Feynman diagrams, operator formalism and conformal field theory

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План

Feynman diagrams (considered here) are graphs with vertices connected by lines labeled by numbers (indeces).

Such graphs visualize Feynman perturbative integrals. Namely, to each vertex of the graph we associate the point in D-dimensional Euclidean space $\mathbb{R}^D$, while the lines (edges) of the graph (with index α) are propagators of massless particles

$$x \quad \overline{\quad\quad\alpha\quad\quad} \quad y \quad\quad = \quad\quad 1/(x-y)^{2\alpha}$$

where $\boxed{(x-y)^{2\alpha} := \left(\sum_{i=1}^{D}(x_i - y_i)(x_i - y_i)\right)^{\alpha}}$, $\alpha \in \mathbb{C}$, $x, y \in \mathbb{R}^D$.

We have 2 types of vertices: the boldface vertices $\bullet$ denote the integration over $\mathbb{R}^D$ and not boldface vertices (not integrated).

Example. Propagator-type massless diagram

$$= \quad \int \frac{d^D z\, d^D u\, d^D y\, d^D w}{(x-z)^{2\alpha_1}\, z^{2\alpha_2}\, (z-u)^{2\alpha_3}\, u^{2\alpha_4}\, (u-y)^{2\alpha_5}\, y^{2\alpha_6} \ldots (w-x)^{2\alpha_9}}$$

These Feynman diagrams are called F.D. in the configuration space.

For certain F.D., results are expressed in terms of **multiple zeta-values and polylogarithms** – subjects for investigations in mathematics.
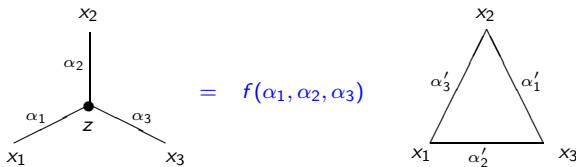
Star-triangle relation:

$$\int \frac{d^D z}{(x_1 - z)^{2\alpha_1} (x_2 - z)^{2\alpha_2} (x_3 - z)^{2\alpha_3}} = \frac{f(\alpha_1, \alpha_2, \alpha_3)}{(x_1 - x_2)^{2\alpha_3'} (x_2 - x_3)^{2\alpha_1'} (x_1 - x_3)^{2\alpha_2'}} ,$$

where $\boxed{\alpha_1 + \alpha_2 + \alpha_3 = D}$, $(x - z)^{2\alpha} = ((x - z)_\mu (x - z)^\mu)^\alpha$ and

$$f(\alpha_1, \alpha_2, \alpha_3) = (2\pi)^{2D} a(\alpha_1) a(\alpha_2) a(\alpha_3),$$

$$a(\alpha) = \frac{\pi^{-D/2} \Gamma(\alpha')}{2^{2\alpha} \Gamma(\alpha)} , \quad \alpha' := D/2 - \alpha .$$

$$= \quad f(\alpha_1, \alpha_2, \alpha_3)$$

E.S. Fradkin and M.Y. Palchik, Phys. Rept. 44 (1978) 249.
A.N.Vasilev, Y.M.Pismak, Y.R.Khonkonen, Theor.Math.Phys. 47 (1981)465.

A.B. Zamolodchikov, "Fishing-net" Diagrams as a Completely Integrable System, Phys.Lett. B 97 (1980) 63-66.   INSPIRE 112 citations

The star-triangle relation can be visualized on $\mathbb{R}^2$ as the Yang-Baxter Equation (faces are painted in a checkerboard pattern).

Fig. **1**

where $\boxed{\theta_1 + \theta_2 + \theta_3 = \pi}$ and

$$= \frac{\Gamma(\alpha(\theta))}{\pi^{\alpha(\theta)}} \frac{1}{(x-x')^{2\alpha(\theta)}}, \qquad \alpha(\theta) := \frac{D}{2\pi}(\pi - \theta).$$

A.Zamolodchikov calculated "fishing-net" planar Feynman diagrams as partition functions of certain integrable statistical model.

The 4-dimensional ϕ^4 field theory (and its multicomponent generalizations) serves the Brout-Englert-Higgs mechanism and thus it is an essential part of the Standard Model of particle physics.

It was shown by explicit evaluation (in MS scheme) of the Gell-Mann-Low β-function in $\phi^4_{D=4}$ theory that special Feynman diagrams – so-called zig-zag diagrams give 44%, 46% and 47% of numerical contributions, respectively, to the 3, 4 and 5 loop orders of β [D.J. Broadhurst and D. Kreimer (1995)].

To evaluate β-function we need to know the residue $\frac{1}{\epsilon} Z(M+1) + const + \epsilon...$ in the expression for the 4-point perturbative Feynman integral (we use dimensional regularization $D = 4 - 2\epsilon$)
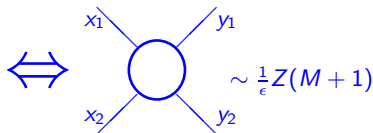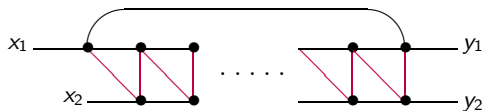
where

$$x_1 \underset{\beta}{\rule{1cm}{0.4pt}} x_2 = \frac{1}{(x_1-x_2)^{2\beta}} , \qquad x_i, y_i \in \mathbb{R}^D, \qquad \bullet = \int d^D x... ,$$

One can show that $Z(M + 1)$ (which is $(M + 1)$-loop contribution to the β-function) is also given by the integral for M-loop 2-point zig-zag diagrams (ZZD):

$G_2(x, y) =$

and it has the general form for $D = 4$:

$$G_2(x, y) = \frac{\pi^{2M}}{(x - y)^2} \, Z(M + 1) \,, \tag{1}$$

where π^{2M} is the normalization factor, $x, y \in \mathbb{R}^4$ and $Z(M + 1)$ is **the same constant** that gives $(M + 1)$-loop order contribution to the β-function in the $\phi^4_{D=4}$ theory.

