Beyond Algebraic Solutions to Stringy Spacetime

Happy birthday, Branko!

(who's counting?)

@ *Nonlinearity, Nonlocality & Ultrametricity*, Beograd, May 2025

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Gauged Linear σ Model

Playbill

- Roadmap
- QFT σ Models
- Worldsheet SuSy
- GLSM → Toric Geometry
- GLSM → Toric Geometry – “Plan B”

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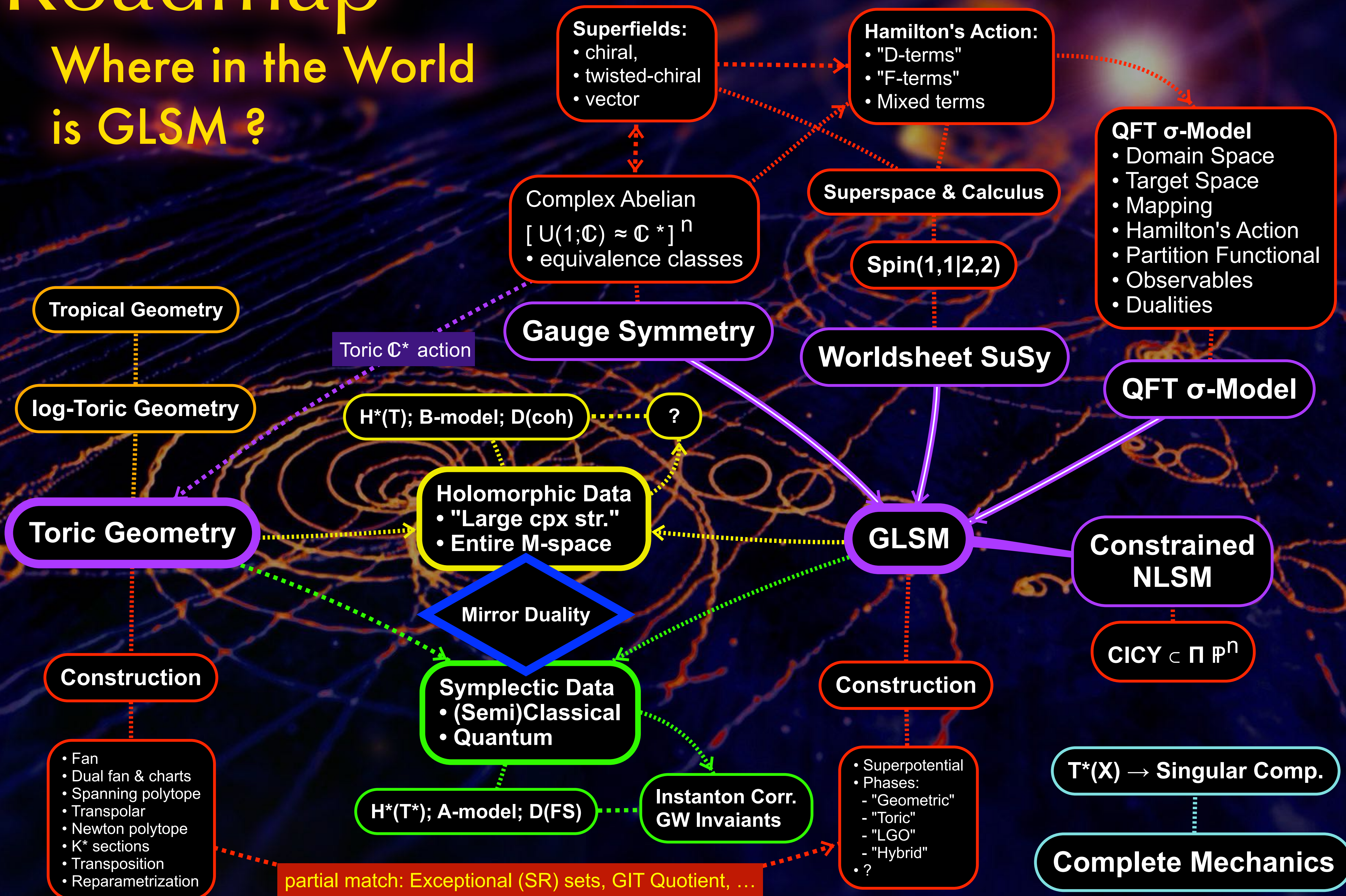
The truth, nothing but the unvarnished truth,
…but by all means — ***not all of it***!

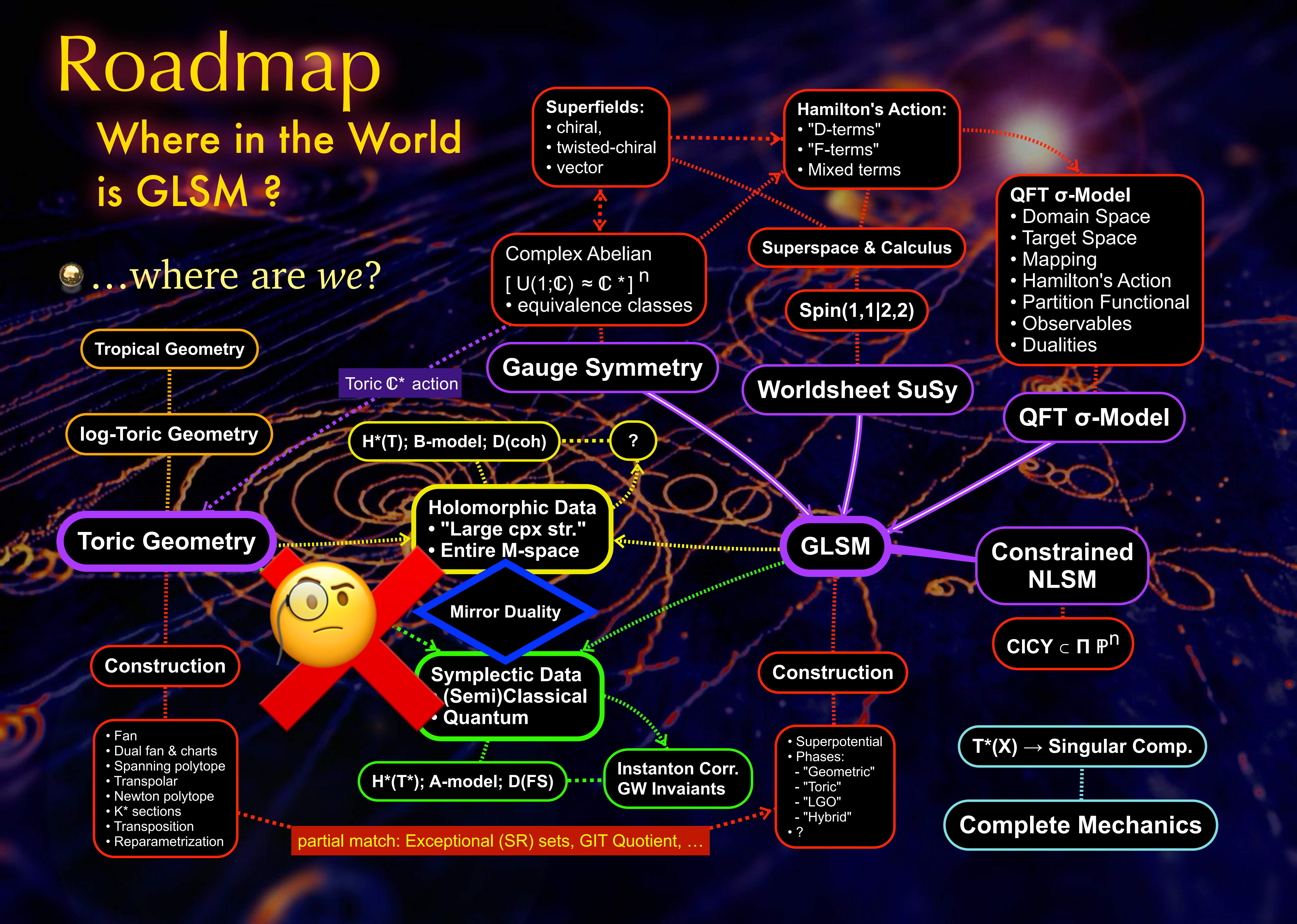
Many thanks to Per Berglund
and Mikiya Masuda

arXiv:2501.11684, arXiv:2502.08002
& refs therein

Roadmap

Where in the World is GLSM ?




Roadmap
Where in the World is GLSM ?
…where are we?
Superfields:
• chiral,
• twisted-chiral
• vector
Hamilton's Action:
• "D-terms"
• "F-terms"
• Mixed terms
QFT σ-Model
• Domain Space
• Target Space
• Mapping
• Hamilton's Action
• Partition Functional
• Observables
• Dualities
Complex Abelian
[U(1;ℂ) ≈ ℂ *] ⁿ
• equivalence classes
Superspace & Calculus
Spin(1,1|2,2)
Tropical Geometry
Toric ℂ* action
Gauge Symmetry
Worldsheet SuSy
QFT σ-Model
log-Toric Geometry
H*(T); B-model; D(coh)
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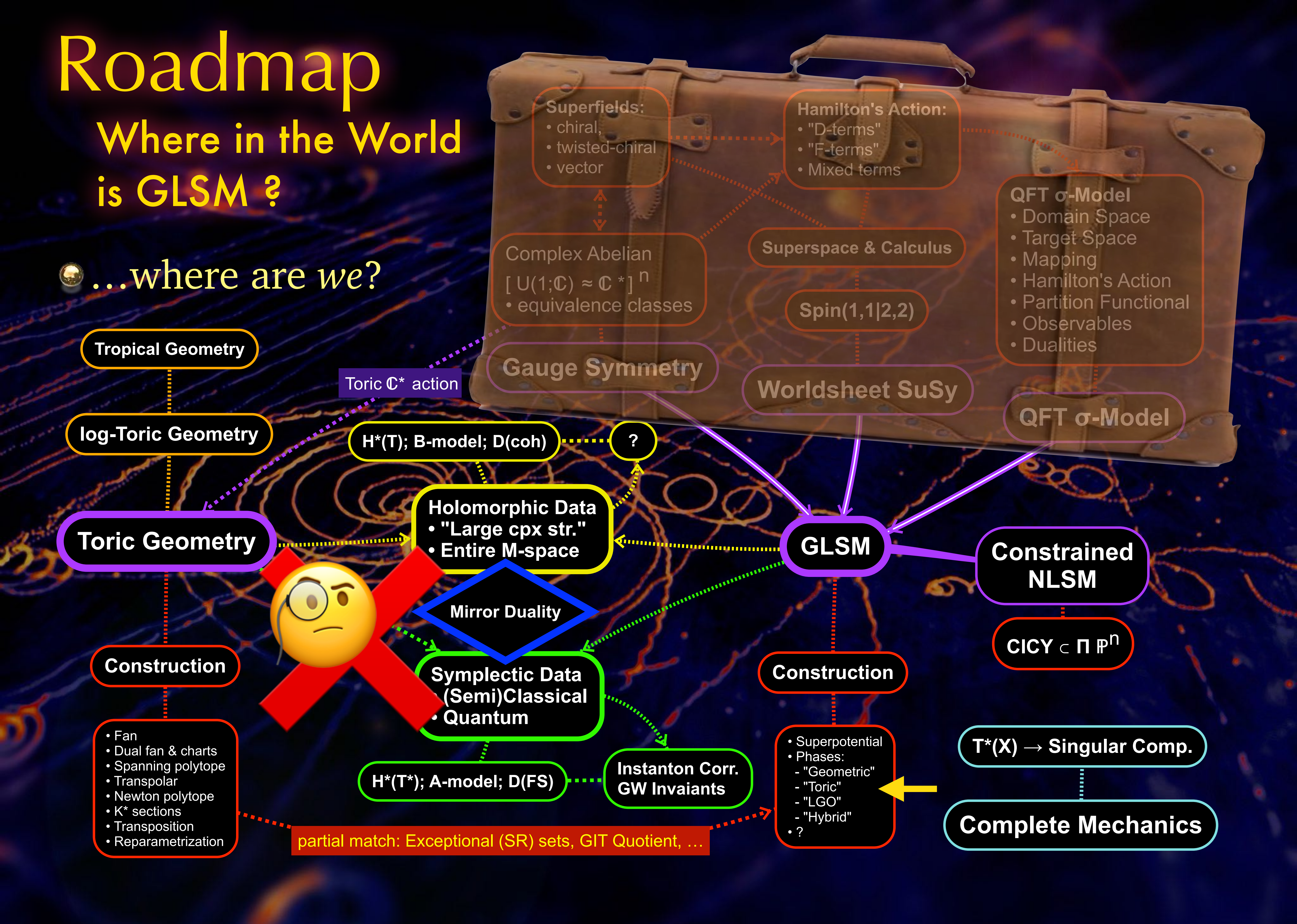
Holomorphic Data
• "Large cpx str."
• Entire M-space
Toric Geometry
GLSM
Constrained NLSM
Mirror Duality
CICY ⊂ Π ℙⁿ
Construction
Symplectic Data
• (Semi)Classical
• Quantum
Construction
• Fan
• Dual fan & charts
• Spanning polytope
• Transpolar
• Newton polytope
• K* sections
• Transposition
• Reparametrization
H*(T*); A-model; D(FS)
Instanton Corr.
GW Invaiants
• Superpotential
• Phases:
- "Geometric"
- "Toric"
- "LGO"
- "Hybrid"
• ?
T*(X) → Singular Comp.
Complete Mechanics
partial match: Exceptional (SR) sets, GIT Quotient, …



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QFT σ Models

A Bird's-Eye View

QFT σ-Model
- Domain Space
- Target Space
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QFT σ-Model

in cl.mech.: $\mathbb{R}^1_\tau$

- Domain space: Riemann surface, Σ_g, locally $\sim\mathbb{R}^{1,1}_{\tau,\sigma}$ w/BC's

QFT σ Models

A Bird's-Eye View

- Domain space: Riemann surface, Σ_g, locally $\sim \mathbb{R}^{1,1}_{\tau,\sigma}$ w/BC's
- Target space: Lorentzian space(time), such as $\mathbb{R}^{1,9}$
- Mapping: "coordinate fields," $X^\mu(\xi) : \Sigma_g \to \mathbb{R}^{1,9}$
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  ↔ Euler-Lagrange EoM

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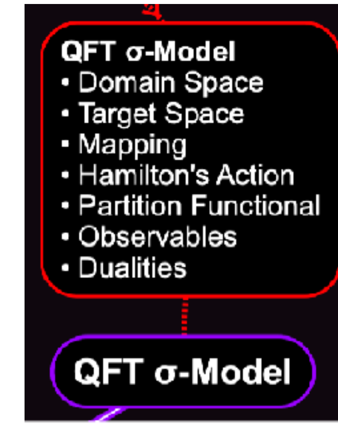

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  - Whence Feynman's "path integral"

$$Z[G_{\mu\nu}, \dots] := \iint_{X^\mu(\xi):\ \Sigma_{g,\gamma} \to \mathscr{X}} \mathbf{D}[X]\, e^{-iS[X^\mu; \gamma_{ij}, G_{\mu\nu}, \dots]/\hbar}$$

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"Dynamically" Determined Target space

$$Z[G_{\mu\nu},\dots] := \iint \mathbf{D}[X]\, e^{-iS[X^\mu;\gamma_{ij},G_{\mu\nu},\dots]/\hbar}$$

$X^\mu(\xi):\ \Sigma_{g,\gamma}\to\mathscr{X}$

Better strategy: specify $S[X;\dots]$ by symmetries & "analytic" properties

QFT σ Models

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- If $Z[G_{\mu\nu},\dots] = e^{-iS_{\text{eff}}[\bar{x}^\mu;\,\bar{\gamma}_{ij}(\xi),\,\widetilde{G}_{\mu\nu}(x),\dots]/\hbar}$, "renormalized" $G \to \widetilde{G}$
  - "Renormalization" is computed iteratively → iterations = "flow"
  - "Renormalization ~~group~~ flow" has fixed points → "quantum stability"

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    - Subsequently generalized, reproduces all gauge interactions in the target-space field theory

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extending Ehrenfest's Theorem

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- & manifestly T-dual (→ mirror-symmetric) formulation of string theory…

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QFT σ Models

The Magic of String Theory

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$\{(\mathscr{F},\mathscr{G}_{AB});\ \ldots\}$

moduli-space QFT

***classical* Equations of Motion**
= quantum stability

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(broken supersymmetric) target-spacetime QFT

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superconformal worldsheet QFT

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  - symmetries & anomaly cancellations
  - (seem to) restrict the space of models to finite-volume spaces
  - UV/IR mixing & pheno predictions [→ arXiv:2407.06207]

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- that may contain our World 😇

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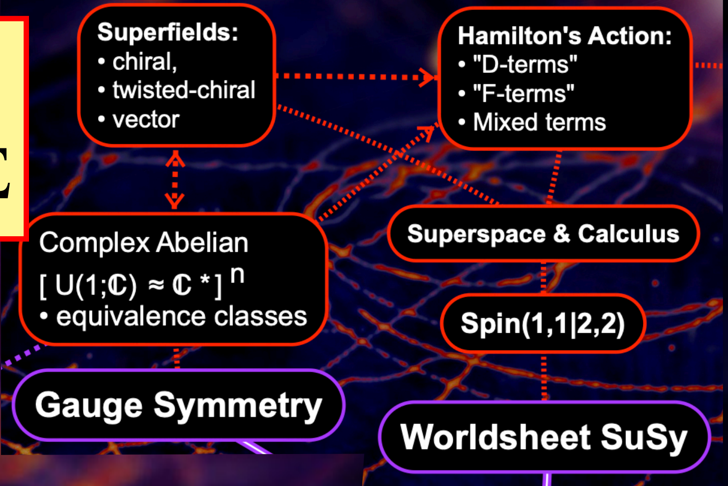
Worldsheet SuSy

A Telegraphic Summary

$$\Phi \to e^{-iq_\Phi \Theta}\Phi$$
$$[V \simeq V + i(\Theta - \overline{\Theta})] \leftrightarrow \Sigma$$



