MAGNETIC FLOWS ON SPHERES

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Lagrangian systems with gyroscopic forces

Let (Q, G) be a Riemannian manifold, and let is given a Lagrangian system (Q, L_1), where $\mathcal{D}$ a nonintegrable distributon of the tangent bundle TQ, and L_1 is Lagrangian

$$L_1(q, \dot{q}) = \frac{1}{2}(\mathrm{G}(\dot{q}), \dot{q}) + (\mathrm{A}, \dot{q}) - V(q),$$

where A is a one-form on Q.
A path $q(t)$ is a *motion of the natural mechanical system* (Q, L_1) if it satisfies the Lagrange-d'Alembert equations

$$\delta L_1 = \big(\frac{\partial L_1}{\partial q} - \frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}}, \delta q\big) = 0, \quad \text{for all} \quad \delta q \in T_q Q.$$

This is equivalent to

$$\delta L = (\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}, \delta q) = \mathsf{F}(\dot{q}, \delta q), \quad \text{for all} \quad \delta q \in T_q Q,$$

where L is the part of the Lagrangian L_1 without the term linear in velocities

$$L(q, \dot{q}) = \frac{1}{2}(\mathsf{G}(\dot{q}), \dot{q}) - V(q).$$

Here the additional force $\mathsf{F}(\dot{q}, \delta q)$ is defined as the exact two-form

$$\mathsf{F} = d\mathsf{A}.$$

One can consider a more general class of systems where an additional force is given as a two-form which is closed, but not need to be exact (magnetic force).

Hamiltonian description

Let $(q_1, \ldots, q_n)$ be local coordinates on Q in which the metric G is given by the quadratic form $\sum_{ij} g_{ij} dq_i \otimes dq_j$,

$$L(q, \dot{q}) = \frac{1}{2} \sum g_{ij} \dot{q}_i \dot{q}_j - V(x),$$
$$\mathsf{F} = \sum_{i<j} f_{ij} dq_i \wedge dq_j.$$

We also introduce the Hamiltonian function

$$H(q, p) = \frac{1}{2}(p, \mathsf{G}^{-1}(p)) + V(q) = \frac{1}{2} \sum G^{ij} p_i p_j + V(q),$$

as the usual Legendre transformation of L. Here $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ are the canonical coordinates of the cotangent bundle T^*Q,

$$p_i = \partial L / \partial \dot{q}_i = \sum_j g_{ij} \dot{q}_j,$$

and $\{g^{ij}\}$ is the inverse of the metric matrix $\{g_{ij}\}$.

In canonical coordinates the equations take the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \sum_{j=1}^{n} g^{ij} p_j, \tag{1}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + \sum_{j=1}^{n} f_{ij}(q) \frac{\partial H}{\partial p_j}. \tag{2}$$

Let $z = (q, p)$. The reduced equations (1), (2) on the cotangent bundle T^*Q can be written in the Hamiltonian form

$$\dot{z} = X_H, \qquad i_{X_H}(\Omega + \mathsf{F}) = -dH, \tag{3}$$

where Ω is the canonical symplectic form on T^*Q:

$$\Omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n \tag{4}$$

The Kirillov-Konstant symplectic form on adjoint orbits.

Consider the (co)adjoint action of G and the G-orbit

$$\mathcal{O}(a) = \{x = Ad_g(a) = g \cdot a \cdot g^{-1} \,|\, g \in G\}$$

through an element $a \in \mathfrak{g}$. The adjoint orbit as a homogeneous space G/G_a, where G_a is the isotropy group of a. Since G is a compact connected Lie group, G_a is also connected. We have

$$ann(a) = \{\xi \in \mathfrak{g}, \, [\xi, a] = 0\} = T_e G_a.$$

By definition, the Kirillov-Konstant symplectic form Ω_{KK} on G/G_a is a G–invariant form, given at the point $\pi(e) \in G/G_a$ by

$$\Omega_{KK}(\xi_1, \xi_2)|_{\pi(e)} = -\langle a, [\xi_1, \xi_2]\rangle, \quad \xi_1, \xi_2 \in ann(a)^{\perp} = [a, \mathfrak{g}], \quad (5)$$

where ξ_1, ξ_2 are considered as tangent vectors to the orbit at $\pi(e)$.

Magnetic flows on adjoint orbit [A. Bolsinov, B.J.]

Consider the system with the kinetic energy given by the normal metric ds_0^2 on $\mathcal{O}(a)$ and the potential function $V(x) = -\langle b, x \rangle$, i.e., with Hamiltonian

$$H(x,p) = \frac{1}{2}\langle [x,p],[x,p] \rangle - \langle b, x \rangle,$$

under the influence of the magnetic force field given by $\epsilon\Omega_{KK}$. Here,

$$T^*\mathcal{O}(a) \subset T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}(x,p).$$

Proposition

The equations of the magnetic pendulum, in redundant variables (x, p)*, are given by*

$$\dot{x} = [x,[p,x]], \tag{6}$$

$$\dot{p} = [p,[p,x]] + \epsilon[x,p] + b - \mathrm{pr}_{ann(x)}\, b, \tag{7}$$

Magnetic spherical pendulum.