In further consideration we use Riemann ζ-numbers:

$$\zeta_k := \sum_{n \geq 1} \frac{1}{n^k} \ .$$

History.

1. The first $Z(3) = 6\zeta_3 \sim \diamond$ and $Z(4) = 20\zeta_5 \sim \triangle$ in (1) were analytically evaluated by [K.G.Chetyrkin, A.L.Kataev, F.V.Tkachov, <u>1980</u>][1] and [K.G.Chetyrkin, F.V.Tkachov, <u>1981</u>], respectively.

2. The constant $Z(5) = \frac{441}{8}\zeta_7$ of the zig-zag diagram (ZZD) with 4 loops $\triangle$ was calculated by D.Kazakov in <u>1983</u>.

3. The 5 loop ZZD $\triangle$ contribution $Z(6) = 168\zeta_9$ to the β-function (in 6-loop order) was found by D.Broadhurst in <u>1985</u>.

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[1] The "two-loop fish diagram" was firstly evaluated in [E.De Rafael, J.L.Rosner, 1974].

4. Then D.Broadhurst and D.Kreimer in **1995** evaluated $Z(M+1)$ numerically up to $(M+1) = 10$ loops, and based on these data they formulated a remarkable conjecture that the constant $Z(M+1)$ is given by the sign alternating sum

$$Z(M+1) = 4\,C_M \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)(M+1)}}{n^{2M-1}} =$$

$$= \begin{cases} 4\,C_M\,\zeta_{2M-1}\,, & \text{for } M = 2N+1\,; \\ 4\,C_M\,(1 - 2^{2(1-M)})\,\zeta_{2M-1}\,, & \text{for } M = 2N\,; \end{cases} \qquad \zeta_k = \sum_{n \geq 1} \frac{1}{n^k}\,, \tag{2}$$

where M is the number of loops in ZZDs and $C_M = \frac{(2M)!}{(M+1)!\,M!}$ is the Catalan number.

5. Finally, the very nontrivial proof of the Broadhurst-Kreimer conjecture was found by [F.Brown and O.Schnetz in **2013,2015**; based on J.M.Drummond (2012)].

In this report, by using methods of D-dimensional CFT, the concise integral presentations for 4-point and 2-point zig-zag Feynman graphs are deduced. It gives a possibility to compute exactly a special class of 2- and 4-point Feynman diagrams (ZZDs for any M) in ϕ_D^4 theory. In particular we find new rather simple proof of the Broadhurst-Kreimer conjecture.

Operator formalism for massless diagrams.

Let $\{\hat{q}_a^\mu, \hat{p}_b^\nu\}$ $(a, b = 1, ..., n)$ be generators of the D-dimensional Heisenberg algebras $\mathcal{H}_a$ (a=1,...,n)

$$[\hat{q}_a^\mu, \hat{q}_b^\nu] = 0 = [\hat{p}_a^\mu, \hat{p}_b^\nu], \qquad [\hat{q}_a^\mu, \hat{p}_b^\nu] = i\,\delta^{\mu\nu}\,\delta_{ab} \qquad (\mu, \nu = 1, ..., D).$$

We introduce states $|x_a\rangle \in V_a$ which diagonalize coordinates $\hat{q}_a^\mu$:

$$\hat{q}_a^\mu |x_a\rangle = x_a^\mu |x_a\rangle.$$

These states form the basis in the representation space V_a of subalgebra $\mathcal{H}_a$. We also introduce the dual states $\langle x_a|$ such that the orthogonality and completeness conditions are valid

$$\langle x_a | x_a' \rangle = \delta^D(x_a - x_a'), \qquad \int d^D x_a\, |x_a\rangle\langle x_a| = I_a,$$

where I_a is the unit operator in V_a and there are no summations over indices a. So, we have the algebra $\mathcal{H}^{(n)} = \otimes_{a=1}^n \mathcal{H}_a$ which acts in the space $V_1 \otimes \cdots \otimes V_n$ with basis vectors $|x_1\rangle \otimes \cdots |x_n\rangle$.

Further we use operators $(\hat{q}_a)^{2\alpha} = (\sum_\mu \hat{q}_a^\mu \hat{q}_a^\mu)^\alpha$ and $(\hat{p}_a)^{2\beta} = (\sum_\mu \hat{p}_a^\mu \hat{p}_a^\mu)^\beta$ with non-integer α and β. These operators are understood as integral operators defined via their integral kernels: $\langle x| (\hat{q})^{-2\alpha} |y\rangle = (x)^{-2\alpha} \delta^D(x-y)$ and

$$\langle x| \frac{1}{(\hat{p})^{2\beta}} |y\rangle = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik(x-y)}}{(k)^{2\beta}} = \frac{a(\beta)}{(x-y)^{2\beta'}} \ ,$$

$$a(\beta) := \frac{2^{-2\beta}}{\pi^{D/2}} \frac{\Gamma(\beta')}{\Gamma(\beta)} \ , \quad \beta' := D/2 - \beta \ .$$