- Superfields = "functions" of $(\xi^{\pm\pm} \,|\, \varsigma^{\pm}, \bar{\varsigma}^{\pm})$, $\varsigma^2 = 0$
  - ***Reducible***: $\Phi : \bar{D}_\pm \Phi = 0$ & $\bar{D}_+ \Sigma = 0 = D_- \Sigma$ ("haploid" = 2×"quartoid")

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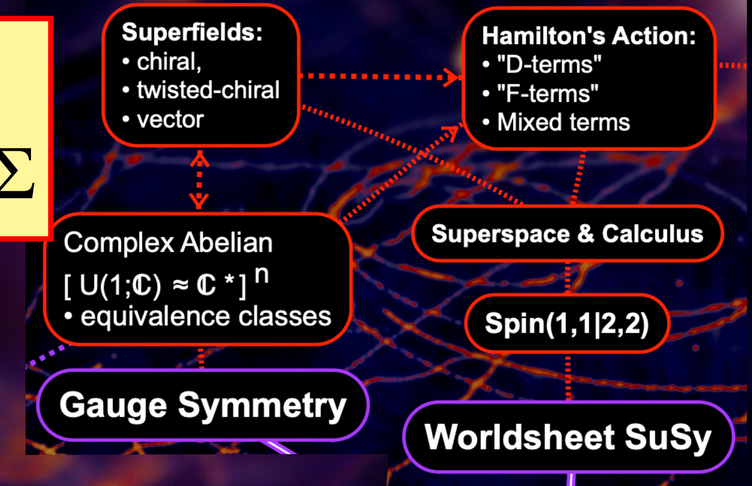

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- ***One more thing***:
  - (Twisted-)chiral superfield close under multiplication
  - If $\bar{D}_{\pm}\Phi = 0$, then $\bar{D}_{\pm}(\Phi_1\Phi_2) = 0$; also, $\bar{D}_{\pm}1 = 0$; — "chiral ring"
  - If $\bar{D}_{+}\Sigma = 0 = D_{-}\Sigma$, then $\bar{D}_{+}(\Sigma_1\Sigma_2) = 0 = D_{-}(\Sigma_1\Sigma_2)$; — "tw.-chiral ring"

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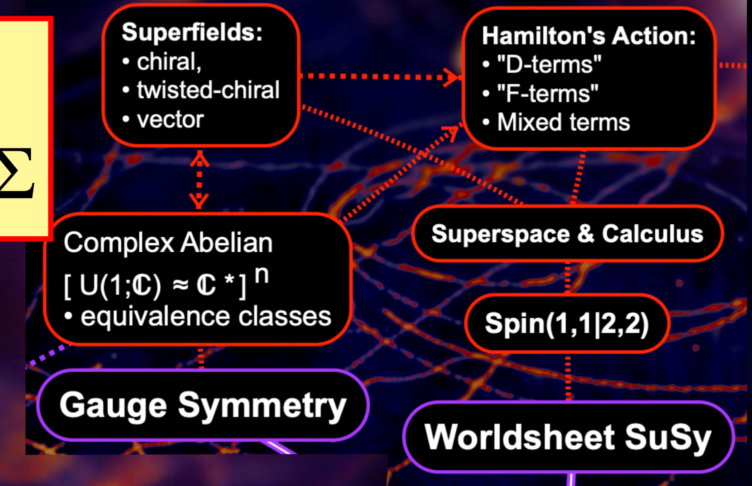

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- ***And another***: $\int \mathrm{d}^2\varsigma\, W(\Phi) + \text{h.c.}$ is supersymmetric — "F-term"
- & $\int \mathrm{d}^4\varsigma\, \overline{\Phi}\, e^{q_\Phi V}\Phi = \int \mathrm{d}^4\varsigma\, \overline{\Phi}\,\Phi + \ldots$ — "D-terms" + "Φ-Σ mixing."

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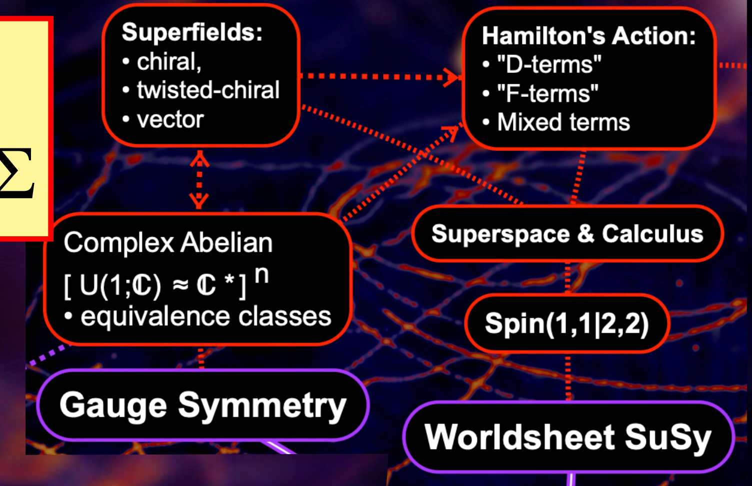

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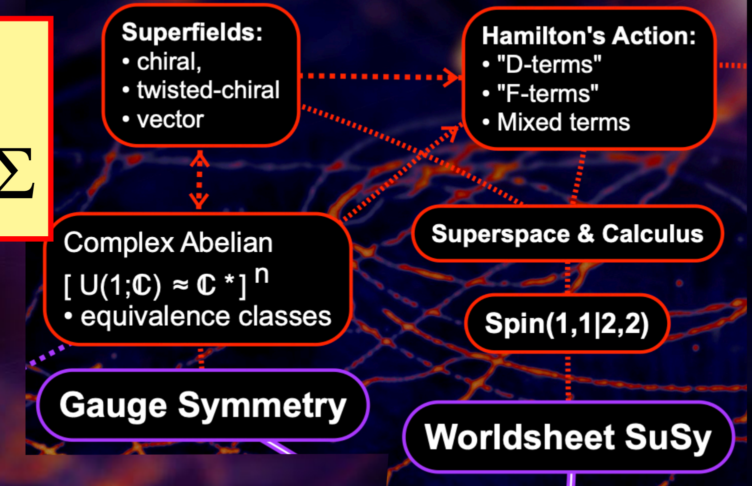
  - ***Reducible***: $\Phi : \bar{D}_{\pm}\Phi = 0$ & $\bar{D}_{+}\Sigma = 0 = D_{-}\Sigma$ ("haploid" = 2×"quartoid")
- ***One more thing***:
  - (Twisted-)chiral superfield close under multiplication
  - If $\bar{D}_{\pm}\Phi = 0$, then $\bar{D}_{\pm}(\Phi_1\Phi_2) = 0$; also, $\bar{D}_{\pm}1 = 0$; — "chiral ring"
  - If $\bar{D}_{+}\Sigma = 0 = D_{-}\Sigma$, then $\bar{D}_{+}(\Sigma_1\Sigma_2) = 0 = D_{-}(\Sigma_1\Sigma_2)$; — "tw.-chiral ring"
- ***And another***: $\int d^2\varsigma\, W(\Phi) + \text{h.c.}$ is supersymmetric — "F-term"
- & $\int d^4\varsigma\, \overline{\Phi}\, e^{q_\Phi V}\Phi = \int d^4\varsigma\, \overline{\Phi}\,\Phi + \ldots$ — "D-terms" + "Φ-Σ mixing."
  - Now, $\int d^2\varsigma\, W(\Phi) + \text{h.c.} = \underline{F}\, W' + \ldots + \text{h.c.}$ & $\int d^4\varsigma\, \overline{\Phi}\,\Phi = \underline{\bar{F}}\, \underline{F} + \ldots$
  - So, $\delta_F\big(\int d^4\varsigma\, \overline{\Phi}\,\Phi + \int d^2\varsigma\, W + \text{h.c.}\big) = 0 \;\Rightarrow\; \underline{\bar{F}} = -W' \to PE \supset |W'|^2$
    $\to$ *Morse theory!*

Worldsheet SuSy

A Telegraphic Summary

$$\Phi \to e^{-iq_\Phi \Theta}\Phi$$
$$\big[V \simeq V + i(\Theta - \overline{\Theta})\big] \leftrightarrow \Sigma$$



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    $\to$ *Morse theory!*
- ***Also***: $\int d\varsigma^{+} d\bar{\varsigma}^{-}\, \widetilde{W}(\Sigma) + \text{h.c.}$ is supersymmetric — "tw. F-term"
  - Simplest: $t\int d\varsigma^{+} d\bar{\varsigma}^{-}\, \Sigma + \text{h.c.} = t_R\underline{\mathscr{D}} + t_I\underline{\mathscr{F}}$, with $\Sigma = [\underline{\sigma}; \bar{\lambda}_{+}, \lambda_{-}; \underline{\mathscr{D}} - i\underline{\mathscr{F}}]$

GLSM

Bear Essentials

- Chiral: $\boldsymbol{X}_i = [x; \psi_\pm; F]_i \ \to$ coordinate fields for X
- & $P = [p; \pi_\pm; F_p] \ \to$ fibre coordinate, line bundle $\mathscr{L}_X$
- Superpotential: $W = P^a f_a(\boldsymbol{X}) \ \to$ (quasi-)homogeneous, $q(p^a) = -q(f_a)$

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- $\mathrm{PE} = \big[\sum_i q_i |x_i|^2 - t_R\big]^2 + |f(x)|^2 + |p|^2 \sum_i \big|\frac{\partial f}{\partial x_i}\big|^2 + t_I^2 + |\sigma|^2 \sum_i q_i^2 |x_i|^2$

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GLSM
Bear Essentials

Systems of algebraic equations

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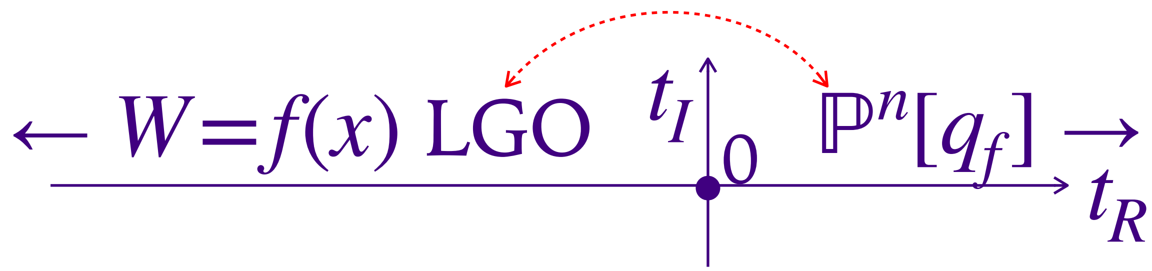
GLSM

→ Toric Geometry

- In pictures, e.g.:



GLSM

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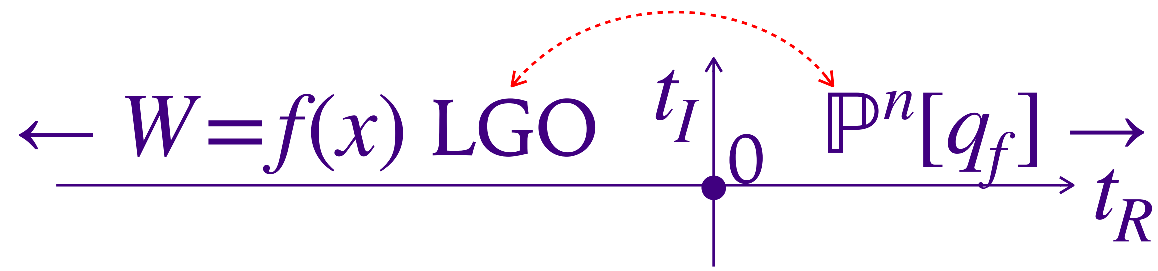
$$|f_a(x)|^2 = 0 \quad \& \quad |p_a|^2 \sum_i \left| \frac{\partial f_a}{\partial x_i} \right|^2 = 0$$



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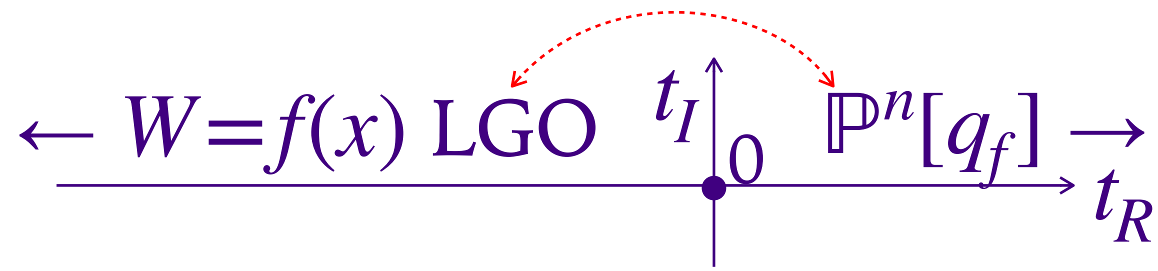

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  - ~ Hirzebruch: $\{p_0(\xi,\eta) := \xi_0\eta_0{}^m + \xi_1\eta_1{}^m = 0\} \subset \mathbb{P}^n \times \mathbb{P}^1$: $H^2(F_m^{(n)}; \mathbb{Z}) = J_1 \oplus_{\mathbb{Z}} J_2$

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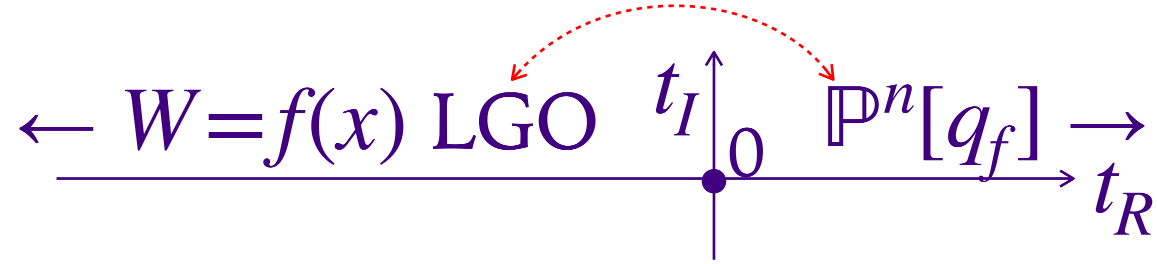

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  - So, $\vec{q}$:

$$\left[\begin{array}{r|ccccc} p & \xi_0 & \cdots & \xi_n & \eta_0 & \eta_1 \\ \hline -(n+1) & 1 & \cdots & 1 & 0 & 0 \\ -2 & 0 & \cdots & 0 & 1 & 1 \end{array}\right] \xrightarrow{\text{toric}} \left[\begin{array}{r|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right] \;\&\; (r_1, r_2)$$

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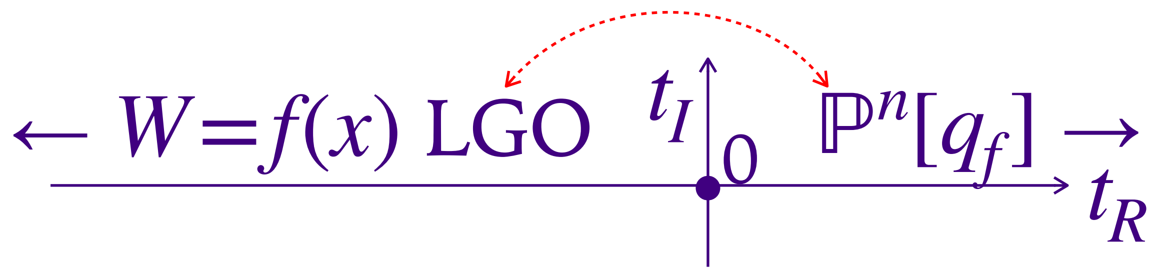

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> The deg-$\binom{1}{-m}$ (equivalence class of) section(s):
>
> $$x_1 := \left[\Big(\frac{\xi_0}{\eta_1{}^m} - \frac{\xi_1}{\eta_0{}^m}\Big) \bmod \frac{p_0(\xi,\eta)}{(\eta_0\eta_1)^m}\right] = \begin{cases} +2\xi_0/\eta_1{}^m, & \eta_1 \neq 0; \\ -2\xi_1/\eta_0{}^m, & \eta_0 \neq 0. \end{cases}$$
>
> (—just like the Wu-Yang magnetic monopole!)

