

Amazing Efficiency of The Harmonic Superspace Methods

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Supersymmetry and superfields

- ▶ Supersymmetry, despite lacking experimental confirmations, is in the heart of the modern of mathematical and quantum physics. It allowed to construct a lot of new theories with remarkable and surprising features: supergravities, superstrings, superbranes, $\mathcal{N} = 4$ super Yang-Mills theory (the first example of the ultraviolet-finite quantum field theory), etc. It also exhibited unexpected relations between these theories, e.g., the “gravity/gauge” duality.
- ▶ The natural approach to supersymmetric theories is the superfield methods.
- ▶ The natural generalization of Minkowski space x^m to supersymmetry is **\mathcal{N} extended Minkowski superspace**

$$\mathcal{M}^{(4|4\mathcal{N})} = \left(x^m, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha} i} \right), \quad i = 1, \dots, \mathcal{N}$$

where $\theta_i^\alpha, \bar{\theta}^{\dot{\alpha} i}$ are anticommuting Grassmann coordinates,
 $\{\theta, \theta\}, \{\theta, \bar{\theta}\} = 0$.

- ▶ The supersymmetric theories are adequately formulated off shell in terms of superfields defined on various superspaces.

Superfields: $\mathcal{N}=1$ - chiral, $\mathcal{N} = 2$ - harmonic

- The fundamental role in $\mathcal{N} = 1$ supersymmetry is played by **chiral** superfields defined by the **chirality** or **anti-chirality** conditions

$$(a) \bar{D}_{\dot{\alpha}} \Phi_L(x, \theta, \bar{\theta}) = 0, \quad \text{or} \quad (b) D_{\alpha} \Phi_R(x, \theta, \bar{\theta}) = 0,$$

where $D_{\alpha}, \bar{D}_{\dot{\alpha}}$ are spinor covariant derivatives. In the proper basis,

$$(a) \Rightarrow \Phi_L(x, \theta, \bar{\theta}) = \varphi_L(x_L, \theta) \quad (b) \Rightarrow \Phi_R(x, \theta, \bar{\theta}) = \varphi_R(x_R, \bar{\theta})$$

- All $\mathcal{N} = 1$ theories of interest, $\mathcal{N} = 1$ matter, SYM and supergravity, can be consistently derived from the principle of preserving the notion of chirality in the interacting case.

- ▶ As for higher \mathcal{N} supersymmetries, in four-dimensions, the self-consistent off-shell superfield formalism is known only for $\mathcal{N} = 2$ and $\mathcal{N} = 3$ cases. It is the harmonic superspace approach discovered in Dubna forty years ago (Galperin, Ivanov, Kalitzin, Ogievetsky, Sokatchev, 1984, 1985).

- ▶ Harmonic $\mathcal{N} = 2$ superspace:

$$Z = (x^m, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}j}, u^{\pm i}), \quad u^{\pm i} \in SU(2)/U(1), \quad u^{+i} u_j^- = 1.$$

- ▶ Analytic harmonic $\mathcal{N} = 2$ superspace:

$$\zeta_A = (x_A^m, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, u^{\pm i}), \quad \theta^{+\alpha, \dot{\alpha}} := \theta^{\alpha, \dot{\alpha}i} u_i^+, \quad x_A^m := x^m - 2i\theta^{(i} \sigma^m \bar{\theta}^{j)} u_i^+ u_j^-$$

- ▶ All basic $\mathcal{N} = 2$ superfields are analytic:

$$\begin{array}{ll} \text{SYM :} & V^{++}(\zeta_A), \text{ matter hypermultiplets : } q^+(\zeta_A), \bar{q}^+(\zeta_A) \\ \text{supergravity :} & H^{++m}(\zeta_A), H^{++\alpha, \dot{\alpha}+}(\zeta_A), H^{++5}(\zeta_A) \end{array}$$

- ▶ They obey $\mathcal{N} = 2$ Grassmann analyticity conditions,

$$D_\alpha^+ \Phi = \bar{D}_{\dot{\alpha}}^+ \Phi = 0 \Rightarrow \Phi = \Phi(\zeta_A),$$

direct analogs of $\mathcal{N} = 1$ chirality.