Star-triangle relation in operator approach [API, Nucl.Phys.B 662 (2003) 461] takes a remarkable form:

$$\langle x|\hat{p}^{2\alpha} \hat{q}^{2(\alpha+\beta)} \hat{p}^{2\beta}|y\rangle = \langle x|\hat{q}^{2\beta} \hat{p}^{2(\alpha+\beta)} \hat{q}^{2\alpha}|y\rangle \quad \Rightarrow$$

$$\boxed{\hat{p}^{2\alpha} \hat{q}^{2(\alpha+\beta)} \hat{p}^{2\beta} = \hat{q}^{2\beta} \hat{p}^{2(\alpha+\beta)} \hat{q}^{2\alpha}} \quad \Rightarrow \quad \hat{q}^{2\alpha}... \ \hat{p}^{2\alpha} \quad \Rightarrow$$

$$\hat{q}^{2\alpha} \hat{p}^{2\alpha} \cdot \hat{q}^{2\gamma} \hat{p}^{2\gamma} = \hat{q}^{2\gamma} \hat{p}^{2\gamma} \cdot \hat{q}^{2\alpha} \hat{p}^{2\alpha} \big|_{\gamma=\alpha+\beta} \ ,$$

which is commutativity condition for operators $\mathcal{H}_\alpha = \hat{q}^{2\alpha} \hat{p}^{2\alpha}$.

Consider the algebra $\mathcal{H}^{(2)} = \mathcal{H}_1 \otimes \mathcal{H}_2$, which acts in $V_1 \otimes V_2$ with basis $|x_1, x_2\rangle := |x_1\rangle \otimes |x_2\rangle$. To evaluate ZZDs in the operator approach we introduce the main object – graph building operator:

$$\hat{Q}_{12}^{(\beta)} := \frac{1}{a(\beta)} \, \mathcal{P}_{12} \, (\hat{p}_1)^{-2\beta} \, (\hat{q}_{12})^{-2\beta} \,,$$

where $(\hat{q}_{12})^2 = (\hat{q}_1^\mu - \hat{q}_2^\mu)(\hat{q}_1^\mu - \hat{q}_2^\mu)$ and $\mathcal{P}_{12}$ is the permutation operator

$$\mathcal{P}_{12}\,\hat{q}_1 = \hat{q}_2\,\mathcal{P}_{12}\,, \quad \mathcal{P}_{12}\,\hat{p}_1 = \hat{p}_2\,\mathcal{P}_{12}\,, \quad \mathcal{P}_{12}|x_1, x_2\rangle = |x_2, x_1\rangle\,, \quad (\mathcal{P}_{12})^2 = I\,.$$

We depict the kernel $\langle x_1, x_2 | \hat{Q}_{12}^{(\beta)} | y_1, y_2 \rangle$ of the graph building operator (GBO) $\hat{Q}_{12}^{(\beta)}$ as

$$\mathcal{P}_{12} \cdot \; \overset{x_1}{\underset{x_2 \cdots}{\Big|}} \overset{\beta'}{\underset{}{\Big]}}_\beta \overset{y_1}{\underset{y_2}{}} = \; \overset{x_2}{\underset{x_1 \cdots}{\Big|}} \overset{\beta'}{\underset{}{\Big]}}_\beta \overset{y_1}{\underset{y_2}{}} = \frac{1}{a(\beta)} \, \langle x_1, x_2 | \, \mathcal{P}_{12} \, (\hat{p}_1)^{-2\beta} \, (\hat{q}_{12})^{-2\beta} \, | y_1, y_2 \rangle =$$

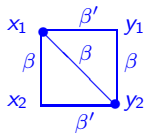$$= \; \frac{1}{(x_2 - y_1)^{2\beta'} \, (y_1 - y_2)^{2\beta}} \, \delta^D(x_1 - y_2)\,,$$

where

$$x_1 \; \cdots\cdots \; x_2 \; = \; \delta^D(x_1 - x_2)\,, \qquad x_1 \; \xrightarrow{\;\;\beta\;\;} \; x_2 \; = \; (x_1 - x_2)^{-2\beta}\,.$$

Now we note that $Q_{12}^{(\beta)}$ is the GBO for the planar zig-zag Feynman graphs.

Example for $(\hat{Q}_{12}^{(\beta)})^2$:

$$\langle x_1, x_2 | \underbrace{\hat{Q}_{12}^{(\beta)} \quad \hat{Q}_{12}^{(\beta)}}_{\int dz_1 dz_2 |z_1, z_2\rangle\langle z_1, z_2|} |y_1, y_2\rangle \quad =$$

(flip upside down)

$$= \int dz_1 dz_2 \quad \mathcal{P}_{12} \cdot \quad \cdots \quad \cdot \mathcal{P}_{12} \cdot \quad \cdots \quad = \quad \cdots \quad = \quad \cdots$$

To obtain 2-loop "fish diagram" we multiply this by $(x_1 - x_2)^{-2\beta}$ and integrate over x_1 and y_1:

For even loops $(2N - 2)$ we have 4-point function

$$= \langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^{2N} | y_1, y_2 \rangle (y_1 - y_2)^{2\beta} =$$

$$\text{;}$$

Here we remove the propagator $1/(y_1 - y_2)^{2\beta}$.

<u>For odd loops</u> $(2N-1)$ we have 4-point function
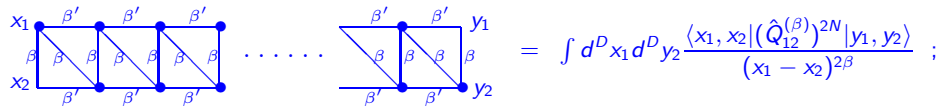
$$= \langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^{2N+1} | y_1, y_2 \rangle (y_1 - y_2)^{2\beta} =$$

.