As an example, consider the Lie group $SO(3)$. The Lie algebra $so(3)$ is isomorphic to the Euclidean space $\mathbb{R}^3$ with bracket operation being the standard vector product. The adjoint orbits are spheres $\langle\gamma,\gamma\rangle = const$. The phase space:

$$T^*S^2 = \{(\gamma, p) \in \mathbb{R}^6 \,|\, \phi_1 = \langle\gamma,\gamma\rangle = 1,\, \phi_2 = \langle\gamma, p\rangle = 0\}.$$

The magnetic spherical pendulum:

$$H = \frac{1}{2}\langle p, p\rangle - \langle b, \gamma\rangle,$$

$$\dot{\gamma} = p, \quad \dot{p} = \epsilon\gamma \times p + b + \mu\gamma.$$

Adding the magnetic term $\epsilon\rho^*\Omega_{KK}$ represents the influence of the magnetic monopole with the force equal to $\epsilon\gamma \times p$.
The system is completely integrable due to the linear first integral $f = \langle b, \gamma \times p + \epsilon\gamma\rangle$.

Exact magnetic field in $\mathbb{R}^n$

Standard symplectic space: $\mathbb{R}^{2n}\{\gamma, p\}$,

$$\Omega = dp_1 \wedge d\gamma_1 + \cdots + dp_n \wedge d\gamma_n$$

and the magnetic two-form F

$$\mathrm{F} = s \sum_{i<j} \kappa_{ij} d\gamma_i \wedge d\gamma_j, \qquad s \in \mathbb{R}.$$

F is exact magnetic: $\mathrm{F} = d\mathrm{A}^\Gamma$, where

$$\mathrm{F} = d\mathrm{A}^\Gamma,\ \mathrm{A}^\Gamma = \frac{s}{2} \sum_{ij} \kappa_{ij}(\gamma_i + \Gamma_i) d\gamma_j,\ \Gamma = (\Gamma_1, \dots, \Gamma_n) \in \mathbb{R}^n.$$

The magnetic Poisson brackets on $\mathbb{R}^{2n}\{\gamma, p\}$

$$\{F, G\}^\kappa = \sum_i \Big(\frac{\partial F}{\partial \gamma_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial \gamma_i}\Big) + s \sum_{i,j} \kappa_{ij} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j}.$$

Motion of a material point in a homogeneous magnetic field

The Hamiltonian:

$$H = \frac{1}{2m}\langle p, p\rangle.$$

The magnetic Hamiltonian flow

$$\begin{aligned}\dot{\gamma} =& \frac{\partial H}{\partial p} = \frac{1}{m}p,\\ \dot{p} =& -\frac{\partial H}{\partial \gamma} + s\kappa\left(\frac{\partial H}{\partial p}\right) = \frac{s}{m}\kappa p.\end{aligned}$$

For $n = 3$, we get the usual Lorentz force

$$\dot{p} = \frac{s}{m}\kappa p = \frac{s}{m}\vec{\kappa} \times p,$$

where

$$\vec{\kappa} = (k_1, k_2, k_3) \longmapsto \kappa = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix}.$$

We can assume that the magnetic form takes the form

$$\mathsf{F} = s(\kappa_{12} d\gamma_1 \wedge d\gamma_2 + \kappa_{34} d\gamma_3 \wedge d\gamma_4 + \cdots + \kappa_{2[n/2]-1,2[n/2]} d\gamma_{2[n/2]-1} \wedge d\gamma_{2[n/2]}$$

If n is even, the system decouples on $k = n/2$ magnetic systems

$$\dot{\gamma}_{2i-1} = \frac{1}{m} p_{2i-1}, \qquad \dot{p}_{2i-1} = \frac{s}{m} \kappa_{2i-1,2i} p_{2i},$$
$$\dot{\gamma}_{2i} = \frac{1}{m} p_{2i}, \qquad \dot{p}_{2i} = -\frac{s}{m} \kappa_{2i-1,2i} p_{2i-1}.$$

They are Hamiltonian equations in $\mathbb{R}^4(\gamma_{2i-1}, \gamma_{2i}, p_{2i-1}, p_{2i})$ with respect to the twisted symplectic forms

$$\Omega_{2i-1,2i} + \mathsf{F}_{2i-1,2i} = dp_{2i-1} \wedge g\gamma_{2i-1} + dp_{2i} \wedge d\gamma_{2i} + s\kappa_{2i-1,2i} d\gamma_{2i-1} \wedge d\gamma_{2i}$$

and the Hamiltonians $H_{2i-1,2i} = \frac{1}{2m}(p_{2i-1}^2 + p_{2i}^2)$.
If $n = 2k+1$ is odd, along with the k systems listed above, there is an additional system of one degree of freedom on $\mathbb{R}^2(\gamma_n, p_n)$ with the standard symplectic form and the Hamiltonian $H_n = \frac{1}{2m} p_n^2$ which generates a uniform motion: $\dot{\gamma}_n = \frac{1}{m} p_n$, $\dot{p}_n = 0$.

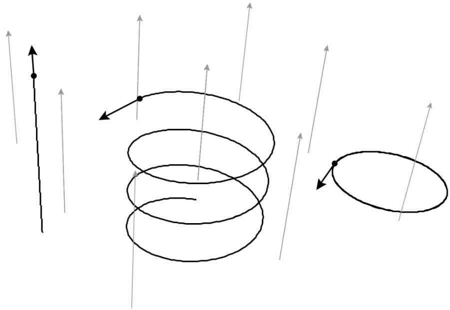

Figure: Three types of trajectories in the homogeneous magnetic field in $\mathbb{R}^3$. The gray lines represent the magnetic field.