Further developments: quantum and geometric

- ▶ One of the first sound applications of $\mathcal{N} = 2$ HSS approach in the quantum domain was a simple general proof of the UV finiteness of $\mathcal{N} = 4$ super YM theory to any order in the loop expansion.
- ▶ Later, quantum harmonic superfield techniques were successfully applied for finding out the complete quantum effective action of gauge theories with extended supersymmetry in four, three, five and six dimensions and exploring the structure of the relevant geometric structure and divergences (I. Buchbinder, A. Budekhina, E.I., S. Kuzenko, B. Merzlikin, I. Samsonov, F. Smilga, K. Stepanyantz, B. Zupnik, ... 1990 - 2025).
- ▶ The $1D$ and $2D$ versions of HSS were worked out by E.I., O. Lechtenfeld, A. Sutulin, 1994 - 2005 and then applied for constructing new models of supersymmetric mechanics and sigma models with extended supersymmetry, including the superconformal models relevant to AdS/CFT correspondence (E.I., S. Sidorov, S. Fedoruk, 2000 - 2024).

- ▶ It is curious that the harmonic methods firstly invented for the off-shell geometric description of theories with extended $\mathcal{N} \geq 2$ supersymmetry proved to shed new light on the purely bosonic geometries.
- ▶ The target space geometry of $\mathcal{N} = 2, 4D$ sigma models is known to be hyper-Kähler in the flat case and quaternion-Kähler in the case of $\mathcal{N} = 2$ supergravity. The HSS approach made it possible to construct the most general action of matter hypermultiplets (A.Galperin , E.I., V.I.Ogievetsky, E. Sokatchev),

$$S_{N=2\sigma} = \int d\zeta^{(-4)} (\bar{q}^+ \mathcal{D}^{++} q^+ - q^+ \mathcal{D}^{++} \bar{q}^+) + \mathcal{L}^{+4}(u^\pm, q^+, \bar{q}^+)$$

- ▶ The interacting Lagrangian $\mathcal{L}^{+4}(u^\pm, q^+, \bar{q}^+)$ is just the hyper-Kähler potential, the true analog of Kähler potential of $\mathcal{N} = 1$ sigma models. It was unknown before. Choosing some \mathcal{L}^{+4} and passing to the component Lagrangian, one necessarily obtains a hyper-Kähler target metric. So the HSS approach provides an efficient way of the **explicit** calculation of the HK (and in fact QK) metrics.

Newest applications: $\mathcal{N} = 2$ higher spins

- ▶ Supersymmetric higher-spin theories provide a bridge between superstring theory and low-energy (super)gauge theories.
- ▶ The component approach to $4D, \mathcal{N} = 1$ supersymmetric free massless higher spin models: Courtright, 1979; Vasiliev, 1980.
- ▶ The complete off-shell $\mathcal{N} = 1$ superfield Lagrangian formulation of $\mathcal{N} = 1, 4D$ free higher spins: Kuzenko et al, 1993, 1994.
- ▶ An off-shell superfield Lagrangian formulation for higher-spin **extended** supersymmetric theories, with all supersymmetries manifest, was unknown for long even for free theories.
- ▶ This gap was filled in I. Buchbinder, E. Ivanov, N. Zaigraev, 2021, 2022, 2023. An off-shell manifestly $\mathcal{N} = 2$ supersymmetric unconstrained formulation of $4D, \mathcal{N} = 2$ integer higher spins and their couplings to hypermultiplets was constructed in the harmonic superspace approach.
- ▶ Quite recently, we generalized the HSS non-conformal construction to $\mathcal{N} = 2$ superconformal multiplets and their hypermultiplet coupling (arXiv:2404.19016 [hep-th], JHEP 08 (2024) 120).