GLSM

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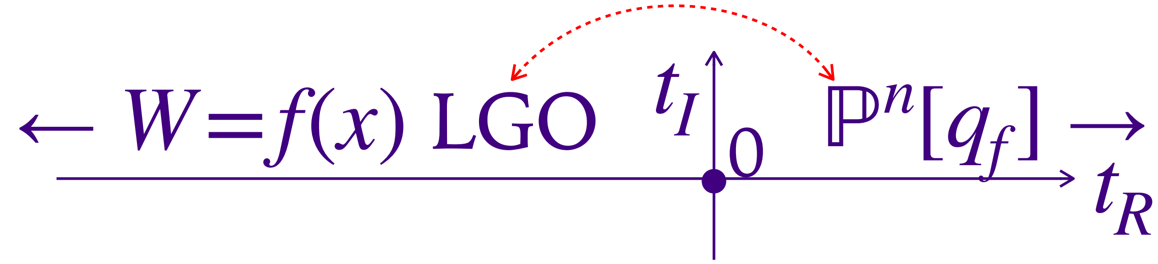

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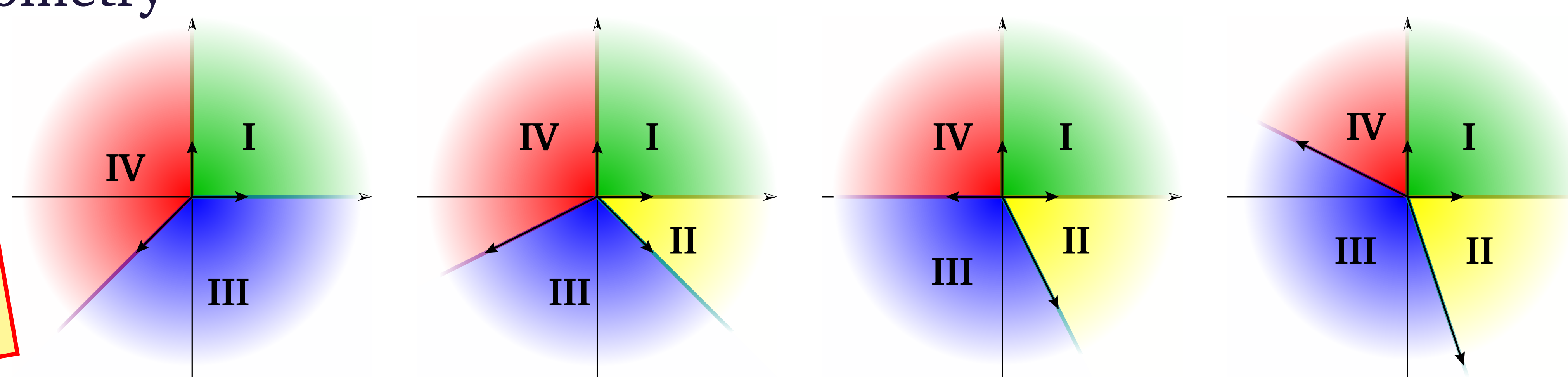
  - I & II = "geometry"
  - III = LGO
  - IV = hybrid fibre→LGO

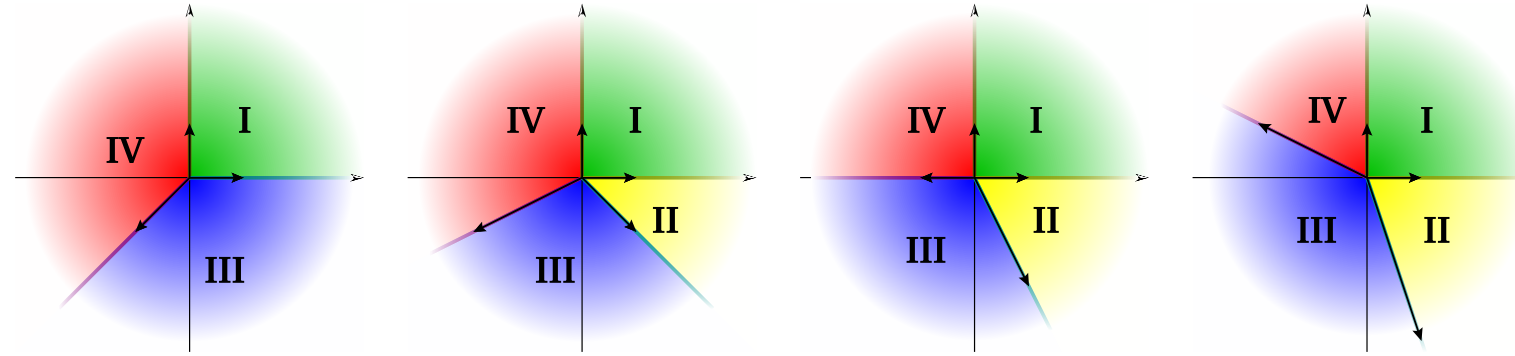
LGO ⊂ GLSM



$m = 0$ $\quad$ $m = 1$ $\quad$ $m = 2$ $\quad$ $m = 3$

GLSM

→ Toric Geometry



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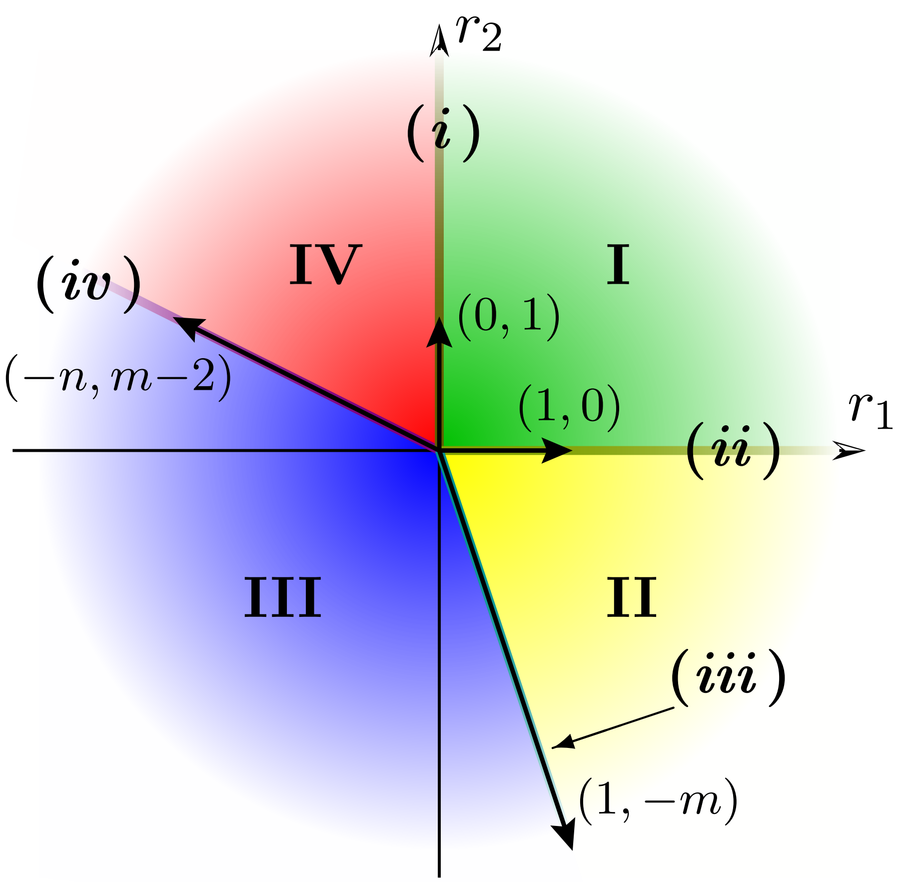
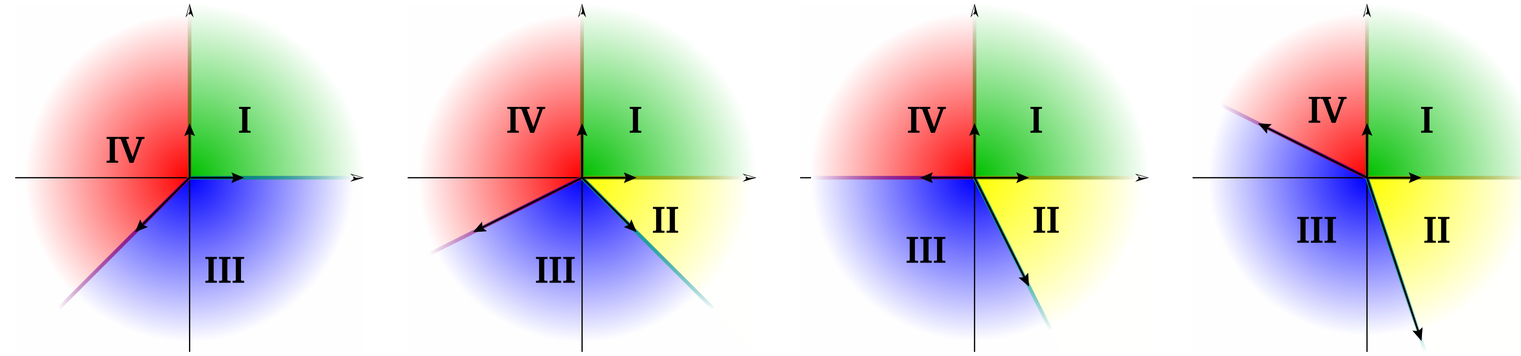
- The "*phase diagram*" = "*secondary fan*"
  - determines the $\langle x_i \rangle$ pattern:



	$\lvert x_0 \rvert$	$\lvert x_1 \rvert$	$\lvert x_2 \rvert$	$\cdots$	$\lvert x_n \rvert$	$\lvert x_{n+1} \rvert$	$\lvert x_{n+2} \rvert$
i	0	0	0	$\cdots$	0	$*$	$*$
I	0	$*$	$*$	$\cdots$	$*$	$*$	$*$
ii	0	0	$*$	$\cdots$	$*$	0	0
II	0	see (2.9)	$*$	$\cdots$	$*$	$*$	$*$
iii	0	$\sqrt{r_1}$	0	$\cdots$	0	0	0
III	$\sqrt{\frac{mr_1+r_2}{(n-1)m+2}}$	$\sqrt{\frac{(m-2)r_1+nr_2}{(n-1)m+2}}$	0	$\cdots$	0	0	0
iv	$\sqrt{-r_1/n}$	0	0	$\cdots$	0	0	0
IV	$\sqrt{-r_1/n}$	0	0	$\cdots$	0	$*$	$*$

GLSM

→ Toric Geometry

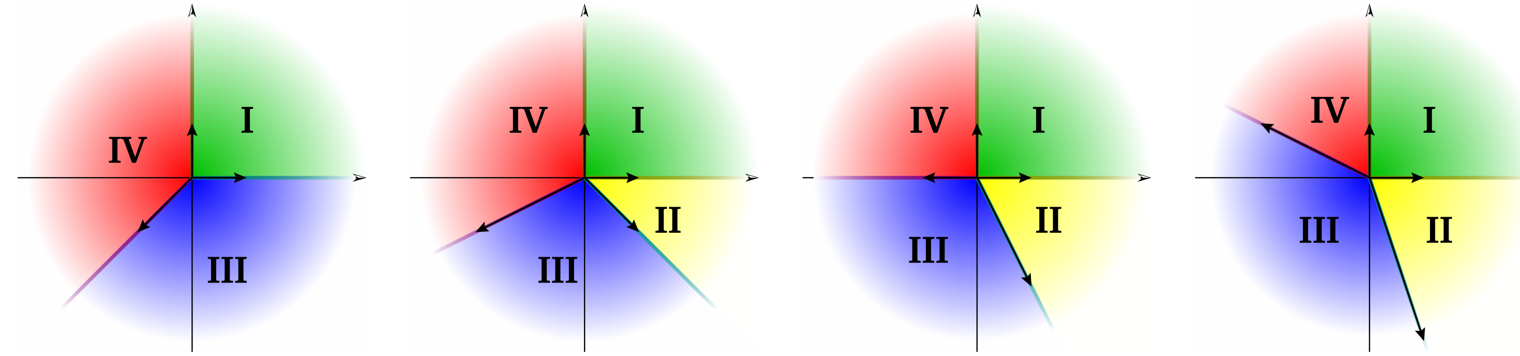


$m = 0$ $m = 1$ $m = 2$ $m = 3$

- The "*phase diagram*" = "*secondary fan*"
  - is the toric rep. of the (enlarged/complete) "Kähler structure"

GLSM

→ Toric Geometry



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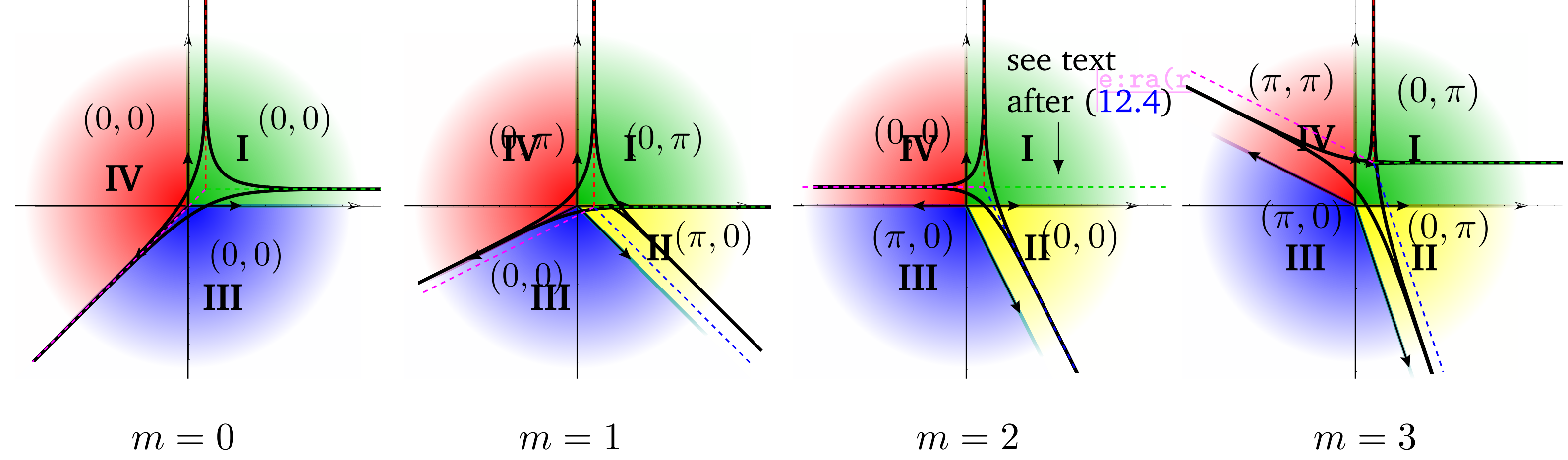
  - "Tropical geometry" ↔ "large cpx str." $\xleftrightarrow{\text{mirror}}$ "large Kähler class"
    - the (semiclassical) space/diagram of GLSM "phases"

GLSM

→ Toric Geometry



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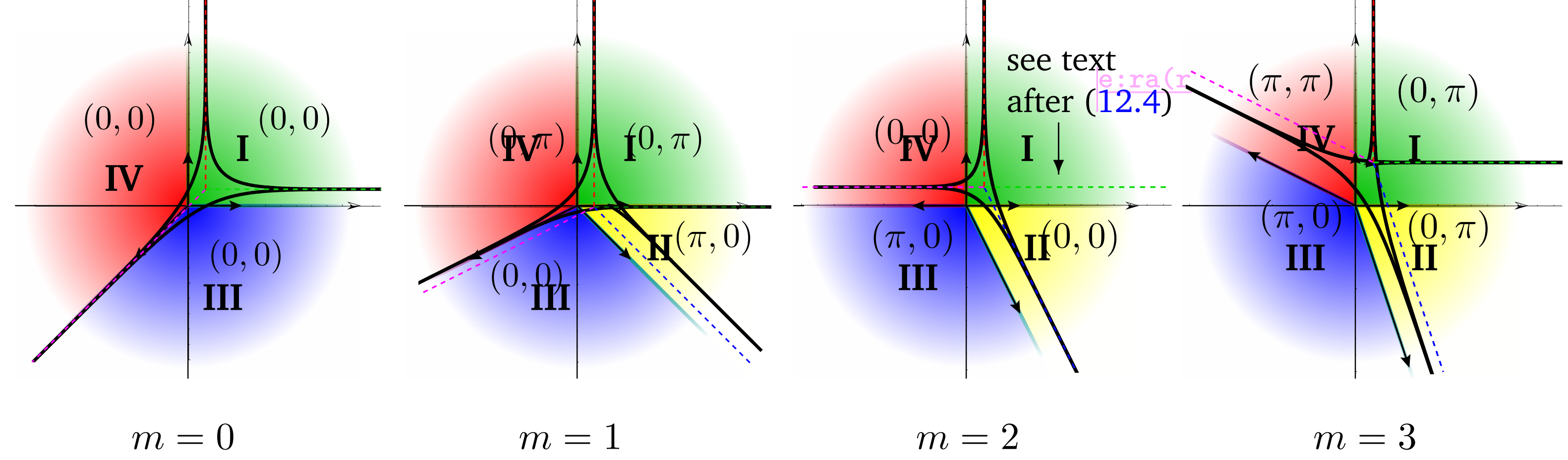
    - the (semiclassical) space/diagram of GLSM “phases”
  - becomes modified by “worldsheet instantons” (cumulative effects)
    - …with shifts and “thickening” of the diagram → “amoebas”
    - (→ “A discriminants,” “Horn uniformization,” … ✅)

GLSM

→ Toric Geometry



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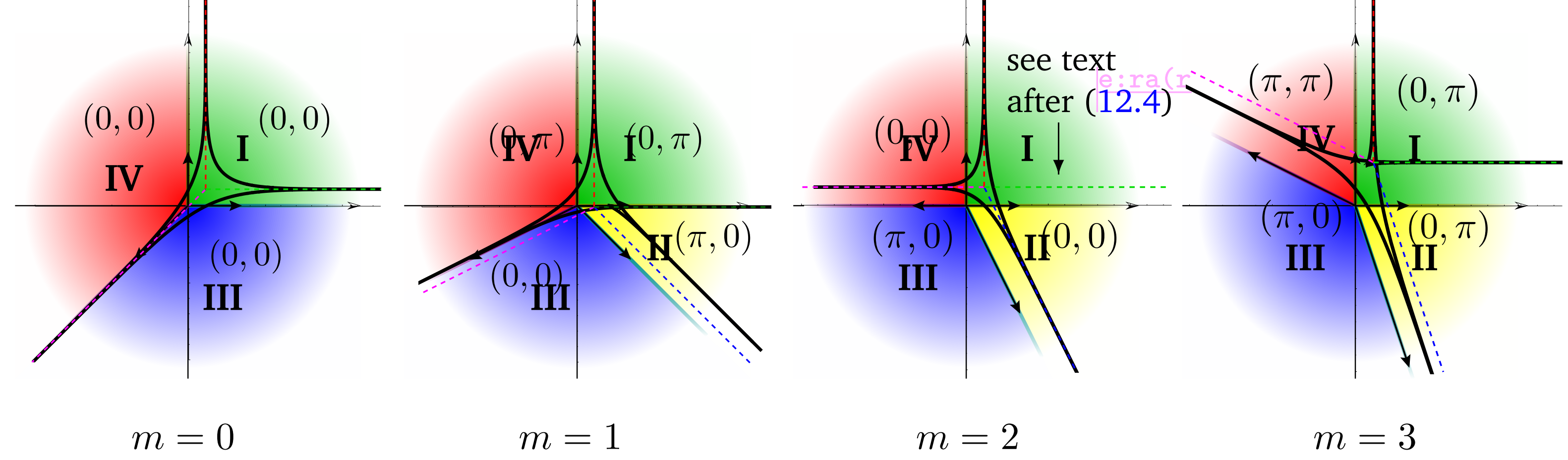
    - (→ “A discriminants,” “Horn uniformization,” … ✅)
  - “log-geometry” ↔ “smallish cpx str.” $\xleftrightarrow{\text{mirror}}$ “small Kähler class”
    - quantum corrections → “quantum cohomology”