For even n, in the case when all $|\kappa_{2i-1,2i}| \neq 0$ are proportional:

$$m_1|\kappa_{1,2}|^{-1} = \cdots = m_{n/2}|\kappa_{n-1,n}|^{-1}, \qquad m_i \in \mathbb{N},$$

such that the common divisor of m_i is 1, all the trajectories are closed with the same period

$$T = \frac{m_1 2\pi m}{|s\kappa_{1,2}|} = \cdots = \frac{m_{n/2} 2\pi m}{|s\kappa_{n-1,n}|}.$$

Gauge Noether integrals

We can take a Lagrangian with a term linear in velocities

$$L_1^\Gamma(\gamma,\dot\gamma) = \frac{m}{2}\langle\dot\gamma,\dot\gamma\rangle + \frac{s}{2}\sum_{i=1}^{[n/2]} \kappa_{2i-1,2i}\big((\gamma_{2i-1}+\Gamma_{2i-1})\dot\gamma_{2i} - (\gamma_{2i}+\Gamma_{2i})\dot\gamma_{2i-1}\big).$$

The Lagrangian L_1^Γ has the following Noether symmetries

$$\xi_i^{\Gamma_{2i-1},\Gamma_{2i}} = (\gamma_{2i-1}+\Gamma_{2i-1})\frac{\partial}{\partial\gamma_{2i}} - (\gamma_{2i}+\Gamma_{2i})\frac{\partial}{\partial\gamma_{2i-1}},$$

modulo the gauge terms

$$-\frac{d}{dt}s\frac{\kappa_{2i-1,2i}}{2}\big((\gamma_{2i-1}+\Gamma_{2i-1})^2 + (\gamma_{2i}+\Gamma_{2i})^2\big),\ i=1,\dots,[n/2].$$

The corresponding gauge Noether first integrals of motion are:

$$\langle\frac{\partial L_1^\Gamma}{\partial\dot\gamma},\xi_i^{\Gamma_{2i-1},\Gamma_{2i}}\rangle + s\frac{\kappa_{2i-1,2i}}{2}\big((\gamma_{2i-1}+\Gamma_{2i-1})^2 + (\gamma_{2i}+\Gamma_{2i})^2\big).$$

In the Hamiltonian description, they take the form

$$\Phi_{2i-1,2i}^{\Gamma_{2i-1},\Gamma_{2i}} = (\gamma_{2i-1}+\Gamma_{2i-1})p_{2i} - (\gamma_{2i}+\Gamma_{2i})p_{2i-1} + s\frac{\kappa_{2i-1,2i}}{2}\big((\gamma_{2i-1}+\Gamma_{2i-1})^2$$

Restrictions to the unit sphere

We work in redundant coordinates and consider the phase space T^*S^{n-1} as a submanifold of $\mathbb{R}^{2n}(\gamma, p)$ given by the equations

$$\phi_1 = \langle \gamma, \gamma \rangle = 1, \qquad \phi_2 = \langle p, \gamma \rangle = 0,$$

and endowed with the twisted symplectic form $\omega + \mathrm{f}$, $\omega = \Omega|_{T^*S^{n-1}}$, $\mathrm{f} = \mathrm{F}|_{T^*S^{n-1}}$. It is convenient to use the Dirac magnetic Poisson brackets on

$$\mathbb{R}^{2n}_* = \{(\gamma, p) \in \mathbb{R}^{2n} \,|\, \phi_1 \neq 0\}$$

defined by

$$\{F, G\}_d = \{F, G\}^\kappa - \frac{\{F, \phi_1\}^\kappa \{G, \phi_2\}^\kappa - \{F, \phi_2\}^\kappa \{G, \phi_1\}^\kappa}{\{\phi_1, \phi_2\}^\kappa}.$$

The constraint functions ϕ_1 and ϕ_2 are Casimir functions of the Dirac brackets. The symplectic leaf $\phi_1 = 1$, $\phi_2 = 0$ within $(\mathbb{R}^{2n}_*(\gamma, p), \{\cdot, \cdot\}_d)$ coincides with $(T^*S^{n-1}, \omega + \mathrm{f})$.

Magnetic geodesic flow

The Hamiltonian:

$$H = \frac{1}{2m}\langle p, p\rangle.$$

By taking $H^* = H - \lambda_1\phi_1 - \lambda_2\phi_2$, we obtain the magnetic Hamiltonian flow

$$\begin{aligned}\dot{\gamma} =& \frac{\partial H^*}{\partial p} = \frac{1}{m}p - \lambda_2\gamma, \\ \dot{p} =& -\frac{\partial H^*}{\partial \gamma} + s\kappa\Big(\frac{\partial H^*}{\partial p}\Big) = 2\lambda_1\gamma + \lambda_2 p + \frac{s}{m}\kappa p - s\lambda_2\kappa\gamma.\end{aligned}$$

Here, from the condition that ϕ_1 and ϕ_2 are first integrals of the flow, the Lagrange multipliers can be calculated to get

$$\lambda_1 = \frac{\frac{s}{m}\langle p, \kappa\gamma\rangle - \frac{1}{m}\langle p, p\rangle}{2\langle\gamma,\gamma\rangle}, \qquad \lambda_2 = \frac{1}{m}\frac{\langle p, \gamma\rangle}{\langle\gamma,\gamma\rangle}.$$

From now on, we consider a basis $[e_1, \ldots, e_n]$ of $\mathbb{R}^n$, such that

$F = s(\kappa_{12} d\gamma_1 \wedge d\gamma_2 + \kappa_{34} d\gamma_3 \wedge d\gamma_4 + \cdots + \kappa_{2[n/2]-1,2[n/2]} d\gamma_{2[n/2]-1} \wedge d\gamma_{2[n/2]}$