- ▶ Recently, it was established that the HSS approach ideally suits for the off-shell description of $\mathcal{N} = 2$ higher-spin gauge multiplets and their hypermultiplet couplings (I. Buchbinder, E. I., N. Zaigraev, 2021 - 2025).
- ▶ Before 2021, it was unknown how to describe $\mathcal{N} = 2$ higher spin ($\mathbf{s} \geq 3$) gauge multiplets off shell in terms of unconstrained superfields - even at the free level. We have shown that the proper formulation is a direct generalization of the HSS $\mathcal{N} = 2$ supergravity. Once again, it is based on the principle of harmonic $\mathcal{N} = 2$ analyticity.
- ▶ The simplest $\mathcal{N} = 2$ supergravity is described by a set of analytic prepotentials with the proper gauge transformation properties (A. Galperin, E. I., V. Ogievetsky, E. Sokatchev, 1987).

$$(h^{++m}, h^{++\alpha, \dot{\alpha}+}, h^{++\alpha, \dot{\alpha}-}, h^{++5}) := h^{++M}(\zeta),$$

$$\delta h^{++M}(\zeta) = \mathcal{D}^{++} \lambda^M(\zeta), \quad \mathcal{D}^{++} = D^{++} + h^{++M} \partial_M$$

- ▶ Surprisingly, this representation rather directly generalizes to the integer higher-spin $\mathbf{s} \geq 3$ $\mathcal{N} = 2$ gauge multiplets

$$h^{++\alpha(s-1)\dot{\alpha}(s-1)}(\zeta), h^{++\alpha(s-2)\dot{\alpha}(s-2)}(\zeta), h^{++\alpha(s-1)\dot{\alpha}(s-2)+}(\zeta), h^{++\dot{\alpha}(s-1)\alpha(s-2)+}(\zeta),$$

where $\alpha(\mathbf{s}) := (\alpha_1 \dots \alpha_s), \dot{\alpha}(\mathbf{s}) := (\dot{\alpha}_1 \dots \dot{\alpha}_s)$

- ▶ The relevant gauge transformations can be defined and shown to leave, in the WZ-like gauge, the physical field multiplet $(\mathbf{s}, \mathbf{s} - \mathbf{1}/2, \mathbf{s} - \mathbf{1}/2, \mathbf{s} - \mathbf{1})$.
- ▶ The on-shell spin contents of $\mathcal{N} = 2$ higher-spin multiplets:

$$\underline{\text{spin } 1} : 1, (1/2)^2, (0)^2$$

$$\underline{\text{spin } 2} : 2, (3/2)^2, 1$$

$$\underline{\text{spin } 3} : 3, (5/2)^2, 2$$

.....

$$\underline{\text{spin } s} : s, (s - 1/2)^2, s - 1$$

- ▶ The general formula for off-shell contents of the spin \mathbf{s} multiplet was found: $8[\mathbf{s}^2 + (\mathbf{s} - \mathbf{1})^2]_B + 8[\mathbf{s}^2 + (\mathbf{s} - \mathbf{1})^2]_F$.

Hypermultiplet couplings

- ▶ Supersymmetric $\mathcal{N} = 1$ generalizations of the bosonic cubic vertices with matter were explored in terms of $\mathcal{N} = 1$ superfields by Gates, Koutrolikos, Kuzenko, I. Buchbinder, E. Buchbinder and many others.
- ▶ In JHEP 05 (2022) 104 we have constructed the off-shell manifestly $\mathcal{N} = 2$ supersymmetric cubic couplings $(\frac{1}{2}, \frac{1}{2}, \mathbf{s})$ of an arbitrary gauge $\mathcal{N} = 2$ multiplet with higher integer spin \mathbf{s} to the hypermultiplet matter in $4D, \mathcal{N} = 2$ harmonic superspace.
- ▶ In our approach $\mathcal{N} = 2$ supersymmetry of cubic vertices is always manifest and off-shell, in contrast, e.g., to the non-manifest light-cone formulations.

