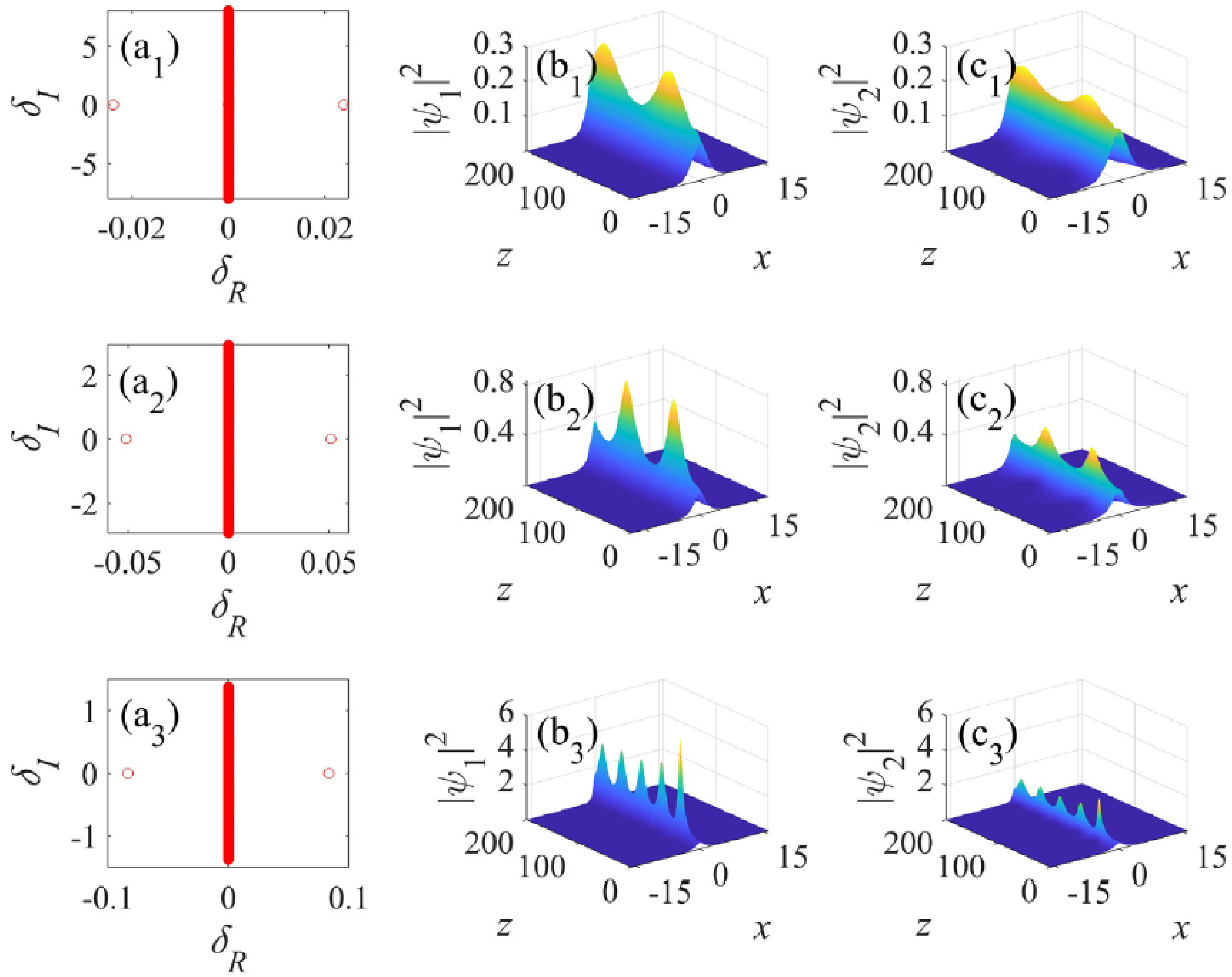
Families of soliton solutions for normalized values of the mismatch, ***Q* = -1, 0, +1**, and examples of stable solitons:



In the case of the **quadratic nonlinearity,** the **1D** system is ***free of the collapse*** in the interval of Lévy indices **0.5 < *α* ≤ 2**. ***Existence and stability*** areas for the solitons:



Examples of unstable solitons for mismatch ***Q* = 1**:



:tra (a_1, a_2, a_3) and perturbed evolution of FF and SH components (b_1, b_2, b_3) and (c_1, c_2, c_3) for VK-unstable solitons with : $\beta_1 = 0.265$. Top, middle, and bottom rows correspond, respectively, to different LI values, *viz.*, $\alpha = 1.5$, $\alpha = 1.0$, and $\alpha = 0.7$.

8. Conclusion

The concept of fractional diffraction was introduced in physics by the Laskin's ***fractional quantum mechanics*** for particles which move, at the classical level, by ***Lévy flights***.

Experimental realization of fractional quantum mechanics was not reported as yet. It was proposed by Longhi to ***emulate*** the fractional quantum mechanics by the light propagation in an optical cavity, implementing the effect of the fractional diffraction by means of ***specific phase shifts*** imparted to separate spectral components of the optical beam.

A real experimental work, using a similar method – ***imparting specific phase shifts to spectral components*** of a temporal optical signal in a fiber cavity – has recently reported ***the first realization of the effective fractional group-velocity dispersion***. This was followed by the ***experimental creation of stable temporal solitons*** in the fiber-optical cavity.

Theoretically, many works have addressed ***dynamics of solitons*** and other self-trapped modes in the framework of the ***fractional nonlinear Schrödinger equation***. In particular, an attempt was made to introduce a nonlinear fractional Gross-Pitaevskii equation for a ***condensate of particles moving, at the classical level, by means of the Lévy flights***.

The remaining challenge to the experiment is realization of the effective ***fractional diffraction*** in the ***spatial domain***, i.e., for planar or bulk waveguides (linear or nonlinear), similar to the recently reported realization of the fractional group-velocity dispersion in optical fibers.

Finally, ***the most challenging objective*** may be the creation of a ***combination*** of fractional dispersion and diffraction for ***spatiotemporal optical pulses***.

Хвала вам на интересовању!

Thank you for your interest!

Copies of this presentation, and/or of articles mentioned in it, can be requested from **malomed@tauex.tau.ac.il**