$\kappa_{2i-1,2i} \geq 0$. Thus, the system becomes

$$\dot{\gamma}_{2i-1} = \frac{1}{m} p_{2i-1}, \qquad \dot{p}_{2i-1} = \frac{s}{m} \kappa_{2i-1,2i} p_{2i} + \mu \gamma_{2i-1},$$
$$\dot{\gamma}_{2i} = \frac{1}{m} p_{2i}, \qquad \dot{p}_{2i} = -\frac{s}{m} \kappa_{2i-1,2i} p_{2i-1} + \mu \gamma_{2i}, \qquad i = 1, \ldots, [n/2]$$

for n even, and, for n odd, there is an additional equation:

$$\dot{\gamma}_n = \frac{1}{m} p_n, \qquad \dot{p}_n = \mu \gamma_n.$$

The multiplier is given by

$$\mu = \frac{s}{m} \sum_{i=1}^{[n/2]} \kappa_{2i-1,2i} (p_{2i-1} \gamma_{2i} - p_{2i} \gamma_{2i-1}) - 2H.$$

Note that the gauge Noether symmetries (14) are tangent to the sphere S^{n-1} for $\Gamma = 0$, leading to the following statement.

Lemma

The functions

$$\Phi_{2i-1,2i} = \Phi^{0,0}_{2i-1,2i} = \gamma_{2i-1}p_{2i} - \gamma_{2i}p_{2i-1} + s\frac{\kappa_{2i-1,2i}}{2}\left(\gamma^2_{2i-1} + \gamma^2_{2i}\right).$$

are first integrals of motion of the magnetic flows. The first integrals of motion Poisson commute:

$$\{\Phi_{2i-1,2i}, \Phi_{2j-1,2j}\}_d = 0, \qquad i, j = 1, \dots, [n/2]$$

on the Poisson manifold $(\mathbb{R}^{2n}_*(\gamma, p), \{\cdot, \cdot\}_d)$.

We thus get the following theorem

Theorem

Assume $\kappa_{12} \neq 0$. *The magnetic flows are completely integrable on* S^2 *and* S^3, *corresponding to* $n = 3$ *and* $n = 4$ *respectively.*

Integral of the third degree in momenta

Lemma

(i) The function J given by

$$J = \frac{s^2}{m^2} \sum_{i=1}^{[n/2]} \kappa_{2i-1,2i}^2 (p_{2i-1}^2 + p_{2i}^2) - \mu^2$$

is the first integral of motion of the magnetic flows on T^*S^{n-1}*.*

(ii) The following commuting relations on the Poisson manifold $(\mathbb{R}_*^{2n}(\gamma, p), \{\cdot,\cdot\}_d)$ *among the first integrals* J, $\Phi_{2i-1,2i}$ *take place:*

$$\{J, \Phi_{2i-1,2i}\}_d = 0, \qquad i = 1, \dots, [n/2].$$

(iii) The functions H, J, $\Phi_{2i-1,2i}$, $i = 1, \dots, [n/2]$ *are functionally independent on* T^*S^{n-1} *for* $n \geq 5$ *for all odd* n *and all* κ *and if* n *is even and* κ *does not satisfy* $\kappa_{12} = \kappa_{34} = \cdots = \kappa_{n-1,n}$*.*

U(2)–symmetry

Lemma

If $\kappa_{2i-1,2i} = \kappa_{2j-1,2j}$, $i < j$, then the following functions

$$\begin{aligned}\Psi^1_{2i-1,2i;2j-1,2j} =& (\gamma_{2i} p_{2j-1} - \gamma_{2j-1} p_{2i}) - (\gamma_{2i-1} p_{2j} - \gamma_{2j} p_{2i-1}) \\ & - s\kappa_{2i-1,2i}(\gamma_{2i-1}\gamma_{2j-1} + \gamma_{2i}\gamma_{2j}) \\ \Psi^2_{2i-1,2i;2j-1,2j} =& (\gamma_{2i-1} p_{2j-1} - \gamma_{2j-1} p_{2i-1}) + (\gamma_{2i} p_{2j} - \gamma_{2j} p_{2i}) \\ & - s\kappa_{2i-1,2i}(\gamma_{2i-1}\gamma_{2j} - \gamma_{2i}\gamma_{2j-1})\end{aligned}$$

are the first integrals of motion of the magnetic geodesic flow on T^*S^{n-1}.

Lemma

The polynomials

$$\Phi_{2i-1,2i},\ \Phi_{2j-1,2j},\ \Psi^1_{2i-1,2i;2j-1,2j},\ \Psi^2_{2i-1,2i;2j-1,2j}$$

generate the following four-dimensional Lie algebra

$$\begin{aligned}
&\{\Phi_{2i-1,2i}, \Phi_{2j-1,2j}\}_d = 0,\\
&\{\Phi_{2i-1,2i}, \Psi^1_{2i-1,2i;2j-1,2j}\}_d = -\Psi^2_{2i-1,2i;2j-1,2j},\\
&\{\Phi_{2j-1,2j}, \Psi^1_{2i-1,2i;2j-1,2j}\}_d = \Psi^2_{2i-1,2i;2j-1,2j},\\
&\{\Phi_{2i-1,2i}, \Psi^2_{2i-1,2i;2j-1,2j}\}_d = \Psi^1_{2i-1,2i;2j-1,2j},\\
&\{\Phi_{2j-1,2j}, \Psi^2_{2i-1,2i;2j-1,2j}\}_d = -\Psi^1_{2i-1,2i;2j-1,2j},\\
&\{\Psi^1_{2i-1,2i;2j-1,2j}, \Psi^2_{2i-1,2i;2j-1,2j}\}_d = 2\Phi_{2j-1,2j} - 2\Phi_{2i-1,2i}.
\end{aligned}$$

on the Poisson manifold $(\mathbb{R}^{2n}_*(\gamma, p), \{\cdot,\cdot\}_d)$*. The Lie algebra is isomorphic to the reductive Lie algebra* $\mathrm{so}(3) \oplus \mathbb{R} \cong \mathrm{u}(2)$*.*