- The couplings to the matter hypermultiplet are accomplished by covariantizing the harmonic derivative in the free action.
- Analytic gauge potentials for any spin \mathbf{s} with the correct transformation rules are recovered by proper gauge-covariantization of the harmonic derivative \mathcal{D}^{++} . The simplest option is gauging of $U(1)$,

$$\begin{aligned}\delta q^{+a} &= -\lambda_0 J q^{+a}, \quad J q^{+a} = i(\tau_3)^a_b q^{+b}, \\ \mathcal{D}^{++} &\Rightarrow \mathcal{D}^{++} + \hat{\mathcal{H}}_{(1)}^{++}, \quad \hat{\mathcal{H}}_{(1)}^{++} = h^{++} J, \\ \delta_\lambda \hat{\mathcal{H}}_{(1)}^{++} &= [\mathcal{D}^{++}, \hat{\Lambda}], \quad \hat{\Lambda} = \lambda J \Rightarrow \delta_\lambda h^{++} = \mathcal{D}^{++} \lambda\end{aligned}$$

- In $\mathcal{N} = 2$ supergravity, that is for $\mathbf{s} = 2$,

$$\begin{aligned}\mathcal{S}_{(2)} &= - \int d\zeta^{(-4)} \frac{1}{2} q^{+a} (\mathcal{D}^{++} + \mathcal{H}_{(2)}) q_a^+, \quad \delta \mathcal{H}_{(2)} = [\mathcal{D}^{++}, \hat{\Lambda}_{(2)}], \\ \mathcal{H}_{(2)} &= h^{++M}(\zeta) \partial_M, \quad \hat{\Lambda}_{(2)} = \lambda^M(\zeta) \partial_M, \quad M := (\alpha\dot{\beta}, 5, \hat{\mu}+)\end{aligned}$$

- For higher \mathbf{s} everything goes analogously. For $\mathbf{s} = 3$

$$\begin{aligned}\mathcal{S}_{(3)} &= - \int d\zeta^{(-4)} \frac{1}{2} q^{+a} (\mathcal{D}^{++} + \mathcal{H}_{(3)} J) q_a^+, \\ \delta \mathcal{H}_{(3)} &= [\mathcal{D}^{++}, \hat{\Lambda}_{(3)}], \quad \mathcal{H}_{(3)} = h^{++\alpha\dot{\alpha}M}(\zeta) \partial_M \partial_{\alpha\dot{\alpha}}, \quad \hat{\Lambda}_{(3)} = \lambda^{\alpha\dot{\alpha}M}(\zeta) \partial_M \partial_{\alpha\dot{\alpha}}\end{aligned}$$

Superconformal couplings

- ▶ (Super)conformal higher-spin theories are considered as a basis for all other types of higher-spin models. Non-conformal ones follow from the superconformal ones through couplings to the **superfield compensators**.
- ▶ In (**Buchbinder, Ivanov, Zaigraev, arXiv:2404.19016 [hep-th]**), we extended the off-shell $\mathcal{N} = 2, 4D$ higher spins and their hypermultiplet couplings to the superconformal case. Rigid $\mathcal{N} = 2, 4D$ superconformal symmetry plays a crucial role in fixing the structure of the theory.
- ▶ $\mathcal{N} = 2, 4D$ SCA preserves harmonic analyticity and is a closure of the rigid $\mathcal{N} = 2$ supersymmetry and special conformal symmetry

$$\delta_\epsilon \theta^{+\hat{\alpha}} = \epsilon^{\hat{\alpha}i} u_i^+, \quad \delta_\epsilon x^{\alpha\dot{\alpha}} = -4i \left(\epsilon^{\alpha i} \bar{\theta}^{+\dot{\alpha}} + \theta^{+\alpha} \bar{\epsilon}^{\dot{\alpha}i} \right) u_i^-, \quad \hat{\alpha} = (\alpha, \dot{\alpha}),$$
$$\delta_k \theta^{+\alpha} = x^{\alpha\dot{\beta}} k_{\beta\dot{\beta}} \theta^\beta, \quad \delta_k x^{\alpha\dot{\alpha}} = x^{\rho\dot{\alpha}} k_{\rho\dot{\rho}} x^{\dot{\rho}\alpha}, \quad \delta_k u^{+i} = (4i \theta^{+\alpha} \bar{\theta}^{+\dot{\alpha}} k_{\alpha\dot{\alpha}}) u^{-i}$$