The vertices $\bullet$ denote the integration over $\mathbb{R}^D$.

Remark 1. We stress that these Feynman integrals represent the contribution to the 4-point correlation functions in bi-scalar D-dimensional "fishnet" theory [V.Kazakov a.o. (2016,2018)]. For clarity, we present the zig-zag diagrams in the form of the spiral graphs having the cylindrical topology. We also stress that integral kernels, shown in the pictures, in the case $D = 4$ and $\beta = 1$, contribute to Green's functions of the standard $\phi_{D=4}^4$ field theory.
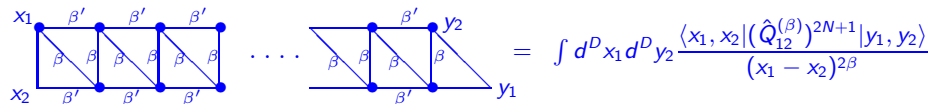
The next important statement is that $Q_{12}^{(\beta)}$ is also the graph building operator for the integrals of the planar zig-zag 2-point Feynman graphs:

1. for even number of loops $2N$

$$= \int d^D x_1 d^D y_2 \frac{\langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^{2N} | y_1, y_2 \rangle}{(x_1 - x_2)^{2\beta}} \ ;$$

2. for odd number of loops $(2N + 1)$
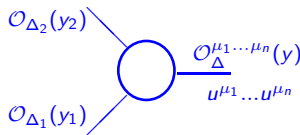
$$= \int d^D x_1 d^D y_2 \frac{\langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^{2N+1} | y_1, y_2 \rangle}{(x_1 - x_2)^{2\beta}}$$

Below we use these representations to evaluate exactly the corresponding classes of 2-point and 4-point Feynman diagrams. For this we need to find eigenvalues and complete set of eigenvectors for GBO $\hat{Q}_{12}^{(\beta)}$.

Remark 2. The elements $H_\beta := \mathcal{P}_{12} \, \hat{Q}_{12}^{(\beta)} \equiv (\hat{p}_1)^{-2\beta} (\hat{q}_{12})^{-2\beta}$ form a commutative set of operators $[H_\alpha, \, H_\beta] = 0 \ (\forall \alpha, \beta)$.

To find eigenvectors for the graph building operator $Q_{12}^{(\beta)}$ we consider the standard 3-point correlation function (in D-dimensional CFT) of three fields $\mathcal{O}_{\Delta_1}$, $\mathcal{O}_{\Delta_2}$ and $\mathcal{O}_\Delta^{\mu_1\cdots\mu_n}$, where $\mathcal{O}_{\Delta_1}$, $\mathcal{O}_{\Delta_2}$ are scalar fields with conf. dimensions Δ_1, Δ_2, while $\mathcal{O}_\Delta^{\mu_1\cdots\mu_n}$ – (symmetric, traceless and transverse) tensor field with conf. dimension Δ. The conformally invariant expression of this correlation function (up to a normalization) is unique and well known [V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova, I.T. Todorov (1976,1977); E.S.Fradkin , M.Y.Palchik (1978);...]

$$= u^{\mu_1}...u^{\mu_n}\,\langle \mathcal{O}_{\Delta_1}(y_1)\,\mathcal{O}_{\Delta_2}(y_2)\,\mathcal{O}_\Delta^{\mu_1\cdots\mu_n}(y)\rangle =$$

$$= \frac{\left(\frac{(u,y-y_1)}{(y-y_1)^2} - \frac{(u,y-y_2)}{(y-y_2)^2}\right)^n}{(y_1-y_2)^{2\eta}(y-y_1)^{2\delta}(y-y_2)^{2\rho}}\,,$$

where $u \in \mathbb{C}^D$ such that $(u,u) = u^\mu u^\mu = 0$ and

$$\eta = \frac{1}{2}(\Delta_1 + \Delta_2 - \Delta + n)\,, \quad \delta = \frac{1}{2}(\Delta_1 + \Delta - \Delta_2 - n)\,, \quad \rho = \frac{1}{2}(\Delta_2 + \Delta - \Delta_1 - n)\,.$$

The eigenvectors for the graph building operator (GBO) are the special form of the 3-point function (conformal triangle) when parameters $\Delta, \Delta_1, \Delta_2$ are expressed via only two numbers $\alpha \in \mathbb{C}$, $\beta \in \mathbb{R}$:

$$\Delta_1 = \frac{D}{2}, \quad \Delta_2 = \frac{D}{2} - \beta, \quad \Delta = D - 2\alpha - \beta + n,$$

so we have for conformal triangle:

$$\langle y_1, y_2 | \Psi_{\alpha,\beta}^{(n,u)}(y) \rangle := \frac{\left(\frac{(u, y - y_1)}{(y - y_1)^2} - \frac{(u, y - y_2)}{(y - y_2)^2}\right)^n}{(y_1 - y_2)^{2\alpha}(y - y_1)^{2\alpha'}(y - y_2)^{2(\alpha+\beta)'}}.$$

Proposition 1. *The wave function* $|\Psi_{\alpha,\beta}^{(n,u)}(y)\rangle = u^{\mu_1} \cdots u^{\mu_n} |\Psi_{\alpha,\beta}^{\mu_1 \cdots \mu_n}(y)\rangle$ *($\forall \alpha, \beta \in \mathbb{C}$) is the eigenvector for the GBO*

$$\hat{Q}_{12}^{(\beta)} |\Psi_{\alpha,\beta}^{(n,u)}(y)\rangle = \tau(\alpha, \beta, n) |\Psi_{\alpha,\beta}^{(n,u)}(y)\rangle,$$

with the eigenvalue

$$\tau(\alpha, \beta, n) = (-1)^n \pi^{D/2} \frac{\Gamma(\beta)\Gamma(\alpha)\Gamma((\alpha+\beta)' + n)}{\Gamma(\beta')\Gamma(\alpha' + n)\Gamma(\alpha + \beta)}.$$

The analogous statement, for $D = 4$ and $\beta = 1$, was made by [N.Gromov, V.Kazakov and G.Korchemsky (2018)].