Theorem (Integrability for $n \leq 6$)

Assume $\kappa_{12} \neq 0$. The magnetic geodesic flows are Liouville integrable on T^*S^4 and T^*S^5, corresponding to $n = 5$ and $n = 6$ respectively, for any κ. Moreover:

(i) *If $n = 5$ and $\kappa_{12} = \kappa_{34}$, then the magnetic system is integrable in the non-commutative sense: generic motions are quasi-periodic over 3-dimensional invariant isotropic submanifolds.*

(ii) *If $n = 6$ and $\kappa_{12} = \kappa_{34} \neq \kappa_{56}$, then the magnetic system is integrable in the non-commutative sense: generic motions are quasi-periodic over 4-dimensional invariant isotropic submanifolds.*

(iii) *If $n = 6$ and $\kappa_{12} = \kappa_{34} = \kappa_{56}$, then the magnetic system is integrable in the non-commutative sense: generic motions are quasi-periodic over 3-dimensional invariant isotropic submanifolds.*

(iv) *For $n = 5$, $\kappa_{34} = 0$ and for $n = 6$, $\kappa_{34} = \kappa_{56} = 0$, the magnetic systems are integrable in the non-commutative sense: generic motions are quasi-periodic over 3-dimensional invariant isotropic submanifolds.*

Rolling of a ball over a sphere with a gyroscope

Let us consider rolling without slipping of a balanced, dynamically nonsymmetric ball over a fixed sphere. Let O_{B}, a, m, $\mathbb{I} = \mathrm{diag}(A, B, C)$, be the center, radius, mass of the system ball+gyroscope and the inertia operator of a ball B, and let b is radius of the sphere. We assume that a gyroscope is placed in a ball B such that the mass center of the system (ball + gyroscope) coincides with the geometric center O_{B} of the ball.
There are three possible configurations:

(i) rolling of B over the outer surface of S and S is outside B;

(ii) rolling of B over the inner surface of S ($b > a$);

(iii) rolling of B over the outer surface of S and S is within B; in this case $b < a$ and the rolling ball B is a spherical shell.

Let $\varepsilon = \frac{b}{b \pm a}$, where we take "+" for the case (i) and "−" in the cases (ii) and (iii) and let $D = ma^2$.

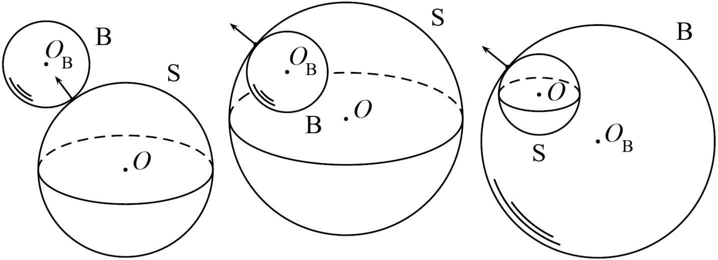

Figure: Rolling of the ball B with center O_B over the sphere S with center O: three scenarios

Borisov Fedorov (1995), Borisov Mamaev (2013)

Vasilije Grigoryevich Demchenko (1898-1972) in 1923 defended PhD thesis

> *Rolling of gyroscopic ball over the sphere*, University of Belgrade, 1923, 94 pages

Committee: Anton Bilimović, Mihailo Petrović Milutin Milanković.

СРПСКА КРАЉЕВСКА АКАДЕМИЈА НАУКА И УМЕТНОСТИ.

ПОСЕБНА ИЗДАЊА
КЊИГА LI

НАУКЕ ПРИРОДНЕ И МАТЕМАТИЧКЕ
КЊИГА 12.

КОТРЉАЊЕ БЕЗ КЛИЗАЊА ГИРОСКОПСКЕ ЛОПТЕ ПО СФЕРИ

ОД

ВАСИЛИЈА ДЕМЧЕНКА

БЕОГРАД
ГРАФИЧКИ ЗАВОД МАКАРИЈЕ А. Д.
1924.

ЦЕНА ____ ДИНАРА

Let the modified inertia operator $\mathsf{I} = \mathbb{I} + D\mathrm{Id}_{so(n)}$ ($D = ma^2$) be equal to the identity operator on $so(n)$ multiplied by a constant τ.

Proposition

The equations of motion of the n-dimensional generalized Demchenko case without twisting are:

$$\tau\dot{\omega} = [\kappa, \omega] + \lambda_0, \qquad \dot{\gamma} = -\varepsilon\omega\gamma,$$

where $\kappa \in so(n)$ is a fixed skew-symmetric matrix and the Lagrange multiplier $\lambda_0 \in (\mathbb{R}^n \wedge \gamma)^\perp$ is determined from the condition that $\omega \in \mathbb{R}^n \wedge \gamma$. The equations of motion reduce to the cotangent bundle of the sphere $\langle\gamma, \gamma\rangle = 1$:

$$\dot{\gamma} = \frac{\varepsilon^2}{\tau}p, \qquad \dot{p} = \frac{1}{\tau}\kappa p + \mu\gamma, \qquad \mu = \frac{1}{\tau}\langle p, \kappa\gamma\rangle - \frac{\varepsilon^2}{\tau}\langle p, p\rangle$$

The magnetic flow with $m = \tau/\varepsilon^2$ and $s = 1/\varepsilon^2$ ($s/m = 1/\tau$).