- The higher-spin analytic prepotentials are introduced analogously to the case of $\mathbf{s} = 2$ where they appear through the proper lengthening of D^{++}

$$\mathcal{D}^{++} \rightarrow \mathcal{D}^{++} + \kappa_2 \hat{\mathcal{H}}_{(s=2)}^{++},$$

$$\hat{\mathcal{H}}_{(s=2)}^{++} := h^{++M} \partial_M = h^{++\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} + h^{++\alpha+} \partial_{\alpha}^{-} + h^{++\dot{\alpha}+} \partial_{\dot{\alpha}}^{-} + h^{(+4)} \partial^{--}$$

For ensuring conformal covariance, it is imperative to introduce the extra potential $h^{(+4)}$. The extended set of potentials embodies $\mathcal{N} = 2$ **Weyl multiplet** ($\mathcal{N} = 2$ conformal SG gauge multiplet).

- For $\mathbf{s} \geq 3$ the gauge-covariantization of the free q^{+a} action requires adding the gauge superfield differential operators of rank $\mathbf{s} - 1$ in ∂_M ,

$$\mathcal{D}^{++} \rightarrow \mathcal{D}^{++} + \kappa_s \hat{\mathcal{H}}_{(s)}^{++} (J)^{P(s)}, \quad P(s) = \frac{1 + (-1)^{s-1}}{2}$$

- For $\mathbf{s} = 3$:

$$\hat{\mathcal{H}}_{(s=3)} = h^{++MN} \partial_N \partial_M + h^{++}, \quad h^{++MN} = (-1)^{P(M)P(N)} h^{++NM}$$

The analytic matrix prepotentials h^{++MN} accommodate the spin $\mathbf{3}$ analog of of the $\mathcal{N} = 2$ Weyl multiplet. $\mathcal{N} = 2$ SCA mixes different entries of h^{++MN} , so we need to take into account all these entries.

Towards AdS background

- It is most interesting to explicitly construct $\mathcal{N} = 2$ higher spins in the AdS background, with the superconformal symmetry $SU(2, 2|2)$ being broken to the AdS supersymmetry $OSp(2|4; R)$.
- The embedding of the $\mathcal{N} = 2$ AdS superalgebra into $SU(2, 2|2)$ is realized through the identification (Bandos, Ivanov, Lukierski, Sorokin, 2002)

$$\begin{aligned}\Psi_{\alpha}^i &= Q_{\alpha}^i + c^{ik} S_{k\alpha}, & \bar{\Psi}_{\dot{\alpha}}^i &= \overline{\Psi_{\alpha}^i} = \bar{Q}_{\dot{\alpha}i} + c_{ik} \bar{S}_{\dot{\alpha}}^k, \\ c^{ik} &= c^{ki} & \overline{c^{ik}} &= c_{ik} = \varepsilon_{il} \varepsilon_{kj} c^{lj}\end{aligned}$$

- The $SU(2, 2|2)$ commutation relations imply for super AdS generators

$$\begin{aligned}\{\Psi_{\alpha}^i, \Psi_{\beta}^k\} &= c^{ik} L_{(\alpha\beta)} + 4i\varepsilon_{\alpha\beta}\varepsilon^{ik} T, & T &:= c_{lm} T^{lm}, & [T, \Psi_{\alpha}^i] &\sim c^{ik} \Psi_{k\alpha}, \\ \{\Psi_{\alpha}^i, \bar{\Psi}_{\dot{\beta}k}\} &= 2\delta_k^i R_{\alpha\dot{\beta}}, & R_{\alpha\dot{\beta}} &= P_{\alpha\dot{\beta}} + \frac{1}{2} c^2 K_{\alpha\dot{\beta}}, & c^2 &:= c^{ik} c_{ik} \sim \frac{1}{R_{AdS}^2}, \\ [R_{\alpha\dot{\alpha}}, R_{\gamma\dot{\gamma}}] &\sim c^2 (\varepsilon_{\alpha\gamma} L_{\dot{\alpha}\dot{\gamma}} + \varepsilon_{\dot{\alpha}\dot{\gamma}} L_{\alpha\gamma}), & [R_{\alpha\dot{\beta}}, \Psi_{\dot{\beta}}^i] &\sim \varepsilon_{\alpha\beta} \bar{\Psi}_{\dot{\beta}}^i \text{ (and c.c.)}\end{aligned}$$