GLSM

→ Toric Geometry



- The "*phase diagram*" = "*secondary fan*"
  - is the toric rep. of the (enlarged/complete) "Kähler structure"
  - "Tropical geometry" ↔ "large cpx str." $\xleftrightarrow{\text{mirror}}$ "large Kähler class"
    - the (semiclassical) space/diagram of GLSM "phases"
  - becomes modified by "worldsheet instantons" (cumulative effects)
    - ...with shifts and "thickening" of the diagram → "amoebas"
    - (→ "A discriminants," "Horn uniformization," … ✅)
  - "log-geometry" ↔ "smallish cpx str." $\xleftrightarrow{\text{mirror}}$ "small Kähler class"
    - quantum corrections → "quantum cohomology"
- Mirrors the "complex structure" w/"discriminant locus" ✅
  - where the ground-state ("target") space singularizes

GLSM

→ Toric Geometry

$$\mathbb{P}^n: \left[\begin{array}{c|cccc} p & x_0 & x_1 & \cdots & x_n \\ \hline -(n+1) & 1 & 1 & \cdots & 1 \end{array}\right]$$

$$F_m^{(n)}: \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right]$$

- From the $U(1;\mathbb{C})^n$-charges, q_i^a, def. $\vec{\nu}_i \in (N \approx \mathbb{Z}^n)$: $\sum_i q_i^a \vec{\nu}_i = 0$
  - $\vec{\nu}_i \in \Sigma$ (spanning) fan, up to $\mathrm{GL}(n;\mathbb{Z})$ lattice automorphisms

$\mathbb{P}^4$	ν_0	ν_1	ν_2	ν_3	ν_4
$\Delta^\star_{\mathbb{P}^4}$	-1	1	0	0	0
	-1	0	1	0	0
	-1	0	0	1	0
	-1	0	0	0	1

$F_m^{(4)}$	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6
$\Delta^\star_{F_m^{(4)}}$	-1	1	0	0	0	$-m$
	-1	0	1	0	0	$-m$
	0	0	0	1	0	$-m$
	0	0	0	0	1	1

$\rightsquigarrow \Sigma \rightsquigarrow \Delta^\star$

GLSM

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- Cox variables: $\vec{\nu}_i \mapsto x_i$, then $f(x) = \sum_{\vec{\mu}_k \in \Delta} \left(a_k \prod_{\vec{\nu}_i \in \Delta^\star} x_i^{\vec{\nu}_i \cdot \vec{\mu}_k + 1}\right)$
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GLSM

→ Toric Geometry

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    - The standard polar operation ***fails*** for non-convex polytopes (non-Fano varieties)

- For non-convex, *transpolar*: $\Delta := (\Delta^\star)^\triangledown$ & $(\Delta)^\triangledown = \Delta^\star$ (in fact, oft-practiced)
- …introduces flip-folded "multifans" that star-subdivide "multitopes" 😱 😱 😱

GLSM

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not unique 😱 🧐 🤔

$\rightsquigarrow \Sigma \rightsquigarrow \Delta^\star$

multiple triangulations

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& other choices

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GLSM

→ Toric Geometry "Plan B"

$$F_m^{(n)}: \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right]$$

$x_{n+j+1} := y_j$

TH & S.-T.Yau, MPLA**7**(1992)3277–3289

- Given the q_i^a, define: $f(x) = a_0\,(\Pi x = \Pi_i x_i) + \sum_k a_k\, \Pi_i x_i^{e_{ik}}$
  - The "fundamental monomial," Πx : Calabi–Yau (Ricci-flat) hypersurface
    - Key to parametrizing the geometry/dynamics of complex structure moduli
    - The transpose-mirror, ${}^{\mathsf{T}}(\Pi x)$, is Tyurin-degenerate $= \mathscr{H} \cup \mathscr{C}$: $c_1(\mathscr{H} \cap \mathscr{C}) = 0$

arXiv:hep-th/9201014, arXiv.:1611.10300, arXiv:2205.12827

GLSM

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- choose a_k and e_{ik} so $\{f(x)=0\} \cap \{df(x)=0\} = \emptyset$ ← transverse
  - $\Pi_i x_i^{e_{ik}}$: smoothing (a_k-param.) deformations…
  - …generated by $\mathfrak{d}^i(x)\,\partial_i$, $\left(\partial_i = \frac{\partial}{\partial x_i},\ \partial_i^{\,2}(\Pi x) \equiv 0\right.$: "distance-1")
    - where $\mathfrak{d}^i(x) = \Pi_{j \neq i}\, x_j^{d_{ij}}$ — just as $\delta\,|\psi_n\rangle = \sum_{m \neq n} c_m\,|\psi_m\rangle$ in QM

GLSM

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  - The monomial "hyperplanes," $\mathfrak{d}^i(x)\,(\partial_i\,\Pi x)$, intersect iteratively
  - …forming the "Newton multitope" of anticanonical monomials, Δ.

GLSM

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  - The monomial "hyperplanes," $\mathfrak{d}^i(x)\,(\partial_i \Pi x)$, intersect iteratively
  - …forming the "Newton multitope" of anticanonical monomials, Δ.
  - The deformations, $\mathfrak{d}^i(x)\,(\partial_i \Pi x)$, span the (multi)fan, $\Sigma \lessdot \Delta^\star = (\Delta)^\triangledown$
    - The transpose-mirror, ${}^{\mathsf{T}}(\Pi x + \ldots) = y_1 \cdot \mathfrak{c}(y) + \ldots$, automatically defines an LGO-like structure with "geometric" and "quantum" symmetries swapped

GLSM

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- Construct $f(x) = a_0 \prod x + \sum_k a_k \prod_i x_i^{e_{ik}}$, with $x_3 = 0 = x_4$:

GLSM

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  - Deformation generators: $\big(\mathfrak{d}^i(x) = \prod_{j\neq i} x_j^{d_{ij}}\big)\partial_i$

$k \in \mathbb{Z}$	x_1-indep.	x_2-indep.	x_5-indep.	x_6-indep.
gen.:	$(x_2 x_5^k x_6^{-k-m})\partial_1$	$(x_1 x_5^k x_6^{m-k})\partial_2$	$(x_1^k x_2^{-k} x_6^{1+km})\partial_5$	$(x_1^k x_2^{-k} x_5^{1+km})\partial_6$
stripe:	$x_2^2 x_5^{1+k} x_6^{1-k-m}$	$x_1^2 x_5^{1+k} x_6^{1-k+m}$	$x_1^{1+k} x_2^{1-k} x_6^{2+km}$	$x_1^{1+k} x_2^{1-k} x_5^{2+km}$

GLSM

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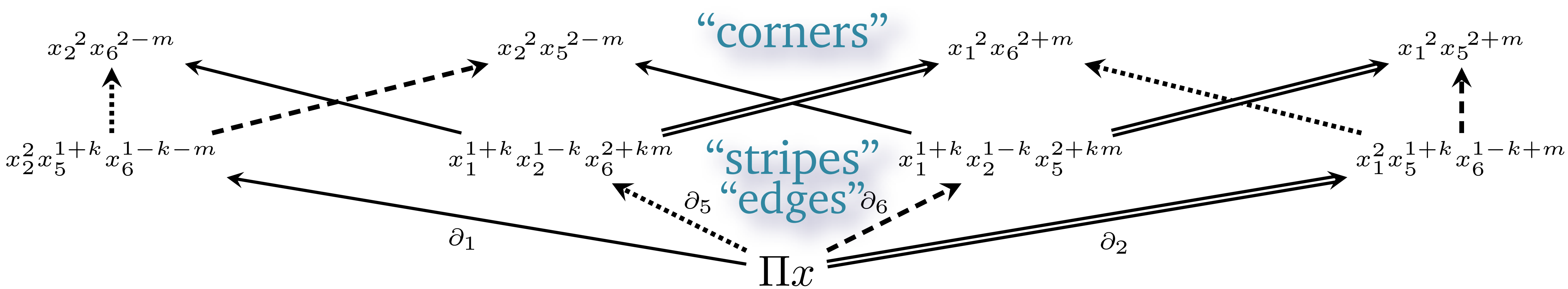
- Poset structure:



- organizes the deformations

GLSM

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$\Delta(F_1^{(2)})$

$\frac{x_1^3 x_5^4}{x_2}$	$\frac{x_1^2 x_5^4}{x_6}$	$\frac{x_1 x_2 x_5^4}{x_6^2}$		
$\frac{x_1^3 x_5^3 x_6}{x_2}$	$x_1^2 x_5^3$	$\frac{x_1 x_2 x_5^3}{x_6}$	$\frac{x_2^2 x_5^3}{x_6^2}$	
$\frac{x_1^3 x_5^2 x_6^2}{x_2}$	$x_1^2 x_5^2 x_6$	$x_1 x_2 x_5^2$	$\frac{x_2^2 x_5^2}{x_6}$	$\frac{x_2^3 x_5^2}{x_1 x_6^2}$
$\frac{x_1^3 x_5 x_6^3}{x_2}$	$x_1^2 x_5 x_6^2$	$\boxed{x_1 x_2 x_5 x_6}$	$x_2^2 x_5$	$\frac{x_2^3 x_5}{x_1 x_6}$
$\frac{x_1^3 x_6^4}{x_2}$	$x_1^2 x_6^3$	$x_1 x_2 x_6^2$	$x_2^2 x_6$	$\frac{x_2^3}{x_1}$
$\frac{x_1^3 x_6^5}{x_2 x_5}$	$\frac{x_1^2 x_6^4}{x_5}$	$\frac{x_1 x_2 x_6^3}{x_5}$	$\frac{x_2^2 x_6^2}{x_5}$	$\frac{x_2^3 x_6}{x_1 x_5}$

$\Delta(F_3^{(2)})$

$\frac{x_1^3 x_5^6 x_6^2}{x_2}$	$\frac{x_1^2 x_5^6}{x_6}$	$\frac{x_1 x_2 x_5^6}{x_6^4}$		
$\frac{x_1^3 x_5^5 x_6^3}{x_2}$	$x_1^2 x_5^5$	$\frac{x_1 x_2 x_5^5}{x_6^3}$		
$\frac{x_1^3 x_5^4 x_6^4}{x_2}$	$x_1^2 x_5^4 x_6$	$\frac{x_1 x_2 x_5^4}{x_6^2}$		
$\frac{x_1^3 x_5^3 x_6^5}{x_2}$	$x_1^2 x_5^3 x_6^2$	$\frac{x_1 x_2 x_5^3}{x_6}$	$\frac{x_2^2 x_5^3}{x_6^4}$	
$\frac{x_1^3 x_5^2 x_6^6}{x_2}$	$x_1^2 x_5^2 x_6^3$	$x_1 x_2 x_5^2$	$\frac{x_2^2 x_5^2}{x_6^3}$	
$\frac{x_1^3 x_5 x_6^7}{x_2}$	$x_1^2 x_5 x_6^4$	$\boxed{x_1 x_2 x_5 x_6}$	$\frac{x_2^2 x_5}{x_6^2}$	$\frac{x_2^3 x_5}{x_1 x_6^5}$
$\frac{x_1^3 x_6^8}{x_2}$	$x_1^2 x_6^5$	$x_1 x_2 x_6^2$	x_2^2/x_6	$\frac{x_2^3}{x_1 x_6^4}$
$\frac{x_1^3 x_6^9}{x_2 x_5}$	$\frac{x_1^2 x_6^6}{x_5}$	$\frac{x_1 x_2 x_6^3}{x_5}$	x_2^2/x_5	$\frac{x_2^3}{x_1 x_5 x_6^3}$
		$\frac{x_1 x_2 x_6^4}{x_5^2}$	$\frac{x_2^2 x_6}{x_5^2}$	$\frac{x_2^3}{x_1 x_5^2 x_6^2}$

GLSM

→ Toric Geometry "Plan B"

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$\frac{x_1^3 x_5^3 x_6}{x_2}$	$x_1^2 x_5^3$	$\frac{x_1 x_2 x_5^3}{x_6}$	$\frac{x_2^2 x_5^3}{x_6^2}$	
$\frac{x_1^3 x_5^2 x_6^2}{x_2}$	$x_1^2 x_5^2 x_6$	$x_1 x_2 x_5^2$	$\frac{x_2^2 x_5^2}{x_6}$	$\frac{x_2^3 x_5^2}{x_1 x_6^2}$
$\frac{x_1^3 x_5 x_6^3}{x_2}$	$x_1^2 x_5 x_6^2$	$\boxed{x_1 x_2 x_5 x_6}$ →	$x_2^2 x_5$	$\frac{x_2^3 x_5}{x_1 x_6}$
$\frac{x_1^3 x_6^4}{x_2}$	$x_1^2 x_6^3$	$x_1 x_2 x_6^2$	$x_2^2 x_6$	$\frac{x_2^3}{x_1}$
$\frac{x_1^3 x_6^5}{x_2 x_5}$	$\frac{x_1^2 x_6^4}{x_5}$	$\frac{x_1 x_2 x_6^3}{x_5}$	$\frac{x_2^2 x_6^2}{x_5}$	$\frac{x_2^3 x_6}{x_1 x_5}$

$\Delta(F_3^{(2)})$

$\frac{x_1^3 x_5^6 x_6^2}{x_2}$	$\frac{x_1^2 x_5^6}{x_6}$	$\frac{x_1 x_2 x_5^6}{x_6^4}$		
$\frac{x_1^3 x_5^5 x_6^3}{x_2}$	$x_1^2 x_5^5$	$\frac{x_1 x_2 x_5^5}{x_6^3}$		
$\frac{x_1^3 x_5^4 x_6^4}{x_2}$	$x_1^2 x_5^4 x_6$	$\frac{x_1 x_2 x_5^4}{x_6^2}$		
$\frac{x_1^3 x_5^3 x_6^5}{x_2}$	$x_1^2 x_5^3 x_6^2$	$\frac{x_1 x_2 x_5^3}{x_6}$	$\frac{x_2^2 x_5^3}{x_6^4}$	
$\frac{x_1^3 x_5^2 x_6^6}{x_2}$	$x_1^2 x_5^2 x_6^3$	$x_1 x_2 x_5^2$	$\frac{x_2^2 x_5^2}{x_6^3}$	
$\frac{x_1^3 x_5 x_6^7}{x_2}$	$x_1^2 x_5 x_6^4$	$\boxed{x_1 x_2 x_5 x_6}$	$\frac{x_2^2 x_5}{x_6^2}$	$\frac{x_2^3 x_5}{x_1 x_6^5}$
$\frac{x_1^3 x_6^8}{x_2}$	$x_1^2 x_6^5$	$x_1 x_2 x_6^2$	x_2^2/x_6	$\frac{x_2^3}{x_1 x_6^4}$
$\frac{x_1^3 x_6^9}{x_2 x_5}$	$\frac{x_1^2 x_6^6}{x_5}$	$\frac{x_1 x_2 x_6^3}{x_5}$	x_2^2/x_5	$\frac{x_2^3}{x_1 x_5 x_6^3}$
		$\frac{x_1 x_2 x_6^4}{x_5^2}$	$\frac{x_2^2 x_6}{x_5^2}$	$\frac{x_2^3}{x_1 x_5^2 x_6^2}$

GLSM

→ Toric Geometry "Plan B"

$$F_m^{(n)}:\ \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right] \qquad x_{n+j+1} := y_j$$

- Construct $f(x) = a_0 \Pi x + \sum_k a_k \Pi_i x_i^{e_{ik}}$ for $m=1$ & $m=3$:

$\Delta(F_1^{(2)})$

$\frac{x_1^3 x_5^4}{x_2}$	$\frac{x_1^2 x_5^4}{x_6}$	$\frac{x_1 x_2 x_5^4}{x_6^2}$		
$\frac{x_1^3 x_5^3 x_6}{x_2}$	$x_1^2 x_5^3$	$\frac{x_1 x_2 x_5^3}{x_6}$	$\frac{x_2^2 x_5^3}{x_6^2}$	
$\frac{x_1^3 x_5^2 x_6^2}{x_2}$	$x_1^2 x_5^2 x_6$	$x_1 x_2 x_5^2$	$\frac{x_2^2 x_5^2}{x_6}$	$\frac{x_2^3 x_5^2}{x_1 x_6^2}$
$\frac{x_1^3 x_5 x_6^3}{x_2}$	$x_1^2 x_5 x_6^2$	$\boxed{x_1 x_2 x_5 x_6}$ →	$x_2^2 x_5$	$\frac{x_2^3 x_5}{x_1 x_6}$
$\frac{x_1^3 x_6^4}{x_2}$	$x_1^2 x_6^3$	$x_1 x_2 x_6^2$	$x_2^2 x_6$	$\frac{x_2^3}{x_1}$
$\frac{x_1^3 x_6^5}{x_2 x_5}$	$\frac{x_1^2 x_6^4}{x_5}$	$\frac{x_1 x_2 x_6^3}{x_5}$	$\frac{x_2^2 x_6^2}{x_5}$	$\frac{x_2^3 x_6}{x_1 x_5}$

$\Delta(F_3^{(2)})$

$\frac{x_1^3 x_5^6 x_6^2}{x_2}$	$\frac{x_1^2 x_5^6}{x_6}$	$\frac{x_1 x_2 x_5^6}{x_6^4}$		
$\frac{x_1^3 x_5^5 x_6^3}{x_2}$	$x_1^2 x_5^5$	$\frac{x_1 x_2 x_5^5}{x_6^3}$		
$\frac{x_1^3 x_5^4 x_6^4}{x_2}$	$x_1^2 x_5^4 x_6$	$\frac{x_1 x_2 x_5^4}{x_6^2}$		
$\frac{x_1^3 x_5^3 x_6^5}{x_2}$	$x_1^2 x_5^3 x_6^2$	$\frac{x_1 x_2 x_5^3}{x_6}$	$\frac{x_2^2 x_5^3}{x_6^4}$	
$\frac{x_1^3 x_5^2 x_6^6}{x_2}$	$x_1^2 x_5^2 x_6^3$	$x_1 x_2 x_5^2$	$\frac{x_2^2 x_5^2}{x_6^3}$	
$\frac{x_1^3 x_5 x_6^7}{x_2}$	$x_1^2 x_5 x_6^4$	$\boxed{x_1 x_2 x_5 x_6}$ →	$\frac{x_2^2 x_5}{x_6^2}$	$\frac{x_2^3 x_5}{x_1 x_6^5}$
$\frac{x_1^3 x_6^8}{x_2}$	$x_1^2 x_6^5$	$x_1 x_2 x_6^2$	x_2^2/x_6	$\frac{x_2^3}{x_1 x_6^4}$
$\frac{x_1^3 x_6^9}{x_2 x_5}$	$\frac{x_1^2 x_6^6}{x_5}$	$\frac{x_1 x_2 x_6^3}{x_5}$	x_2^2/x_5	$\frac{x_2^3}{x_1 x_5 x_6^3}$
		$\frac{x_1 x_2 x_6^4}{x_5^2}$	$\frac{x_2^2 x_6}{x_5^2}$	$\frac{x_2^3}{x_1 x_5^2 x_6^2}$