Three-dimensional Demchenko case without twisting

The reduced equations of motion on T^*S^2, for

$$\kappa = \kappa_{12} \mathsf{e}_1 \wedge \mathsf{e}_2,$$

become

$$\begin{aligned}
\dot\gamma_1 &= \frac{\varepsilon^2}{\tau} p_1, & \dot p_1 &= \frac{1}{\tau}\kappa_{12} p_2 + \mu\gamma_1,\\
\dot\gamma_2 &= \frac{\varepsilon^2}{\tau} p_2, & \dot p_2 &= -\frac{1}{\tau}\kappa_{12} p_1 + \mu\gamma_2,\\
\dot\gamma_3 &= \frac{\varepsilon^2}{\tau} p_3, & \dot p_3 &= \mu\gamma_3,
\end{aligned}$$

$$\mu = \frac{\kappa_{12}}{\tau}(p_1\gamma_2 - p_2\gamma_1) - \frac{\varepsilon^2}{\tau}(p_1^2 + p_2^2 + p_3^2),$$

They are Hamiltonian with respect to the magnetic Poisson structure and the Hamiltonian is

$$h = \frac{\varepsilon^2}{2\tau}(p_1^2 + p_2^2 + p_3^2).$$

Theorem

The reduced equations of the Demchenko case without twisting are Liouville integrable on T^*S^2 with the first integrals h, Φ, where

$$\Phi(\gamma, p) = \gamma_1 p_2 - \gamma_2 p_1 + \frac{\kappa_{12}}{2\varepsilon^2}(\gamma_1^2 + \gamma_2^2).$$

Theorem

The reduced equations of the three-dimensional Demchenko case without twisting can be explicitly integrated in elliptic functions and their degenerations.

Saksida (2002)

Let us introduce polar coordinates r, φ by

$$\gamma_1 = r\cos\varphi, \;\; \gamma_2 = r\sin\varphi.$$

In the new coordinates the first integrals can be rewritten as:

$$h = \frac{\tau}{2\varepsilon^2}\big(\dot r^2 + r^2\dot\varphi^2 + \frac{r^2\dot r^2}{1-r^2}\big), \tag{8}$$

$$\Phi = \frac{\tau}{\varepsilon^2} r^2\dot\varphi + \frac{\kappa_{12}}{2\varepsilon^2} r^2. \tag{9}$$

We have

$$\dot\varphi = \frac{2\varepsilon^2\Phi - \kappa_{12} r^2}{2\tau r^2}.$$

Introducing $u = r^2$, one derives

$$\dot u^2 = Q_3(u),$$

$$Q_3(u) := \frac{\kappa_{12}^2}{\tau^2}(u-1)\big(u^2 - \frac{4\varepsilon^2}{\kappa_{12}^2}(2h\tau + \kappa_{12}\Phi)u + \frac{4\varepsilon^4\Phi^2}{\kappa_{12}^2}\big)$$

$$= \frac{\kappa_{12}^2}{\tau^2}(u-1)(u-u_1)(u-u_2).$$

From Vieta's formulas, it follows that $u_1 u_2 > 0$, or in other words, the remaining two roots u_1, u_2 of Q_3 are of the same sign.

(A) $0 < u_1 < u_2 < 1$; Case (A) happens when the discriminant of the polynomial $Q_2(u) = (u - u_1)(u - u_2)$ is greater then zero, the minimum of $Q_2(u)$ is between 0 and 1, and $Q_2(1) > 0$. This yields conditions:

$$h\tau + \kappa_{12}\Phi > 0,$$
$$2h\tau + \kappa_{12}\Phi < \frac{\kappa_{12}^2}{2\varepsilon^2},$$
$$2h\tau + \kappa_{12}\Phi - \varepsilon^2\Phi < \frac{\kappa_{12}^2}{4\varepsilon^2}$$

(B) $0 < u_1 < 1 < u_2$. Case (B) happens when $Q_2(1) < 0$, that is

$$2h\tau + \kappa_{12}\Phi - \varepsilon^2\Phi > \frac{\kappa_{12}^2}{4\varepsilon^2}$$

In both cases r belongs to an annulus:

$$\text{Case (A)} \quad \sqrt{u_1} \leqslant r \leqslant \sqrt{u_2}; \qquad \text{Case (B)} \quad \sqrt{u_1} \leqslant r \leqslant 1.$$

When the discriminant of the polynomial Q_3 vanishes, the corresponding elliptic functions degenerate. It happens if $u_1 = u_2$, or when one of the roots u_1, u_2 is equal to 1. Direct calculations show that the discriminant of the polynomial Q_3 vanishes when

$$h\tau + \kappa_{12}\Phi = 0, \qquad \text{or} \qquad 2h\tau + \kappa_{12}\Phi - \varepsilon^2\Phi = \frac{\kappa_{12}^2}{4\varepsilon^2}.$$