- ▶ The first step toward constructing an off-shell $\mathcal{N} = 2$ AdS higher spin theory is to define the super AdS invariant Lagrangian of hypermultiplet, such that it respect no full superconformal invariance, but only the super AdS one.
- ▶ One needs to define the AdS covariant version of the analyticity-preserving harmonic derivative \mathcal{D}^{++} . The appropriate \mathcal{D}_{AdS}^{++} acting on $q^{+a} = (q^+, \tilde{q}^+)$ has the structure

$$\begin{aligned}\mathcal{D}_{AdS}^{++} &= \partial^{++} - 4i\hat{\theta}^{+\alpha}\hat{\theta}^{+\dot{\alpha}}\nabla_{\alpha\dot{\alpha}} + h^{++}\hat{T} + \mathcal{O}(c) \\ \nabla_{\alpha\dot{\alpha}} &= (1 + \frac{1}{2}c^2x^2)\partial_{\alpha\dot{\alpha}}, \quad h^{++} = i|c|[(\hat{\theta}^+)^2 - (\hat{\tilde{\theta}}^+)^2] + \mathcal{O}(c), \\ \hat{T}(q^+, \tilde{q}^+) &= (q^+, -\tilde{q}^+),\end{aligned}$$

where $\hat{\theta}_\alpha^+, \hat{\theta}_{\dot{\alpha}}^+$ are some redefinitions of the original Grassmann coordinates and $\mathcal{O}(c)$ stand for terms vanishing in the limit $c^{ik} \rightarrow 0$.

- ▶ An extra term $\sim \hat{T}$ in \mathcal{D}_{AdS}^{++} is necessary for breaking superconformal invariance and it produces a mass of q^+ proportional to $1/R_{AdS}^2$. In the properly defined flat limit this term becomes the central charge extension of flat \mathcal{D}^{++} and \hat{T} goes just into the derivative ∂_5 .
- ▶ More details on the AdS invariant q^+ Lagrangians will be given in our work with Nikita Zagraev (in progress).

Outlook

The theory of $\mathcal{N} = 2$ supersymmetric higher spins $s \geq 3$ opens a new direction of applications of the harmonic superspace approach which earlier proved to be indispensable for description of more conventional $\mathcal{N} = 2$ theories with maximal spins $s \leq 2$. Once again, the basic property underlying these new higher-spin theories is the preservation of harmonic Grassmann analyticity (all basic gauge potentials are unconstrained analytic superfields). The theory of conformal higher spins already embodies that of AdS higher spins because the supergroup underlying the latter is a subgroup of $\mathcal{N} = 2$ superconformal group.

Under way:

- ▶ The linearized actions of conformal higher-spin $\mathcal{N} = 2$ multiplets ($\mathcal{N} = 2$ analogs of the square of Weyl tensor) and their AdS descendants (Ivanov & Zaigraev, to appear).
- ▶ Quantization, induced actions,...
- ▶ $\mathcal{N} = 2$ supersymmetric half-integer spins (Ivanov & Zaigraev, 2025) ?
- ▶ From the linearized theory to its full nonlinear version? At present, the latter is known only for $s \leq 2$ ($\mathcal{N} = 2$ super Yang - Mills and $\mathcal{N} = 2$ supergravities). This problem will seemingly require accounting for **ALL** higher $\mathcal{N} = 2$ superspins simultaneously. New supergeometries?

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THANK YOU !

DEAR BRANKO, MY WARMEST CONGRATULATIONS!!