Now we modify the scalar product in $V_1 \otimes V_2$

$$\langle \overline{\Psi}|\Phi\rangle := \langle\Psi|\, U\, |\Phi\rangle = \int d^4x_1 d^4x_2 \frac{\Psi^*(x_2\,,x_1)\,\Phi(x_1\,,x_2)}{(x_1-x_2)^{2\beta}}\,,$$

where $\beta \equiv D - \Delta_1 - \Delta_2$, since with respect to this new scalar product the operator $\hat{Q}_{12}^{(\beta)}$ is Hermitian and U is the new metric in $V_1 \otimes V_2$.

Moreover we should impose additional conditions on the parameters α, β and Δ:

$$2(\alpha^* + \alpha) = 2n + D - 2\beta \quad \Rightarrow \quad \boxed{\alpha = \frac{1}{2}\left(n + D/2 - \beta\right) - i\nu}, \quad \nu \in \mathbb{R}\,.$$

Thus, the eigenvalue problem for $\hat{Q}_{12}^{(\beta)}$ is characterized by two real numbers $\beta, \nu \in \mathbb{R}$ and we have $\Delta = \frac{D}{2} + 2i\nu$.

Remarkable fact: under these conditions, the GBO eigenvalue is real

$$\tau(\alpha, \beta, n) = (-1)^n \frac{\pi^{D/2}\Gamma(\beta)\,\Gamma(\frac{D}{4} + \frac{n}{2} - \frac{\beta}{2} + i\nu)\,\Gamma(\frac{D}{4} + \frac{n}{2} - \frac{\beta}{2} - i\nu)}{\Gamma(\beta')\,\Gamma(\frac{D}{4} + \frac{n}{2} + \frac{\beta}{2} + i\nu)\,\Gamma(\frac{D}{4} + \frac{n}{2} + \frac{\beta}{2} - i\nu)} \in \mathbb{R}.$$

In view of conditions on α, β, we introduce concise notation

$$|\Psi_{\nu,\beta,y}^{(n,u)}\rangle := |\Psi_{\alpha,\beta}^{(n,u)}(y)\rangle = u^{\mu_1}\cdots u^{\mu_n}|\Psi_{\alpha,\beta}^{\mu_1\cdots\mu_n}(y)\rangle,$$

$$\Psi_{\nu,\beta,y}^{(n,u)}(x_1, x_2) := \langle x_1, x_2|\Psi_{\nu,\beta,y}^{(n,u)}\rangle.$$

Since the eigenvalue τ is real (it is invariant under the transformation $\nu \to -\nu$), two eigenvectors $|\Psi_{\nu,\beta,x}^{(n,u)}\rangle$ and $|\Psi_{\lambda,\beta,y}^{(m,v)}\rangle$, having different eigenvalues τ (e.g. $n \neq m$ and $\lambda \neq \pm\nu$), should be orthogonal to each other with respect to the new scalar product. Indeed, we have the following orthogonality condition for two conformal triangles (see, e.g., [V.K. Dobrev, G. Mack, I.T.Todorov, M.C.Mintchev, V.B.Petkova (1976-1978); N. Gromov, V. Kazakov, and G. Korchemsky (2019)])

$$\overline{\langle\Psi_{\lambda,\beta,y}^{(m,v)}|}\Psi_{\nu,\beta,x}^{(n,u)}\rangle = \int d^D x_1\, d^D x_2\, \langle\Psi_{\lambda,\beta,y}^{(m,v)}|U|x_1 x_2\rangle\langle x_1 x_2|\Psi_{\nu,\beta,x}^{(n,u)}\rangle =$$

$$= \int d^D x_1 \, d^D x_2 \frac{(\Psi_{\lambda,\beta,y}^{(m,v)}(x_2\,,x_1))^* \, \Psi_{\nu,\beta,x}^{(n,u)}(x_1\,,x_2)}{(x_1-x_2)^{2(D-\Delta_1-\Delta_2)}} =$$

$$= \delta_{nm} C_1(n\,,\nu)\, \delta_{nm}\, \delta(\nu-\lambda)\, \delta^D(x-y)\,(u,v)^n +$$

$$+ \, C_2(n\,,\nu)\, \delta_{nm}\, \delta(\nu+\lambda)\, \frac{\left((u,v) - 2\frac{(u,x-y)(v,x-y)}{(x-y)^2}\right)^n}{(x-y)^{2(D/2+2i\nu)}}, \quad (3)$$

where $(u,v) = u^\mu v^\mu$, $\beta = D - \Delta_1 - \Delta_2 = \Delta_1 - \Delta_2$ and

$$C_1(n\,,\nu) = \frac{(-1)^n \, 2^{1-n} \, \pi^{3D/2+1} \, n! \, \Gamma(2i\nu)\, \Gamma(-2i\nu)}{\Gamma\left(\frac{D}{2}+n\right) \left(\left(\frac{D}{2}+n-1\right)^2 + 4\nu^2\right) \Gamma\left(\frac{D}{2}+2i\nu-1\right) \Gamma\left(\frac{D}{2}-2i\nu-1\right)} \quad (4)$$