Four-dimensional Demchenko case without twisting

Let

$$\kappa = \kappa_{12} e_1 \wedge e_2 + \kappa_{34} e_3 \wedge e_4.$$

The reduced equations are:

$$\dot\gamma_1 = \frac{\varepsilon^2}{\tau} p_1, \qquad \dot p_1 = \frac{1}{\tau}\kappa_{12} p_2 + \mu\gamma_1,$$

$$\dot\gamma_2 = \frac{\varepsilon^2}{\tau} p_2, \qquad \dot p_2 = -\frac{1}{\tau}\kappa_{12} p_1 + \mu\gamma_2,$$

$$\dot\gamma_3 = \frac{\varepsilon^2}{\tau} p_3, \qquad \dot p_3 = \frac{1}{\tau}\kappa_{34} p_4 + \mu\gamma_3,$$

$$\dot\gamma_4 = \frac{\varepsilon^2}{\tau} p_4, \qquad \dot p_4 = -\frac{1}{\tau}\kappa_{34} p_3 + \mu\gamma_4,$$

$$\mu = \frac{1}{\tau}\big(\kappa_{12}(p_1\gamma_2 - p_2\gamma_1) + \kappa_{34}(p_3\gamma_4 - p_4\gamma_3)\big) - \frac{\varepsilon^2}{\tau}(p_1^2 + p_2^2 + p_3^2 + p_4^2).$$

The Hamiltonian is

$$h = \frac{\varepsilon^2}{2\tau}(p_1^2 + p_2^2 + p_3^2 + p_4^2).$$

Theorem

The reduced equations of generalized Demchenko case for $n = 4$ *are Liouville integrable on* T^*S^3 *with the three first integrals* h, Φ_{12}, *and* Φ_{34} *in involution, where*

$$\Phi_{12}(p,\gamma) = \gamma_1 p_2 - \gamma_2 p_1 + \frac{\kappa_{12}}{2\varepsilon^2}(\gamma_1^2 + \gamma_2^2),$$
$$\Phi_{34}(p,\gamma) = \gamma_3 p_4 - \gamma_4 p_3 + \frac{\kappa_{34}}{2\varepsilon^2}(\gamma_3^2 + \gamma_4^2).$$

Theorem

The reduced equations of generalized Demchenko case without twisting for $n = 4$ *can be explicitly integrated in elliptic functions and their degenerations.*

Let us introduce new coordinates $\rho_1, \rho_3, \varphi_1, \varphi_3$ by

$$\gamma_1 = \rho_1 \cos\varphi_1, \quad \gamma_2 = \rho_1 \sin\varphi_1, \quad \gamma_3 = \rho_3 \cos\varphi_3, \quad \gamma_4 = \rho_3 \sin\varphi_3.$$

In the new coordinates the first integrals become

$$\begin{aligned} h &= \frac{\tau}{2\varepsilon^2}\left(\dot\rho_1^2 + \rho_1^2\dot\varphi_1^2 + \dot\rho_3^2 + \rho_3^2\dot\varphi_3^2\right), \\ \Phi_{12} &= \frac{\tau}{\varepsilon^2}\rho_1^2\dot\varphi_1 + \frac{\kappa_{12}}{2\varepsilon^2}\rho_1^2, \\ \Phi_{34} &= \frac{\tau}{\varepsilon^2}\rho_3^2\dot\varphi_3 + \frac{\kappa_{34}}{2\varepsilon^2}\rho_3^2. \end{aligned}$$

We have

$$\dot\varphi_1 = \frac{2\varepsilon^2\Phi_{12} - \kappa_{12}\rho_1^2}{2\tau\rho_1^2}, \qquad \dot\varphi_3 = \frac{2\varepsilon^2\Phi_{34} - \kappa_{34}\rho_3^2}{2\tau\rho_3^2}.$$

Introducing $u = \rho_1^2$, it follows

$$\dot{u}^2 = P_3(u).$$

Here, P_3 is a polynomial in u of the degree not greater than three:

$$P_3(u) := a_0 u^3 + a_1 u^2 + a_2 u + a_3,$$

where

$$a_0 = \frac{\kappa_{12}^2 - \kappa_{34}^2}{\tau^2}, \qquad a_3 = -\frac{4\varepsilon^4 \Phi_{12}^2}{\tau^2},$$

$$a_1 = -\frac{8\varepsilon^2 h}{\tau} - \frac{2\kappa_{34}}{\tau^2}(2\varepsilon^2\Phi_{34} - \kappa_{34}) - \frac{\kappa_{12}^2}{\tau^2} - \frac{4\varepsilon^2\kappa_{12}\Phi_{12}}{\tau^2},$$

$$a_2 = \frac{8\varepsilon^2 h}{\tau} - \frac{(2\varepsilon^2\Phi_{34} - \kappa_{34})^2}{\tau^2} + \frac{4\varepsilon^2\kappa_{12}\Phi_{12}}{\tau^2} + \frac{4\varepsilon^4\Phi_{12}^2}{\tau^2}.$$

Let us express the variable ρ_1^2 in terms of the Weierstrass $\wp$-function in a generic case: $\kappa_{12}^2 \neq \kappa_{34}^2$ and the polynomial $P_3(u)$ has all roots distinct. Introducing z such that

$$u = \frac{4}{a_0} z - \frac{a_1}{3a_0},$$

we have

$$\dot{z}^2 = 4z^3 - g_2 z - g_3,$$

where

$$g_2 = \frac{a_1^2}{12} - \frac{a_0 a_2}{4}, \qquad g_3 = \frac{a_0 a_1 a_2}{4} - \frac{a_1^3}{216} - \frac{a_0^2 a_3}{16}.$$

We get

$$\int_z^\infty \frac{d\xi}{\sqrt{4\xi^3 - g_2\xi - g_3}} - \int_{z_0}^\infty \frac{d\xi}{\sqrt{4\xi^3 - g_2\xi - g_3}} = \pm(t - t_0).$$

Finally, using the Weierstrass $\wp$-function one obtains

$$z = \wp(A \pm (t - t_0)), \quad z_0 = \wp(A).$$

Qualitative analysis

Let us consider the case $\kappa_{12}^2 \neq \kappa_{34}^2$. Then $P_3(u)$ is a degree three polynomial. The coordinates ρ_1, φ_1 and ρ_3, φ_3 are polar coordinates on the projections of the sphere $\langle \gamma, \gamma \rangle = 1$ to the coordinate planes $O\mathrm{e}_1\mathrm{e}_2$ and $O\mathrm{e}_3\mathrm{e}_4$, respectively. Hence, ρ_1 and ρ_3, and consequently u can take values between 0 and 1.
Since

$$P_3(0) = -\frac{4\varepsilon^4 \Phi_{12}^2}{\tau^2} < 0,$$

and

$$P_3(1) = -\frac{4\varepsilon^4 \Phi_{34}^2}{\tau^2} < 0,$$

one concludes that on interval $(0, 1)$ the polynomial $P_3(u)$ has (i) no real roots; (ii) two distinct real roots; or (iii) one double real root.

(i)

If the number of real roots is zero, then the polynomial $P_3(u)$ takes negative values on the whole interval $(0,1)$. Thus, the case (i) does not correspond to a real motion.

(ii)

In the case (ii) when the polynomial $P_3(u)$ has two distinct real roots $u_1 < u_2$ on the interval $(0,1)$, the projection of a trajectory to the $O\mathrm{e}_1\mathrm{e}_2$ and $O\mathrm{e}_3\mathrm{e}_4$ planes belong, respectively, to the annuli

$$\sqrt{u_1} \leqslant \rho_1 \leqslant \sqrt{u_2} \qquad \text{and} \qquad \sqrt{1-u_2^2} \leqslant \rho_3 = \sqrt{1-\rho_1^2} \leqslant \sqrt{1-u_1^2}.$$

There are three types of the trajectories in this case. Let

$$\hat{u} = \frac{2\varepsilon^2\Phi_{12}}{\kappa_{34}}.$$

If $\hat{u}$ belongs to (u_1, u_2) then $\dot{\varphi}_1$ changes the sign and trajectories are presented on Figures 3 and 4. If $\hat{u}$ is equal to u_1 or u_2, then the trajectories are presented on Figures 6 and 5 respectively. Otherwise, the trajectories are presented on Figure 7.

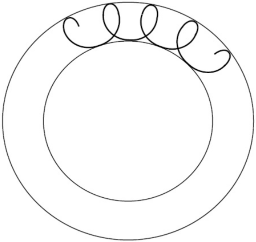

Figure: The case $u_1 < \hat{u} < u_2$

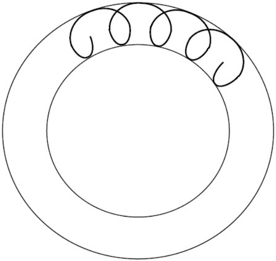

Figure: The case $u_1 < \hat{u} < u_2$

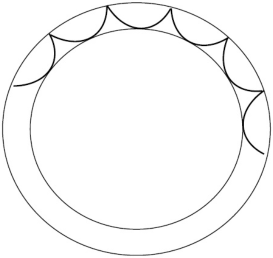

Figure: The case $\hat{u} = u_2$

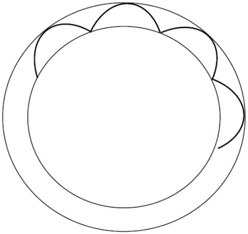

Figure: The case $\hat{u} = u_1$

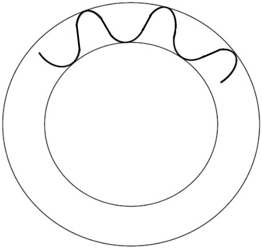

Figure: The case when $\hat{u}$ do not belongs to the interval $[u_1, u_2]$

The case of a double root $u_1 = u_2$ corresponds to the stationary motion

$$\begin{aligned} \rho_1 &= const, & \varphi_1 &= \alpha_1 t + \varphi_{10}, \\ \rho_3 &= \sqrt{1-\rho_1^2} = const, & \varphi_3 &= \alpha_3 t + \varphi_{20}, \end{aligned}$$

where

$$\alpha_1 = \frac{2\varepsilon^2\Phi_{12} - \kappa_{12}u_1}{2\tau u_1} = const, \quad \alpha_3 = \frac{2\varepsilon^2\Phi_{34} - \kappa_{34}(1-u_1)}{2\tau(1-u_1)} = const.$$

From the equations of motion it follows that the constants α_1 and α_3 should satisfy:

$$\kappa_{12}\alpha_1 - \kappa_{34}\alpha_3 + \tau(\alpha_1^2 - \alpha_3^2) = 0.$$

Since the roots u_1 and u_2 of the polynomial $P_3(u)$ coincide, the discriminant of the polynomial $P_3(u)$ is equal to zero.

THANK YOU FOR YOUR ATTENTION

We congratulate Professor Branko Dragović on his 80th anniversary!!!