We note that the coefficient C_1 is independent on β and plays the important role as the inverse of the Plancherel measure used in the completeness condition (resolution of unity); see below. In contrast to this, the coefficient C_2 in (3) depends on β, but the explicit form for C_2 will not be important for us.

$$C_2(n,\nu) = 2\pi^{D+1} \frac{n!}{2^n} \frac{\Gamma\left(\frac{D}{4} - \frac{\Delta_1 - \Delta_2}{2} + \frac{n}{2} - i\nu\right)}{\Gamma\left(\frac{D}{4} - \frac{\Delta_1 - \Delta_2}{2} + \frac{n}{2} + i\nu\right)} \frac{\Gamma\left(\frac{D}{4} + \frac{\Delta_1 - \Delta_2}{2} + \frac{n}{2} - i\nu\right)}{\Gamma\left(\frac{D}{4} + \frac{\Delta_1 - \Delta_2}{2} + \frac{n}{2} + i\nu\right)} \cdot$$

$$\frac{\Gamma\left(2i\nu\right)\Gamma\left(\frac{D}{2} + 2i\nu - 1 + n\right)}{\Gamma\left(\frac{D}{2} + n - 2i\nu\right)\Gamma\left(\frac{D}{2} + 2i\nu - 1\right)\Gamma\left(\frac{D}{2} + n\right)} \quad (5)$$

Completeness (or resolution of unity I) for the basis of the eigenfunctions $|\Psi_{\nu,\beta,x}^{\mu_1\cdots\mu_n}\rangle$ is written as [V.K. Dobrev, G. Mack, I.T.Todorov, M.C.Mintchev, V.B.Petkova (1976-1978); N. Gromov, V. Kazakov, and G. Korchemsky (2019)]

$$I = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\nu}{C_1(n,\nu)} \int d^D x \, |\Psi_{\nu,\beta,x}^{\mu_1\cdots\mu_n}\rangle \langle \overline{\Psi_{\nu,\beta,x}^{\mu_1\cdots\mu_n}}| =$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\nu}{C_1(n,\nu)} \int d^D x \, |\Psi_{\nu,\beta,x}^{\mu_1\cdots\mu_n}\rangle \langle \Psi_{\nu,\beta,x}^{\mu_1\cdots\mu_n}| \, U \, .$$

This is main formula needed to evaluation of ZZDs.

Substitution of this resolution of unity into expressions (via GBO) for zig-zag 4-point Feynman graphs gives (here M is a number of loops)

$$G_4^{(M)}(x_1, x_2; y_1, y_2) = \langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^M \uparrow | y_1, y_2 \rangle (y_1 - y_2)^{2\beta} =$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\nu}{C_1(n, \nu)} \int d^D x \, \langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^M | \Psi_{\nu, \beta, x}^{\mu_1 \cdots \mu_n} \rangle \langle \Psi_{\nu, \beta, x}^{\mu_1 \cdots \mu_n} | U | y_1, y_2 \rangle (y_1 - y_2)^{2\beta} =$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} d\nu \, \frac{(\tau(\alpha, \beta, n))^M}{C_1(n, \nu)} \int d^D x \, \langle x_1, x_2 | \Psi_{\nu, \beta, x}^{\mu_1 \cdots \mu_n} \rangle \langle \Psi_{\nu, \beta, x}^{\mu_1 \cdots \mu_n} | y_2, y_1 \rangle, \quad (6)$$

where the integral over x in the right hand side of (6) is evaluated in terms of conformal blocks [F.A.Dolan, H.Osborn (2001,2004); H.Osborn, A.Petkou (1994)] (in four-dimensional case, this integral was considered in detail by [N. Gromov, V. Kazakov, and G. Korchemsky (2019)]).

Further we use the expression for 2-point zig-zag functions $G_2^{(M)}(x_2, y_1)$

$$G_2^{(M)}(x_2, y_1) = \int d^D x_1 d^D y_2 \frac{\langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^M | y_1, y_2 \rangle}{(x_1 - x_2)^{2\beta}} =$$

and make the same procedure as for 4-point ZZ functions: $G_2^{(M)}(x_2, y_1) =$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d\nu}{C_1(n,\nu)} \int d^D x_1 d^D y_2 \, d^D x \, \frac{\langle x_1, x_2 | (\hat{Q}_{12}^{(\beta)})^M | \Psi_{\nu,\beta,x}^{\mu_1 \cdots \mu_n} \rangle \langle \Psi_{\nu,\beta,x}^{\mu_1 \cdots \mu_n} | U | y_1, y_2 \rangle}{(x_1 - x_2)^{2\beta}} =$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} d\nu \frac{(\tau(\alpha,\beta,n))^M}{C_1(n,\nu)} \int d(x_1, y_2, x) \frac{\langle x_1, x_2 | \Psi_{\nu,\beta,x}^{\mu_1 \cdots \mu_n} \rangle \langle \Psi_{\nu,\beta,x}^{\mu_1 \cdots \mu_n} | y_2, y_1 \rangle}{(x_1 - x_2)^{2\beta} (y_1 - y_2)^{2\beta}} =$$

$$= \frac{1}{(x_2 - y_1)^{2\beta}} \frac{\Gamma(D/2 - 1)}{\Gamma(D - 2)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n + D - 2)}{2^n \Gamma(n + D/2 - 1)} \int_0^{\infty} d\nu \frac{\tau^{M+3}(\alpha,\beta,n)}{C_1(n,\nu)}, \quad (7)$$

where we apply the integral

$$\int d^D x_1 d^D y_2 \, d^D x \, \frac{\langle x_1, x_2 | \Psi_{\nu,\beta,x}^{\mu_1 \cdots \mu_n} \rangle \langle \Psi_{\nu,\beta,x}^{\mu_1 \cdots \mu_n} | y_2, y_1 \rangle}{(x_1 - x_2)^{2\beta} (y_1 - y_2)^{2\beta}} =$$

$$= \frac{(-1)^n \Gamma(n + D - 2) \Gamma(D/2 - 1)}{2^n \Gamma(n + D/2 - 1) \Gamma(D - 2)} \frac{\tau^3(\alpha,\beta,n)}{(x_2 - y_1)^{2\beta}}. \quad (8)$$

The integral over ν in the right hand side of (7) for $\beta = 1$ and even $D > 2$ can be evaluated explicitly and gives the linear combination of ζ-values with rational coefficients.

To prove Broadhurst and Kreimer conjecture we need to consider the special case $\beta = 1$, $D = 4$. In this case $\alpha = \frac{n+1}{2} - i\nu$ and GBO eigenvalue is simplified

$$\tau(\nu, n) := \tau(\alpha, \beta, n)|_{D=4, \beta=1} = \frac{(-1)^n (2\pi)^2}{(1+n)^2 + 4\nu^2} \ .$$

The coefficient C_1 in the definition of the Plancherel mesure for $\beta = 1$, $D = 4$ is reduced to

$$C_1(n, \nu) = \frac{\pi^5}{2^{n+3}(1+n)\,\nu^2} \, \tau(\nu, n) \ .$$

Finally we substitute $\tau(\nu, n)$, $C_1(n, \nu)$ into (7), integrate over ν and obtain

$$G_2(x_2, y_1)|_{D=4, \beta=1} \quad = \quad \frac{4\pi^{2M}}{(x_2 - y_1)^2} \, C_M \sum_{n=0}^{\infty} (-1)^{n(M+1)} \frac{1}{(n+1)^{2M-1}} \quad , \quad (9)$$

where $C_M = \frac{1}{(M+1)} \binom{2M}{M}$ is a Catalan number. The relation (9) is equivalent the Broadhurst and Kreimer formula for the M loop zig-zag diagram (it corresponds to the $(M+1)$ loop contribution to the β-function in $\phi_{D=4}^4$ theory).

The generalization of the graph building operator is

$$Q_{12}^{(\zeta,\kappa,\gamma)} := \frac{1}{a(\kappa)a(\gamma)}\, \mathcal{P}_{12}\, \hat{q}_{12}^{-2\zeta}\, \hat{p}_1^{-2\kappa}\, \hat{p}_2^{-2\gamma}\, \hat{q}_{12}^{-2\beta}\,, \qquad \zeta+\beta = \kappa+\gamma\,.$$

We depict the integral kernel of the D-dimensional operator $Q_{12}^{(\zeta,\kappa,\gamma)}$ as follows
($\kappa' := D/2 - \kappa,\ \gamma' := D/2 - \gamma$)

$$= \langle x_1, x_2 | Q_{12}^{(\zeta,\kappa,\gamma)} | y_1, y_2 \rangle =$$

$$= \frac{1}{a(\kappa)a(\gamma)} \cdot \langle x_1, x_2 | \, \mathcal{P}_{12}\, \hat{q}_{12}^{-2\zeta}\, \hat{p}_1^{-2\kappa}\, \hat{p}_2^{-2\gamma}\, \hat{q}_{12}^{-2\beta}\, | y_1, y_2 \rangle =$$

$$= \frac{1}{(x_1-x_2)^{2\zeta}(x_2-y_1)^{2\kappa'}(x_1-y_2)^{2\gamma'}(y_1-y_2)^{2\beta}}\,.$$

Thus, the operator $Q_{12}^{(\zeta,\kappa,\gamma)}$ is the GBO for the ladder diagrams

$$= (x_1-x_2)^{2\zeta}\, \langle x_1, x_2 | (\hat{Q}_{12}^{(\zeta,\kappa,\gamma)})^{2N} | y_1, y_2 \rangle (y_1-y_2)^{2\beta},$$

Proposition 2. The eigenfunction for the operator $Q_{12}^{(\zeta,\kappa,\gamma)}$ is given by 3-point correlation function (conformal triangle)

$$\langle y_1, y_2 | \Psi_{\delta,\rho}^{(n,u)}(y)\rangle \;=\; \begin{array}{c}\text{(triangle diagram)}\end{array} \cdot \left(\frac{(u,y-y_1)}{(y-y_1)^2} - \frac{(u,y-y_2)}{(y-y_2)^2}\right)^n \;\equiv\; \begin{array}{c}\text{(triangle diagram)}\end{array}$$

where we depict the nontrivial rank-n tensor numerator as arrows on the lines (the rank is fixed by indices on the lines: $\rho \to (\rho, n)$, etc) and denote

$$2\alpha = \Delta_1 + \Delta_2 - (\Delta - n)\,, \quad 2\delta = \Delta_1 - \Delta_2 + (\Delta - n)\,, \quad 2\rho = \Delta_2 - \Delta_1 + (\Delta - n)\,,$$

i.e., conformal dimensions $\Delta, \Delta_1, \Delta_2$ are arbitrary parameters in this case. Thus, we have

$$Q_{12}^{(\zeta,\kappa,\gamma)}\, |\Psi_{\delta,\rho}^{(n,u)}(y)\rangle = \bar{\tau}(\kappa,\gamma; \delta, \alpha; n)\, |\Psi_{\delta,\rho}^{(n,u)}(y)\rangle\,.$$

where $\alpha + \rho = \kappa'$, $\alpha + \delta = \gamma'$ and $\bar{\tau}(\kappa,\gamma; \delta, \alpha; n)$ is an eigenvalue

$$\bar{\tau}(\kappa,\gamma; \delta, \alpha; n) = (-1)^n \cdot \tau(\delta', \kappa, n) \cdot \tau(\alpha, \gamma, n)\,,$$

$$\tau(\alpha, \beta, n) = (-1)^n\, \frac{\pi^{D/2}\Gamma(\beta)\Gamma(\alpha)\Gamma(\alpha' - \beta + n)}{\Gamma(\beta')\Gamma(\alpha' + n)\Gamma(\alpha + \beta)}$$

Remark 1. We introduce new notation $\beta + \zeta = -2u$ – spectral parameter, and express parameters α, δ, ρ via conf. dimensions $\Delta_{1,2}$:

$$\beta - \zeta = D - \Delta_1 - \Delta_2 \,, \quad \gamma - \zeta = D/2 - \Delta_1 \,, \quad \kappa - \zeta = D/2 - \Delta_2 \,.$$

In this case the general GBO dependes on u, Δ_1, Δ_2 and is equal (up to a normalization factor) to the R-operator [D. Chicherin, S. Derkachov, A. P. Isaev (2013)]

$$R_{\Delta_1 \Delta_2}(u) = a(\kappa)a(\gamma)Q_{12}^{(\zeta,\kappa,\gamma)} =$$

$$= \mathcal{P}_{12} \, \hat{q}_{12}^{2(u + \frac{D - \Delta_1 - \Delta_2}{2})} \hat{p}_1^{2(u + \frac{\Delta_2 - \Delta_1}{2})} \, \hat{p}_2^{2(u + \frac{\Delta_1 - \Delta_2}{2})} \, \hat{q}_{12}^{2(u + \frac{\Delta_1 + \Delta_2 - D}{2})}$$

which is conformal invariant solution of the Yang-Baxter equation

$$R_{\Delta_1 \Delta_2}(u - v) \, R_{\Delta_1 \Delta_3}(u) R_{\Delta_2 \Delta_3}(v) = R_{\Delta_2 \Delta_3}(v) \, R_{\Delta_1 \Delta_3}(u) R_{\Delta_1 \Delta_2}(u - v) \,.$$

The operator $R_{\Delta_1\Delta_2}(u)$ intertwines two spaces $V_{\Delta_1} \otimes V_{\Delta_2} \to V_{\Delta_2} \otimes V_{\Delta_1}$, where V_{Δ_i} is the space of scalar conf. fields with dimensions Δ_i. Let we have $V_{\Delta_1} \otimes V_{\Delta_2} = \sum_{\Delta,n} V_\Delta^{(n)}$, where $V_\Delta^{(n)}$ – is the space of tensor fields of rank n. Formally we have

$$R_{\Delta_1\Delta_2}(u) = \int d\mu(y) \sum_{n,\Delta} C(n,\Delta) |\Psi_{\Delta_1,\Delta_2,\Delta}^{(n,u)}(y)\rangle\langle\Psi_{\Delta_1,\Delta_2,\Delta}^{(n,u)}(y)|$$

Thus, eigenfunctions of $R_{\Delta_1\Delta_2}(u) = a(\kappa)a(\gamma)Q_{12}^{(\zeta,\kappa,\gamma)}$ should describe the fusion of two scalar conformal fields with dimensions Δ_1, Δ_2 into the composite tensor field with dimension Δ. Thus, conformal triangles are Clebsch-Gordan coefficients which produce this fusion.

Remark 2. The special case (for $D = 1$ and $\Delta_1 = \Delta_2 \equiv \frac{D}{2} - \xi$) of this R-operator $R_{\Delta_1 \Delta_2}(u)$ underlies Lipatov's integrable model of the high-energy asymptotics of multicolor QCD. Indeed, we have
(see D.Chicherin, S.Derkachov, API, JHEP 04 (2013) 020)

$$\mathcal{P}_{12} R_{12}^{(\kappa, \xi)} = \hat{q}_{12}^{2(u+\xi)} \, \hat{p}_1^{2u} \, \hat{p}_2^{2u} \, \hat{q}_{12}^{2(u-\xi)} \overset{u \to 0}{\to} 1 + u \, h_{12}^{(\xi)} + ... ,$$

$$h_{12}^{(\xi)} = 2 \ln q_{12}^2 + \hat{q}_{12}^{2\xi} \ln(\hat{p}_1^2 \, \hat{p}_2^2) \, \hat{q}_{12}^{-2\xi} ,$$

where $h_{12}^{(\xi)}$ is a local density of the Lipatov's Hamiltonian (holomorphic part of)

$$\mathcal{H} = \sum_{i=1}^{N-1} h_{i,i+1}^{(\xi)} .$$

Conclusion.

1.) In this report, we demonstrated how the progress in the investigations of the multidimensional CFT can be applied, e.g., in the analytical evaluations of massless Feynman diagrams.

2.) We believe that the approach described here gives the universal method of the evaluation of contributions into the special class of correlation functions and critical exponents in various CFT.

3.) We also wonder if it is possible to apply our D-dimensional generalizations to evaluation similar 4-points functions (with fermions) that arise in the generalized "fishnet" model, in double scaling limit of γ-deformed $N = 4$ SYM theory.

4.) Very recent paper M. Kade, M. Staudacher, Supersymmetric brick wall diagrams and the dynamical fishnet, arXiv:2408.05805 [hep-th]. Supersymmetric generalizations of Basso-Dixon fishnet and brick wall diagrams.

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