GLSM

→ Toric Geometry "Plan B"

$$F_m^{(n)}: \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right] \qquad x_{n+j+1} := y_j$$

- Construct $f(x) = a_0 \Pi x + \sum_k a_k \Pi_i x_i^{e_{ik}}$ for $m = 1$ & $m = 3$:

$\Delta(F_1^{(2)})$

$\frac{x_1^3 x_5^4}{x_2}$	$\frac{x_1^2 x_5^4}{x_6}$	$\frac{x_1 x_2 x_5^4}{x_6^2}$		
$\frac{x_1^3 x_5^3 x_6}{x_2}$	$x_1^2 x_5^3$	$\frac{x_1 x_2 x_5^3}{x_6}$	$\frac{x_2^2 x_5^3}{x_6^2}$	
$\frac{x_1^3 x_5^2 x_6^2}{x_2}$	$x_1^2 x_5^2 x_6$	$x_1 x_2 x_5^2$	$\frac{x_2^2 x_5^2}{x_6}$	$\frac{x_2^3 x_5^2}{x_1 x_6^2}$
$\frac{x_1^3 x_5 x_6^3}{x_2}$	$x_1^2 x_5 x_6^2$	$\boxed{x_1 x_2 x_5 x_6}$	$x_2^2 x_5$	$\frac{x_2^3 x_5}{x_1 x_6}$
$\frac{x_1^3 x_6^4}{x_2}$	$x_1^2 x_6^3$	$x_1 x_2 x_6^2$	$x_2^2 x_6$	$\frac{x_2^3}{x_1}$
$\frac{x_1^3 x_6^5}{x_2 x_5}$	$\frac{x_1^2 x_6^4}{x_5}$	$\frac{x_1 x_2 x_6^3}{x_5}$	$\frac{x_2^2 x_6^2}{x_5}$	$\frac{x_2^3 x_6}{x_1 x_5}$

$\Delta(F_3^{(2)})$

$\frac{x_1^3 x_5^6 x_6^2}{x_2}$	$\frac{x_1^2 x_5^6}{x_6}$	$\frac{x_1 x_2 x_5^6}{x_6^4}$		
$\frac{x_1^3 x_5^5 x_6^3}{x_2}$	$x_1^2 x_5^5$	$\frac{x_1 x_2 x_5^5}{x_6^3}$		
$\frac{x_1^3 x_5^4 x_6^4}{x_2}$	$x_1^2 x_5^4 x_6$	$\frac{x_1 x_2 x_5^4}{x_6^2}$		
$\frac{x_1^3 x_5^3 x_6^5}{x_2}$	$x_1^2 x_5^3 x_6^2$	$\frac{x_1 x_2 x_5^3}{x_6}$	$\frac{x_2^2 x_5^3}{x_6^4}$	
$\frac{x_1^3 x_5^2 x_6^6}{x_2}$	$x_1^2 x_5^2 x_6^3$	$x_1 x_2 x_5^2$	$\frac{x_2^2 x_5^2}{x_6^3}$	
$\frac{x_1^3 x_5 x_6^7}{x_2}$	$x_1^2 x_5 x_6^4$	$\boxed{x_1 x_2 x_5 x_6}$	$\frac{x_2^2 x_5}{x_6^2}$	$\frac{x_2^3 x_5}{x_1 x_6^5}$
$\frac{x_1^3 x_6^8}{x_2}$	$x_1^2 x_6^5$	$x_1 x_2 x_6^2$	x_2^2/x_6	$\frac{x_2^3}{x_1 x_6^4}$
$\frac{x_1^3 x_6^9}{x_2 x_5}$	$\frac{x_1^2 x_6^6}{x_5}$	$\frac{x_1 x_2 x_6^3}{x_5}$	x_2^2/x_5	$\frac{x_2^3}{x_1 x_5 x_6^3}$
		$\frac{x_1 x_2 x_6^4}{x_5^2}$	$\frac{x_2^2 x_6}{x_5^2}$	$\frac{x_2^3}{x_1 x_5^2 x_6^2}$

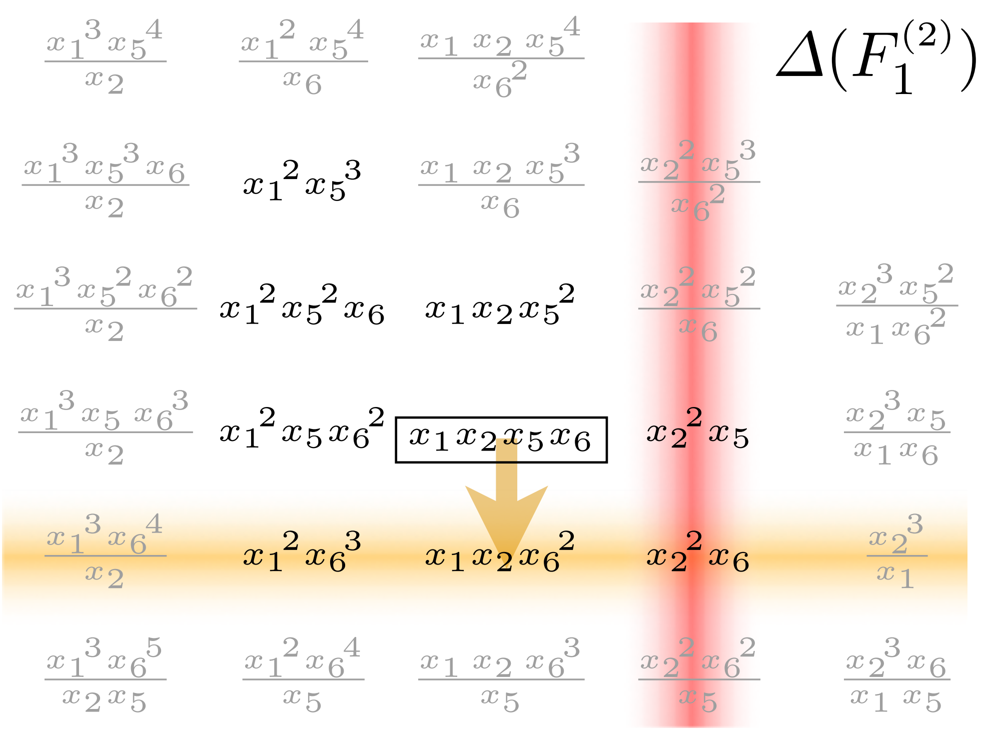
GLSM

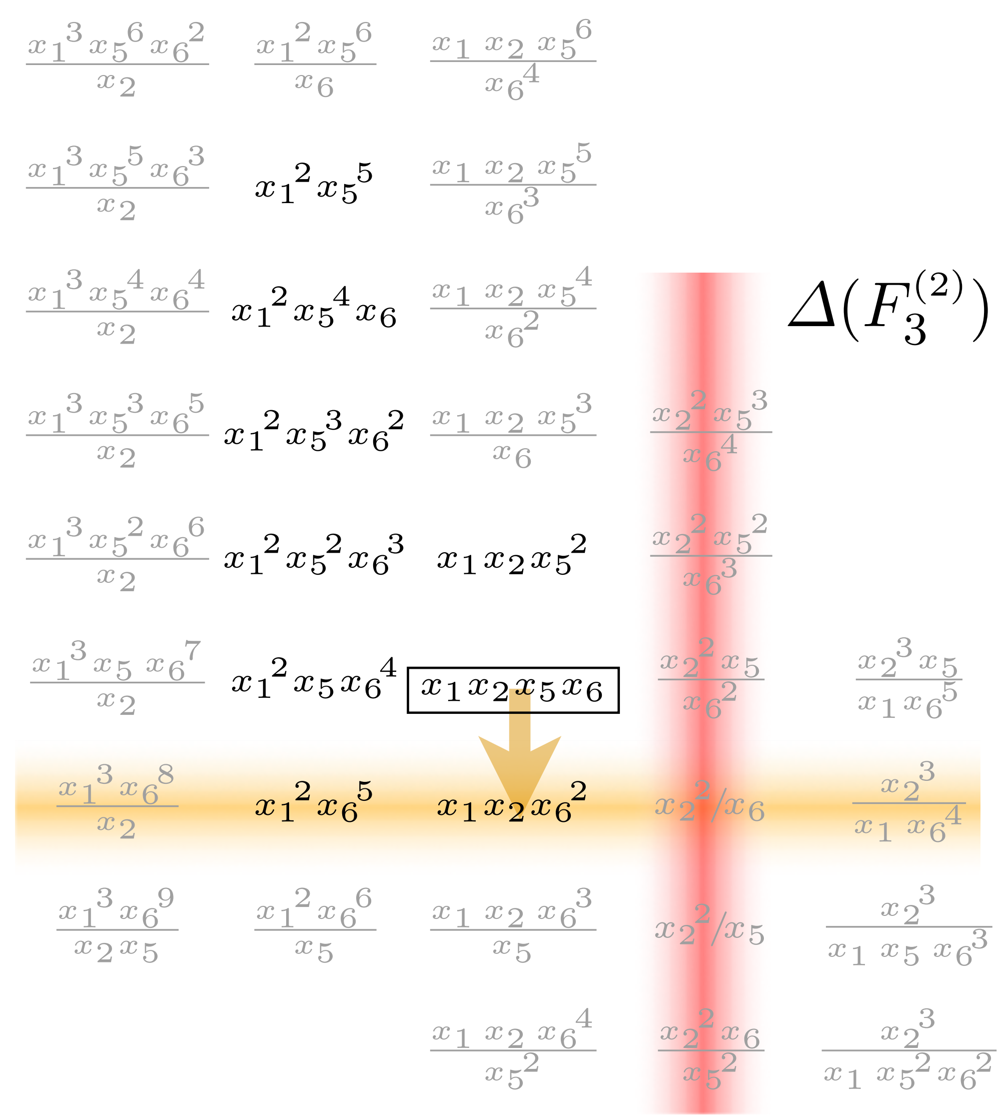
→ Toric Geometry "Plan B"

$$F_m^{(n)}: \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right]$$

$x_{n+j+1} := y_j$

- Construct $f(x) = a_0 \Pi x + \sum_k a_k \Pi_i x_i^{e_{ik}}$ for $m=1$ & $m=3$:





GLSM

→ Toric Geometry "Plan B"

$$F_m^{(n)}:\ \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right] \qquad x_{n+j+1} := y_j$$

- Construct $f(x) = a_0 \Pi x + \sum_k a_k \Pi_i x_i^{e_{ik}}$ for $m=1$ & $m=3$:

$\Delta(F_1^{(2)})$

$\frac{x_1^3 x_5^4}{x_2}$	$\frac{x_1^2 x_5^4}{x_6}$	$\frac{x_1 x_2 x_5^4}{x_6^2}$		
$\frac{x_1^3 x_5^3 x_6}{x_2}$	$x_1^2 x_5^3$	$\frac{x_1 x_2 x_5^3}{x_6}$	$\frac{x_2^2 x_5^3}{x_6^2}$	
$\frac{x_1^3 x_5^2 x_6^2}{x_2}$	$x_1^2 x_5^2 x_6$	$x_1 x_2 x_5^2$	$\frac{x_2^2 x_5^2}{x_6}$	$\frac{x_2^3 x_5^2}{x_1 x_6^2}$
$\frac{x_1^3 x_5 x_6^3}{x_2}$	$x_1^2 x_5 x_6^2$	$\boxed{x_1 x_2 x_5 x_6}$	$x_2^2 x_5$	$\frac{x_2^3 x_5}{x_1 x_6}$
$\frac{x_1^3 x_6^4}{x_2}$	$x_1^2 x_6^3$	$x_1 x_2 x_6^2$	$x_2^2 x_6$	$\frac{x_2^3}{x_1}$
$\frac{x_1^3 x_6^5}{x_2 x_5}$	$\frac{x_1^2 x_6^4}{x_5}$	$\frac{x_1 x_2 x_6^3}{x_5}$	$\frac{x_2^2 x_6^2}{x_5}$	$\frac{x_2^3 x_6}{x_1 x_5}$

$\Delta(F_3^{(2)})$

$\frac{x_1^3 x_5^6 x_6^2}{x_2}$	$\frac{x_1^2 x_5^6}{x_6}$	$\frac{x_1 x_2 x_5^6}{x_6^4}$		
$\frac{x_1^3 x_5^5 x_6^3}{x_2}$	$x_1^2 x_5^5$	$\frac{x_1 x_2 x_5^5}{x_6^3}$		
$\frac{x_1^3 x_5^4 x_6^4}{x_2}$	$x_1^2 x_5^4 x_6$	$\frac{x_1 x_2 x_5^4}{x_6^2}$		
$\frac{x_1^3 x_5^3 x_6^5}{x_2}$	$x_1^2 x_5^3 x_6^2$	$\frac{x_1 x_2 x_5^3}{x_6}$	$\frac{x_2^2 x_5^3}{x_6^4}$	
$\frac{x_1^3 x_5^2 x_6^6}{x_2}$	$x_1^2 x_5^2 x_6^3$	$x_1 x_2 x_5^2$	$\frac{x_2^2 x_5^2}{x_6^3}$	
$\frac{x_1^3 x_5 x_6^7}{x_2}$	$x_1^2 x_5 x_6^4$	$\boxed{x_1 x_2 x_5 x_6}$	$\frac{x_2^2 x_5}{x_6^2}$	$\frac{x_2^3 x_5}{x_1 x_6^5}$
$\frac{x_1^3 x_6^8}{x_2}$	$x_1^2 x_6^5$	$x_1 x_2 x_6^2$	x_2^2/x_6	$\frac{x_2^3}{x_1 x_6^4}$
$\frac{x_1^3 x_6^9}{x_2 x_5}$	$\frac{x_1^2 x_6^6}{x_5}$	$\frac{x_1 x_2 x_6^3}{x_5}$	x_2^2/x_5	$\frac{x_2^3}{x_1 x_5 x_6^3}$
		$\frac{x_1 x_2 x_6^4}{x_5^2}$	$\frac{x_2^2 x_6}{x_5^2}$	$\frac{x_2^3}{x_1 x_5^2 x_6^2}$

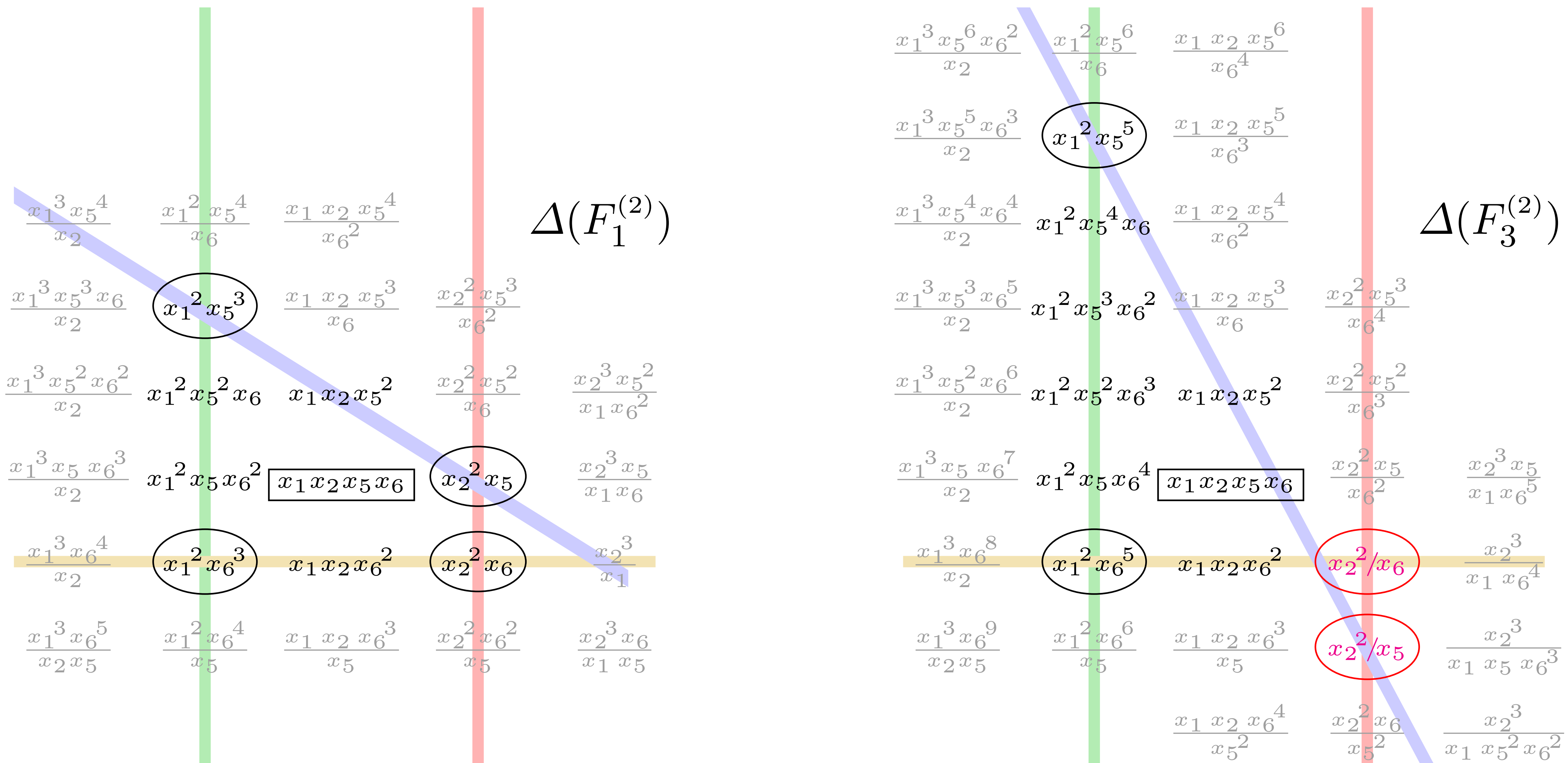
GLSM

→ Toric Geometry “Plan B”

$$F_m^{(n)}:\ \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right]$$

$x_{n+j+1} := y_j$

- Construct $f(x) = a_0\,\Pi x + \sum_k a_k\,\Pi_i x_i^{e_{ik}}$ for $m=1$ & $m=3$:



GLSM

→ Toric Geometry "Plan B"

$$F_m^{(n)}: \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right]$$

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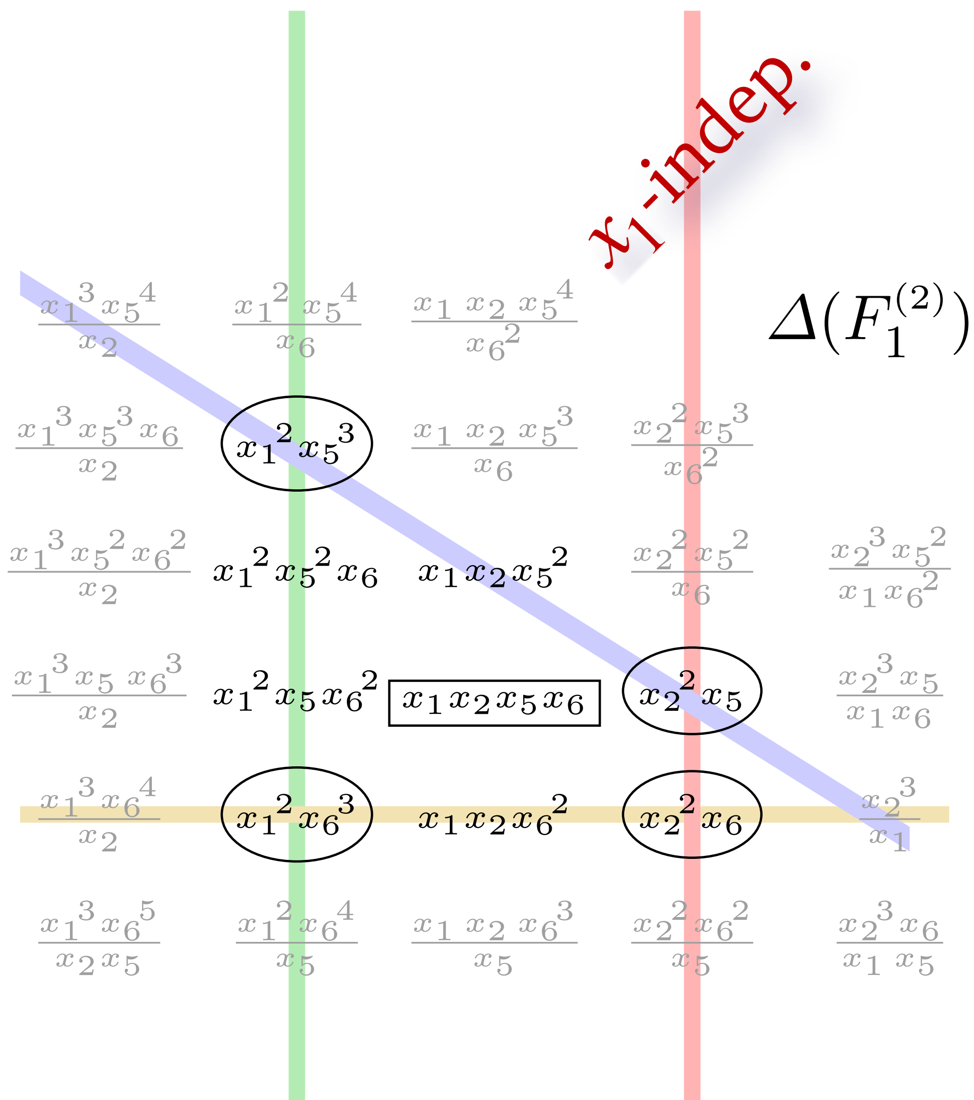
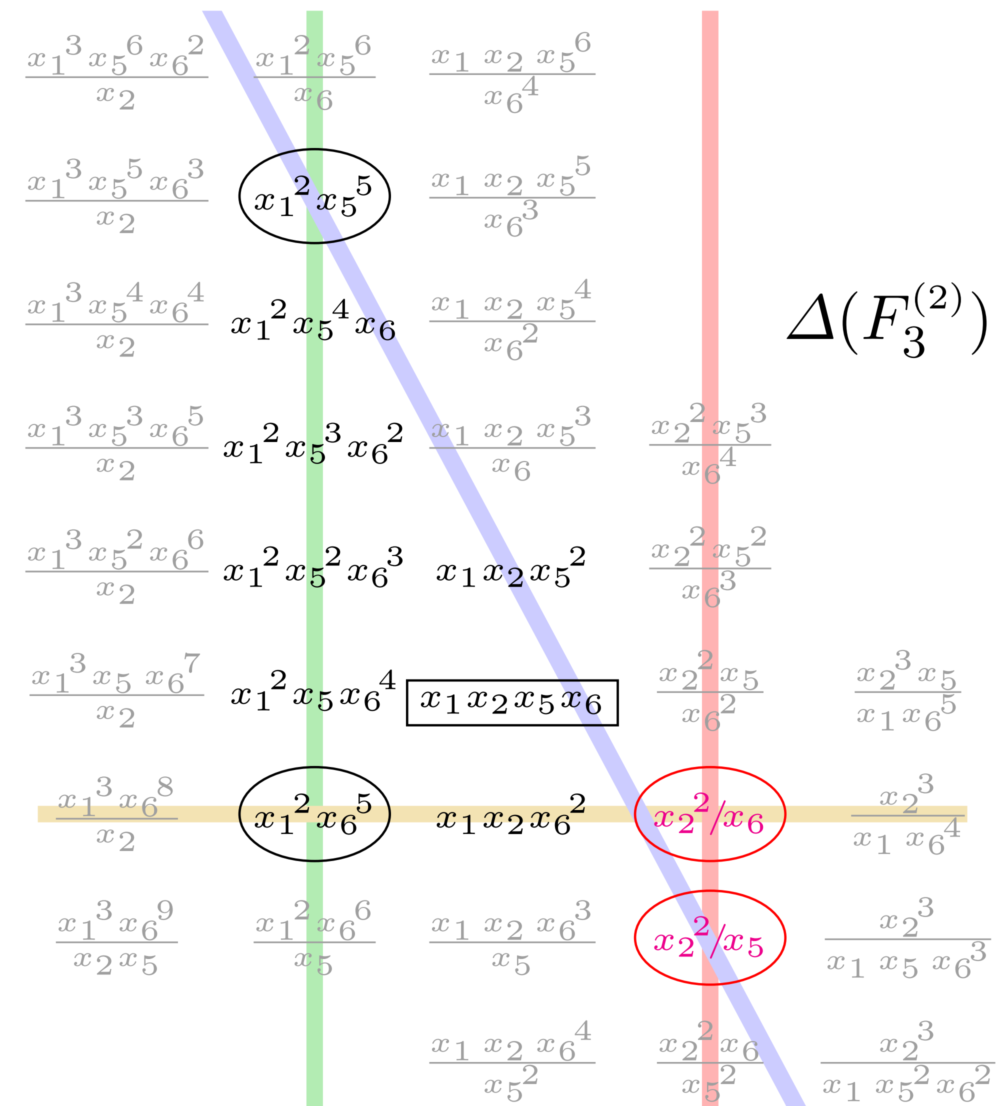
- Construct $f(x) = a_0 \Pi x + \sum_k a_k \Pi_i x_i^{e_{ik}}$ for $m=1$ & $m=3$:



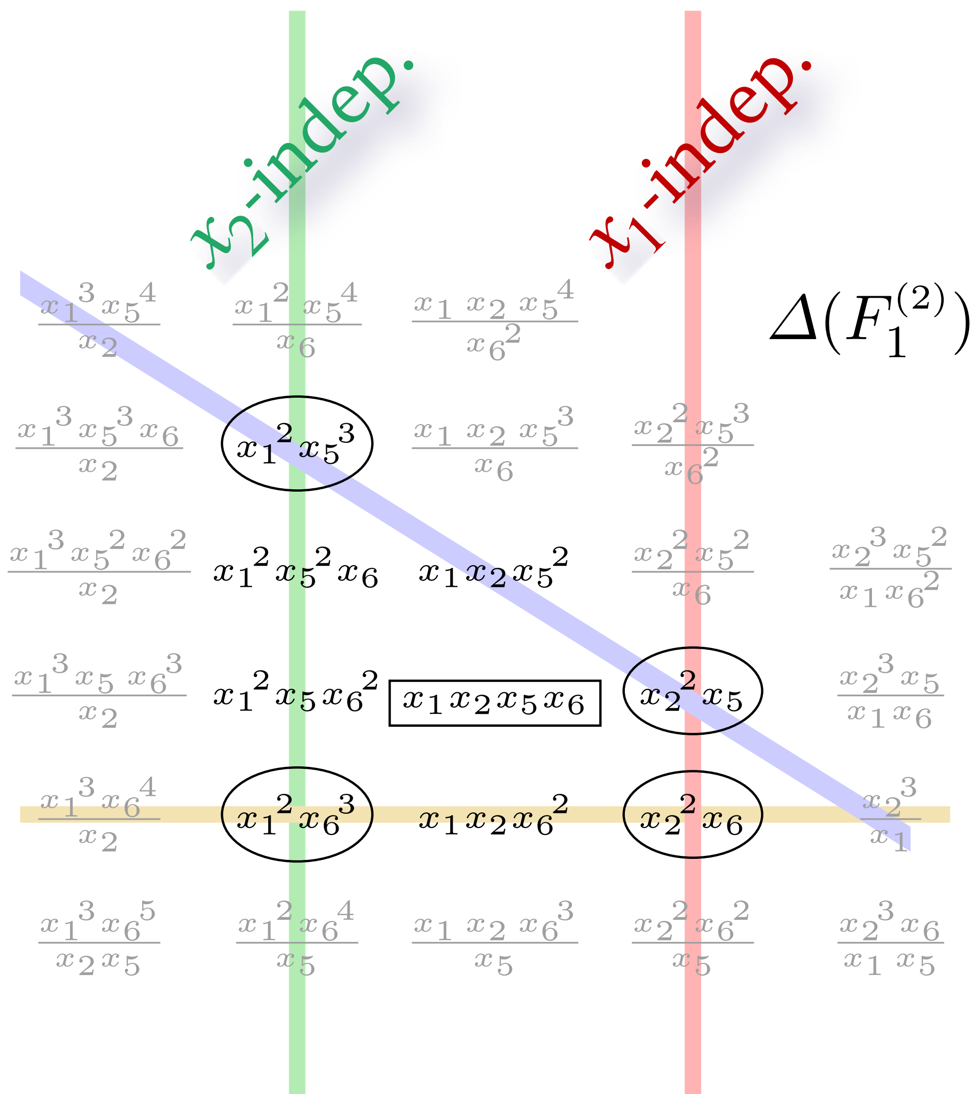


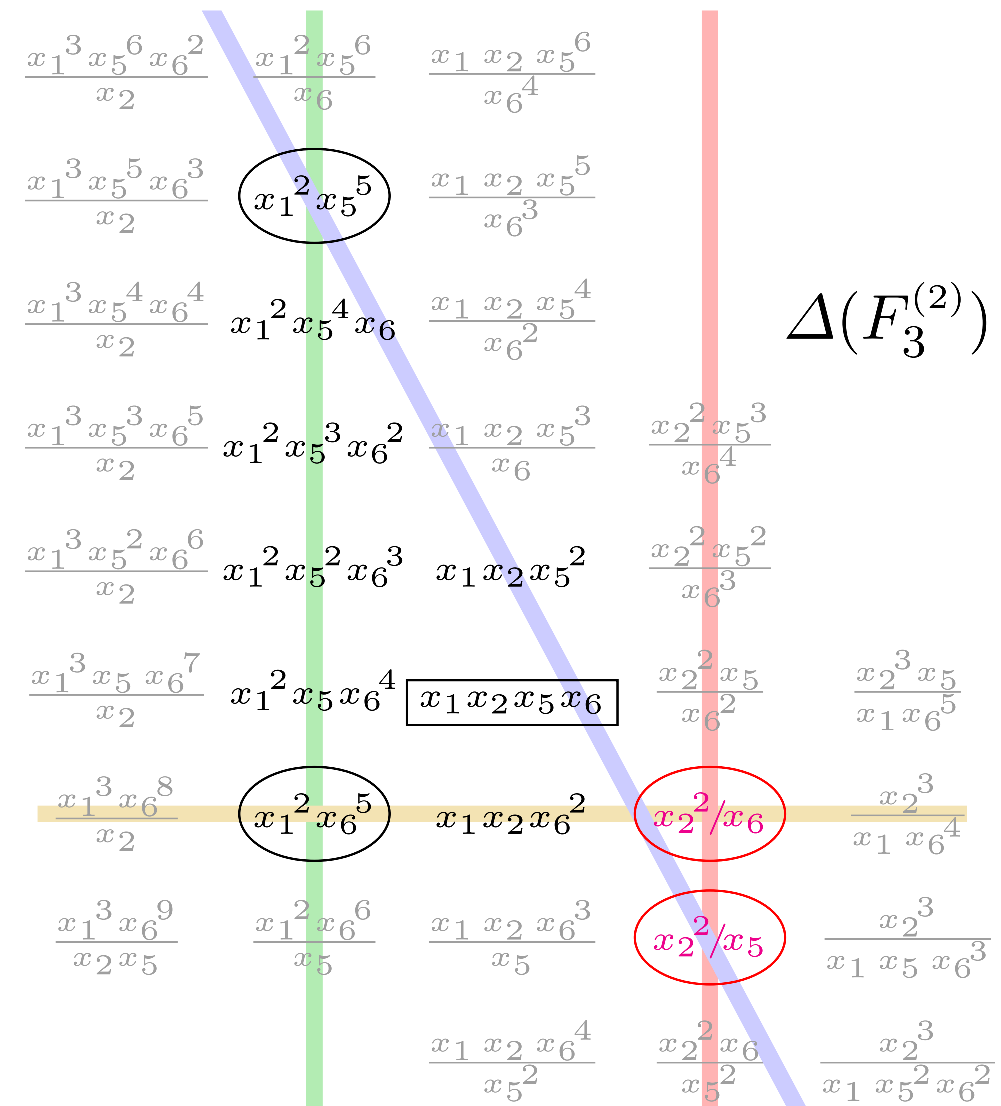
GLSM

→ Toric Geometry "Plan B"

$$F_m^{(n)}: \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right] \qquad x_{n+j+1} := y_j$$

- Construct $f(x) = a_0 \prod x + \sum_k a_k \prod_i x_i^{e_{ik}}$ for $m = 1$ & $m = 3$:





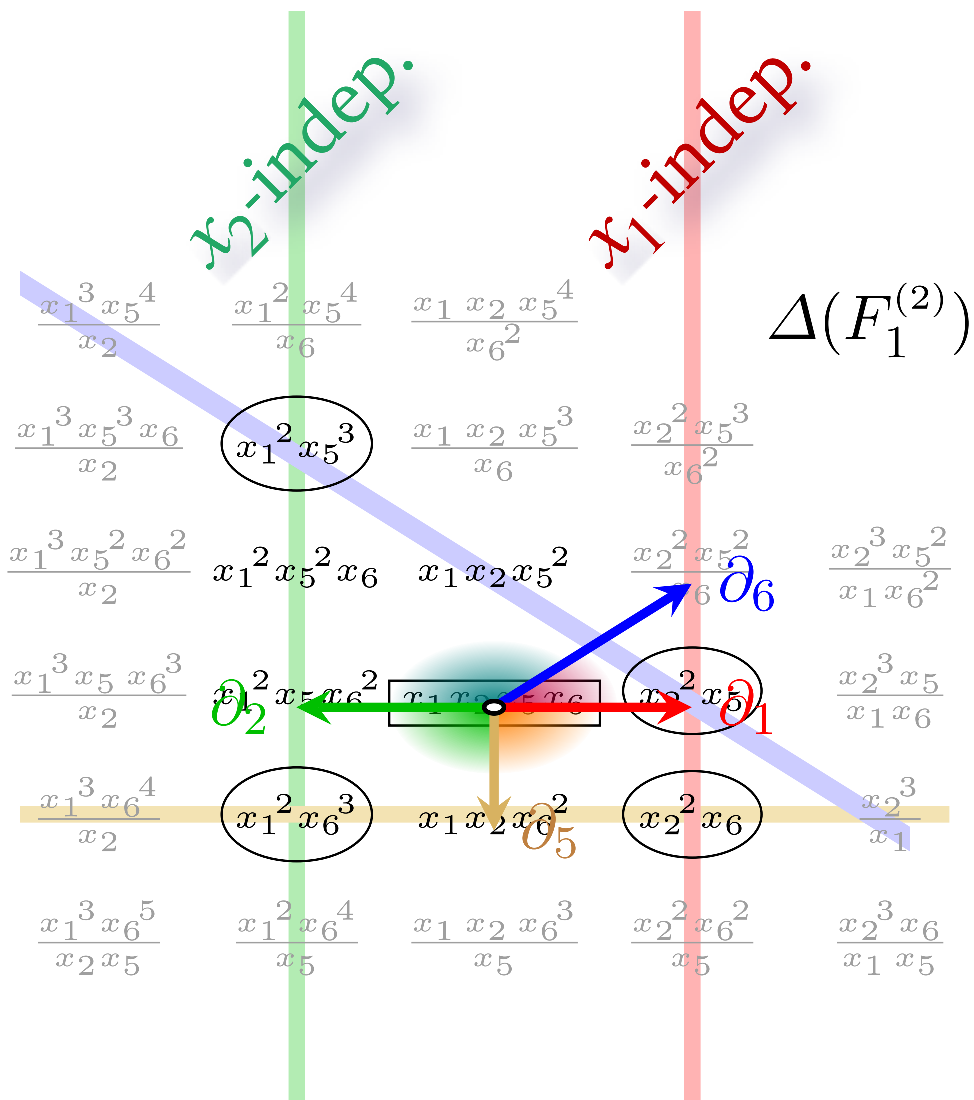
GLSM

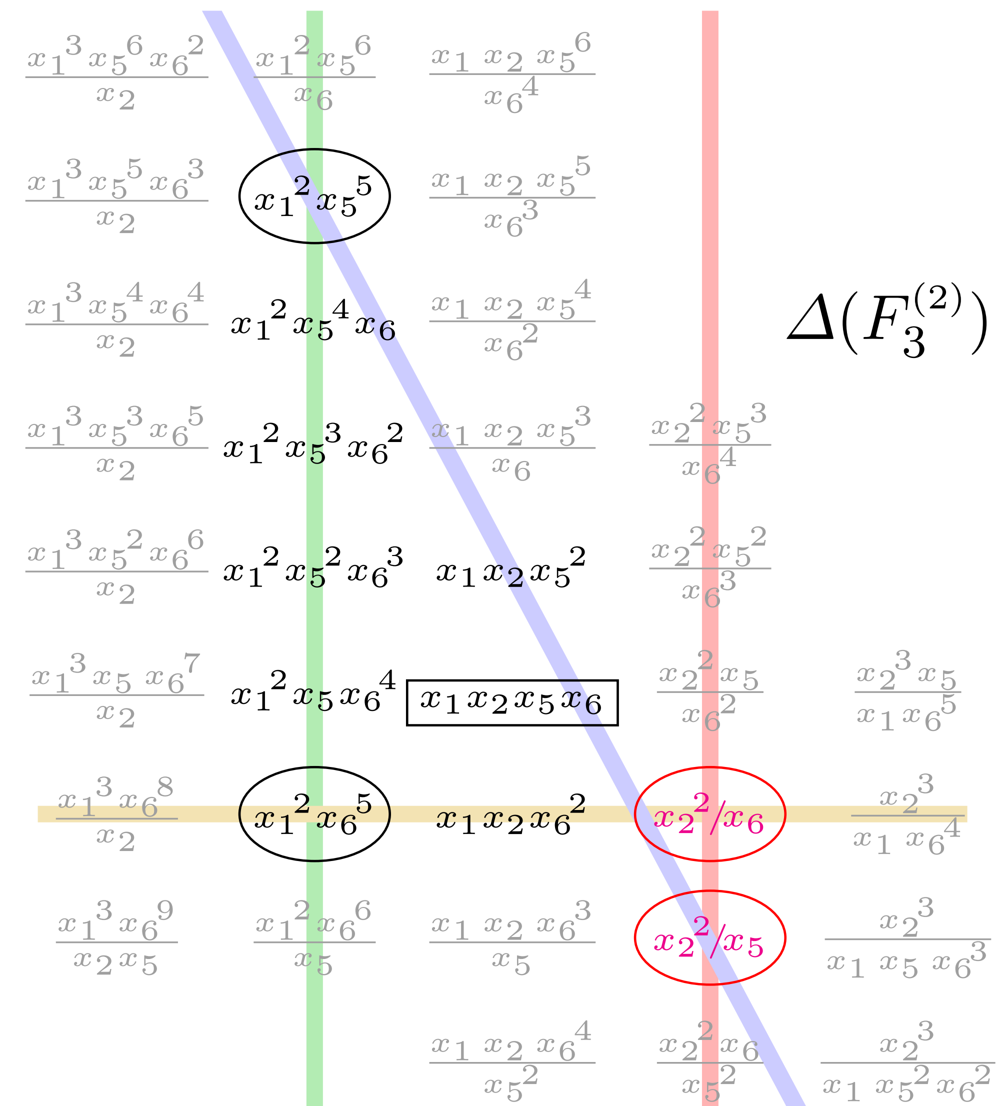
→ Toric Geometry "Plan B"

$$F_m^{(n)}:\left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right]$$

$x_{n+j+1} := y_j$

- Construct $f(x) = a_0 \Pi x + \sum_k a_k \Pi_i x_i^{e_{ik}}$ for $m=1$ & $m=3$:





GLSM

→ Toric Geometry “Plan B”

$$F_m^{(n)}:\left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right]$$

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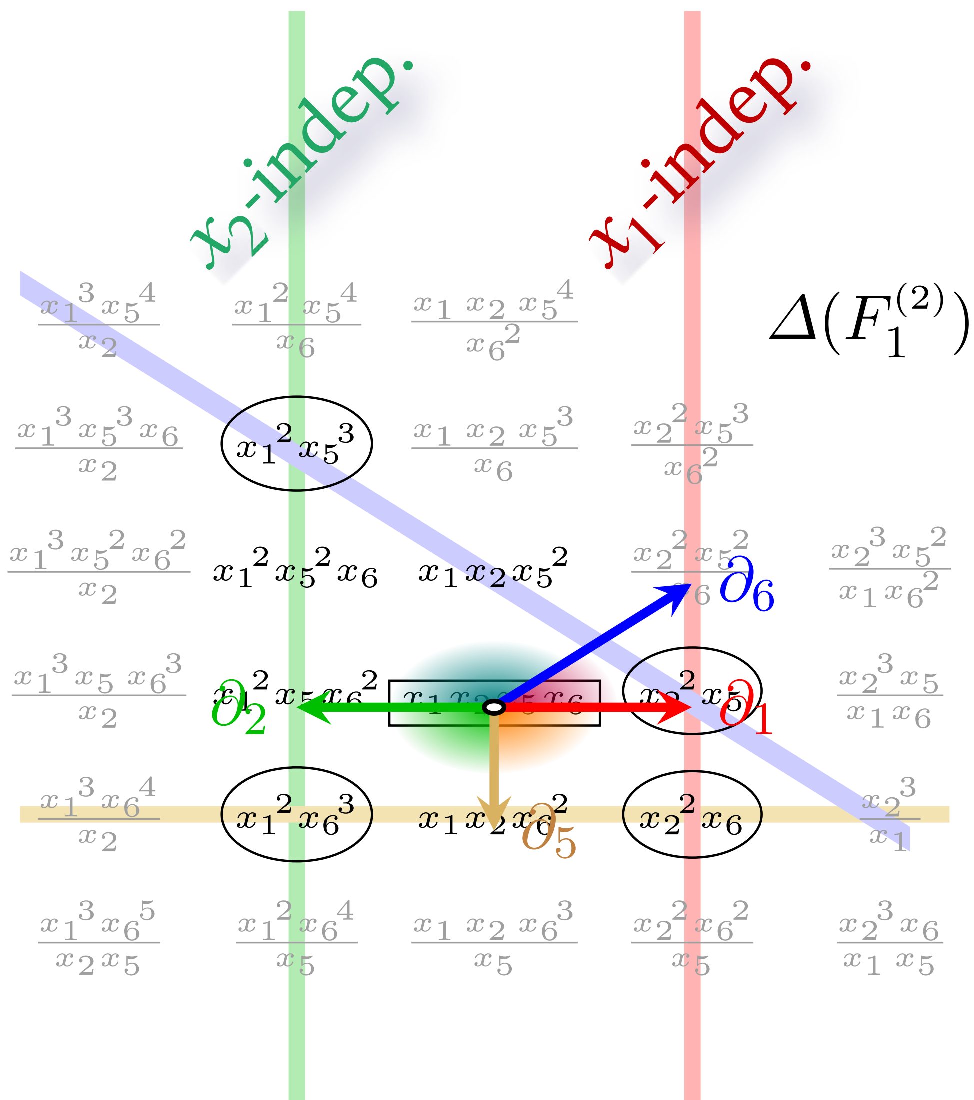
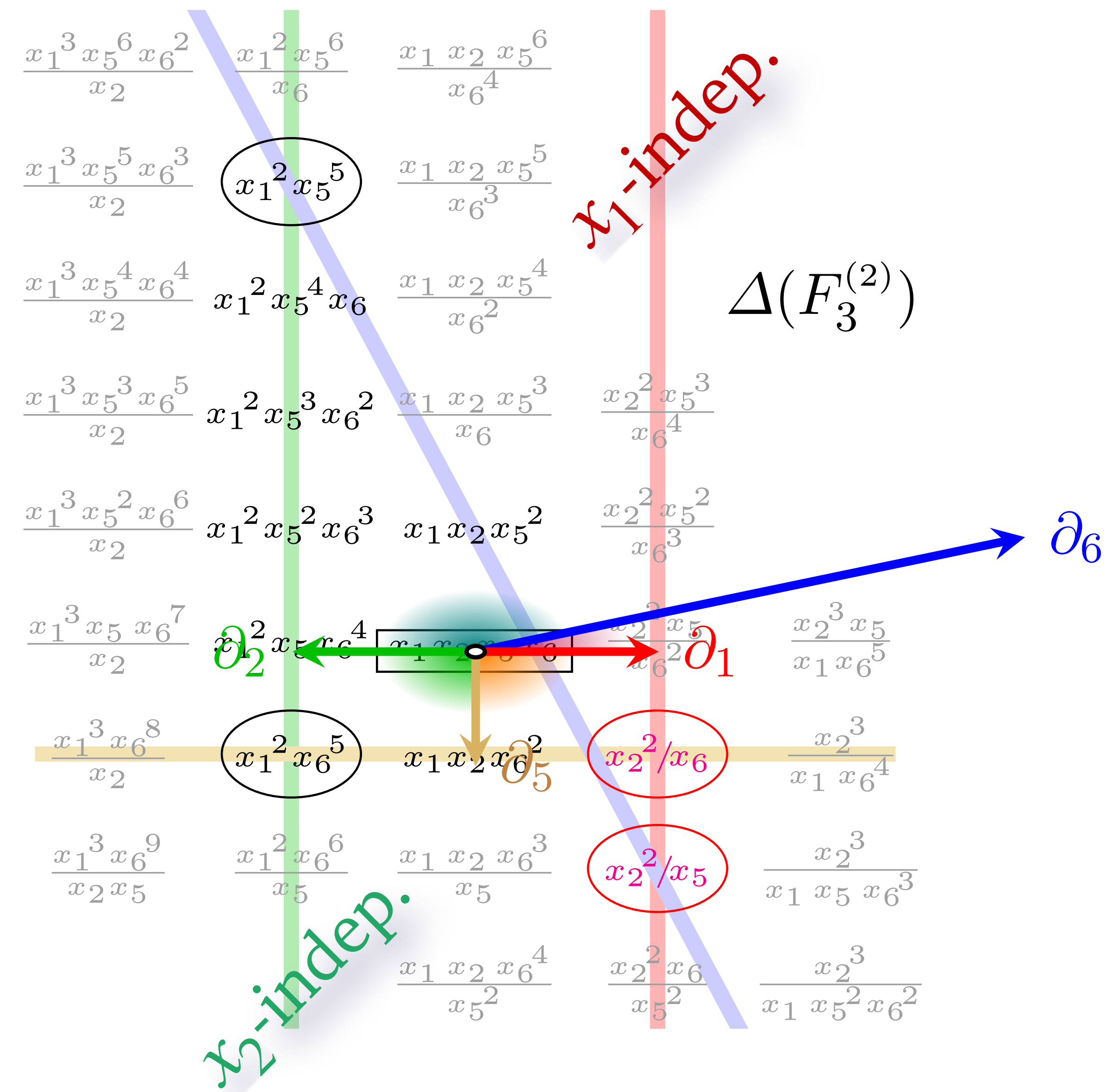
- Construct $f(x) = a_0 \Pi x + \sum_k a_k \Pi_i x_i^{e_{ik}}$ for $m=1$ & $m=3$:





GLSM

→ Toric Geometry “Plan B”

$$F_m^{(n)}:\ \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right]$$

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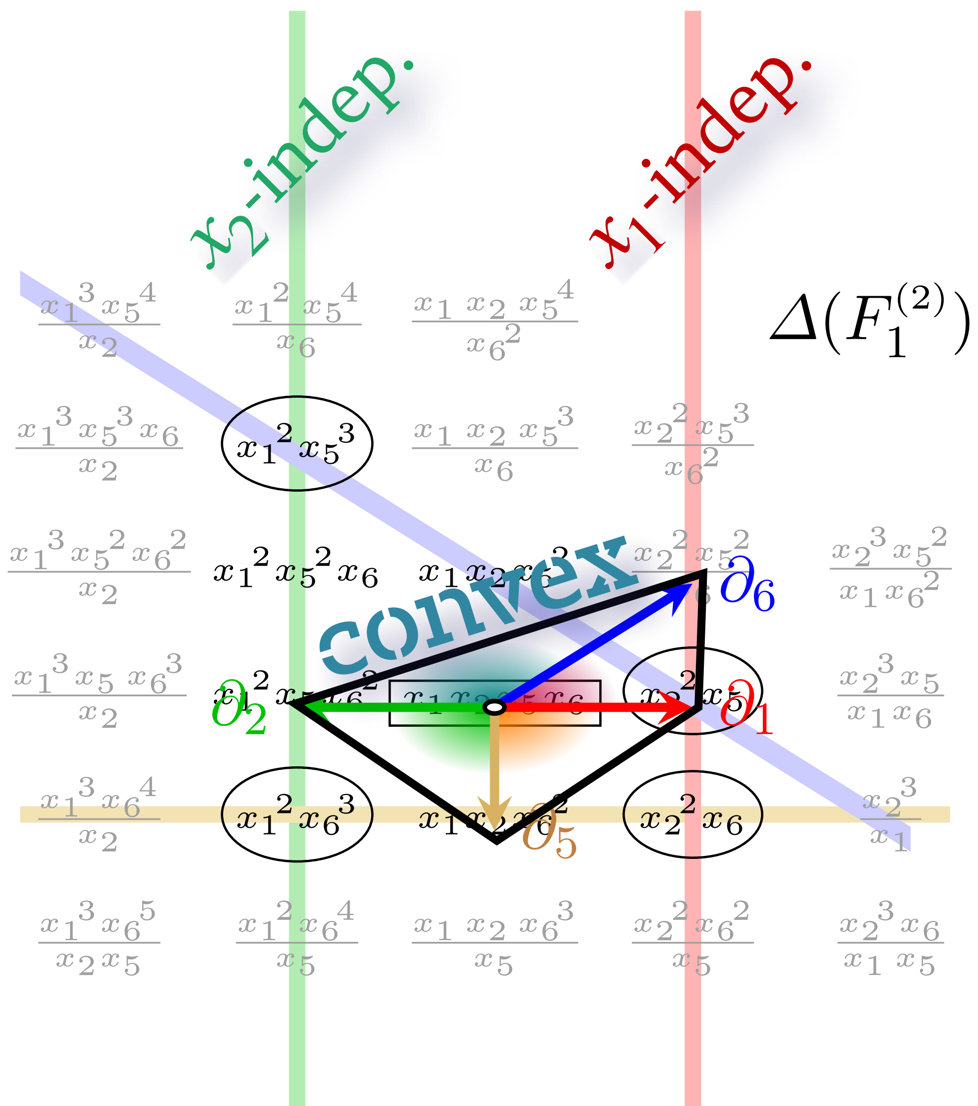
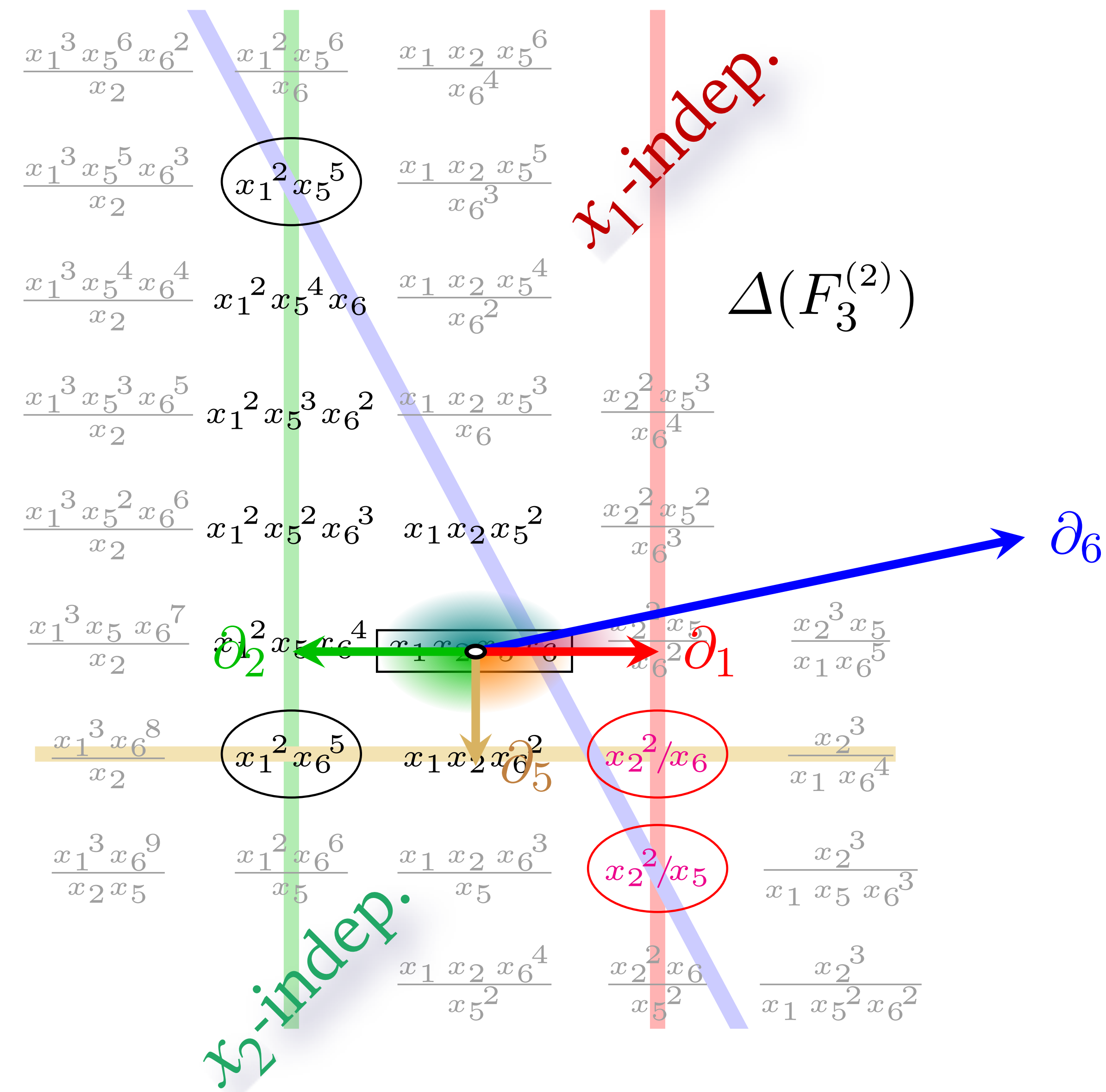
- Construct $f(x) = a_0 \Pi x + \sum_k a_k \Pi_i x_i^{e_{ik}}$ for $m=1$ & $m=3$:





GLSM

→ Toric Geometry "Plan B"

$$F_m^{(n)}: \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m{-}2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right] \quad x_{n+j+1} := y_j$$

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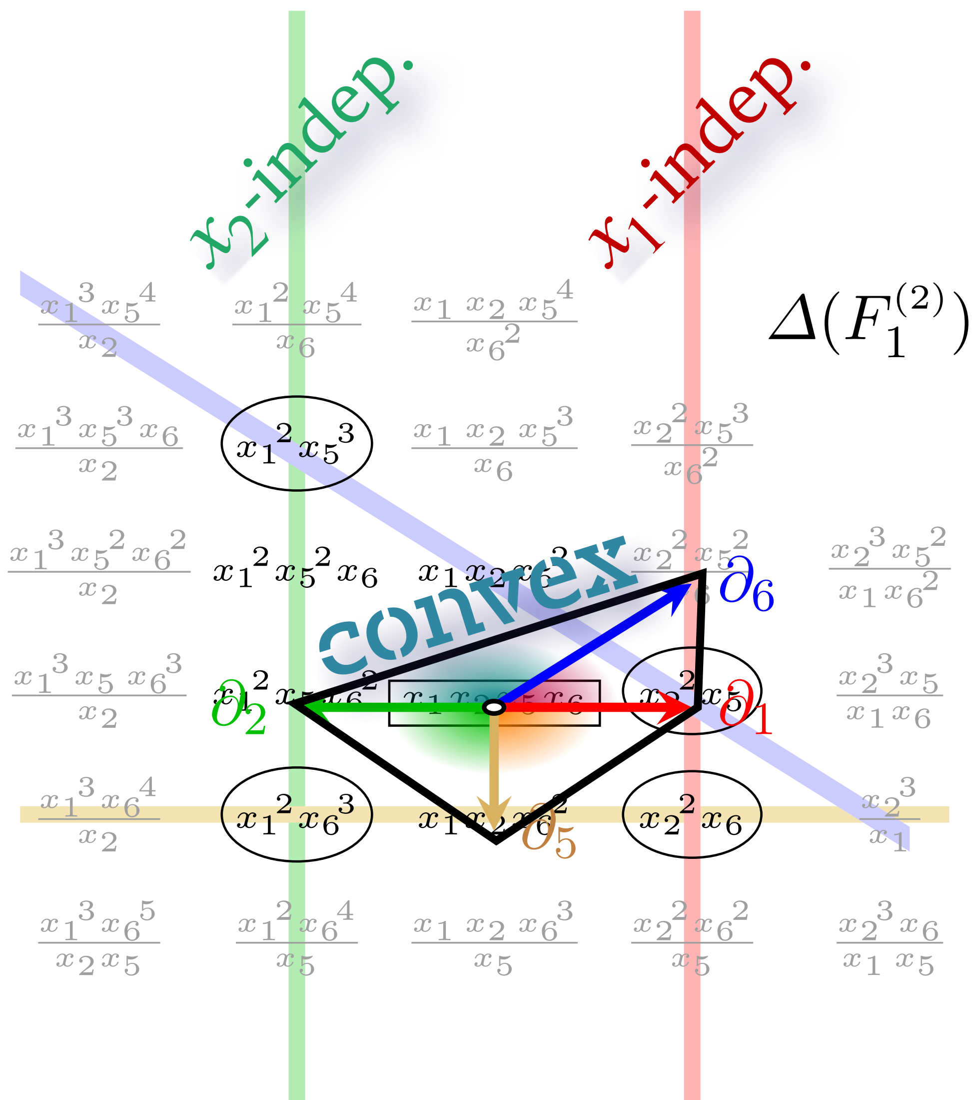
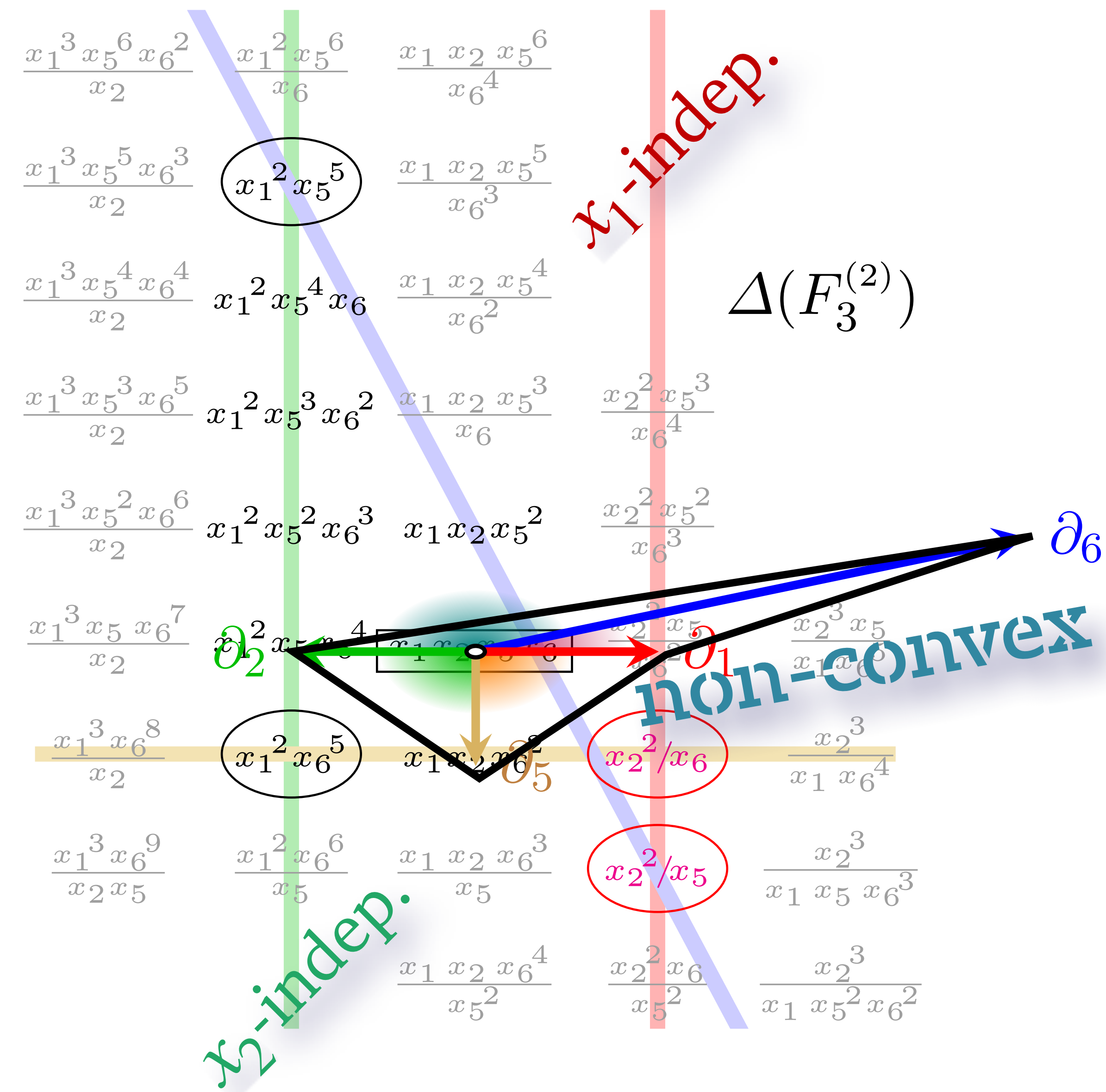




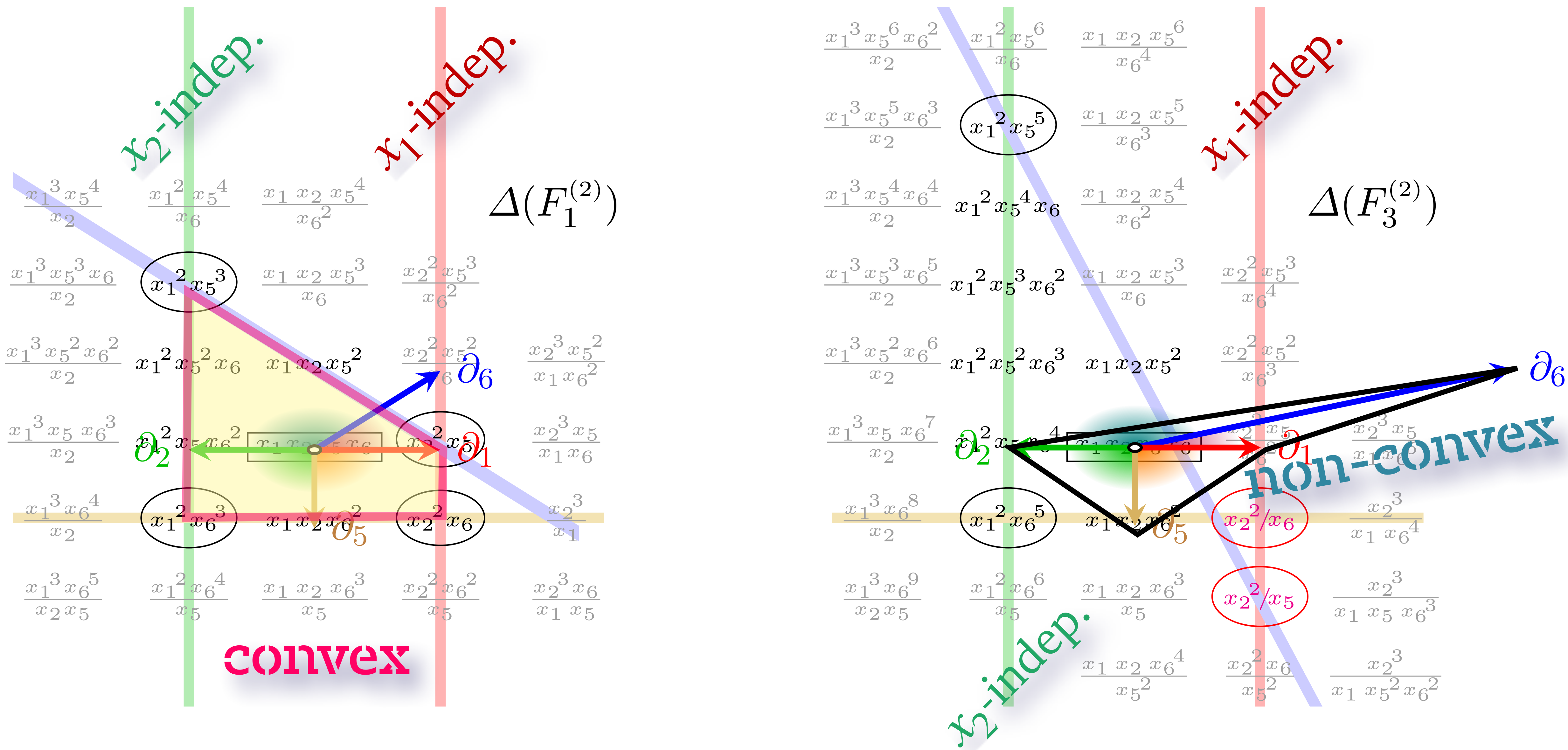
GLSM

→ Toric Geometry "Plan B"

$$F_m^{(n)}: \left[\begin{array}{c|cccccc} p & x_1 & x_2 & \cdots & x_n & y_0 & y_1 \\ \hline -n & 1 & 1 & \cdots & 1 & 0 & 0 \\ m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array}\right]$$

$x_{n+j+1} := y_j$

- Construct $f(x) = a_0 \Pi x + \sum_k a_k \Pi_i x_i^{e_{ik}}$ for $m = 1$ & $m = 3$:



GLSM

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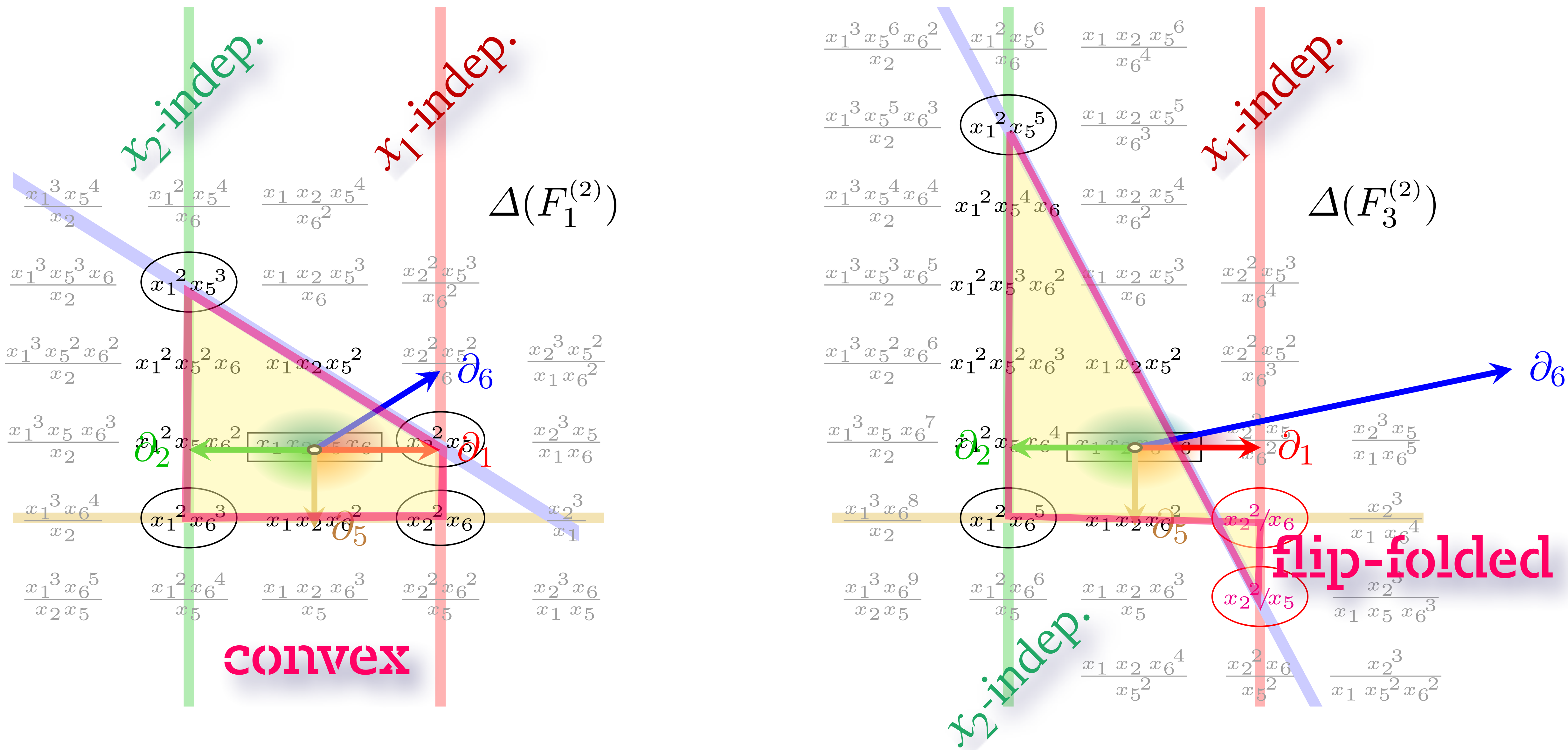

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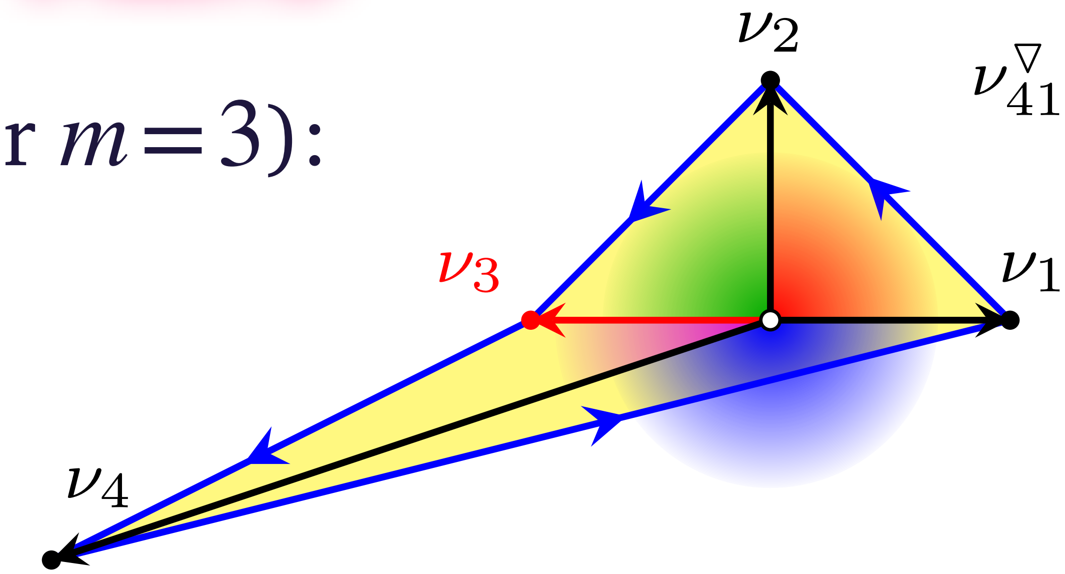
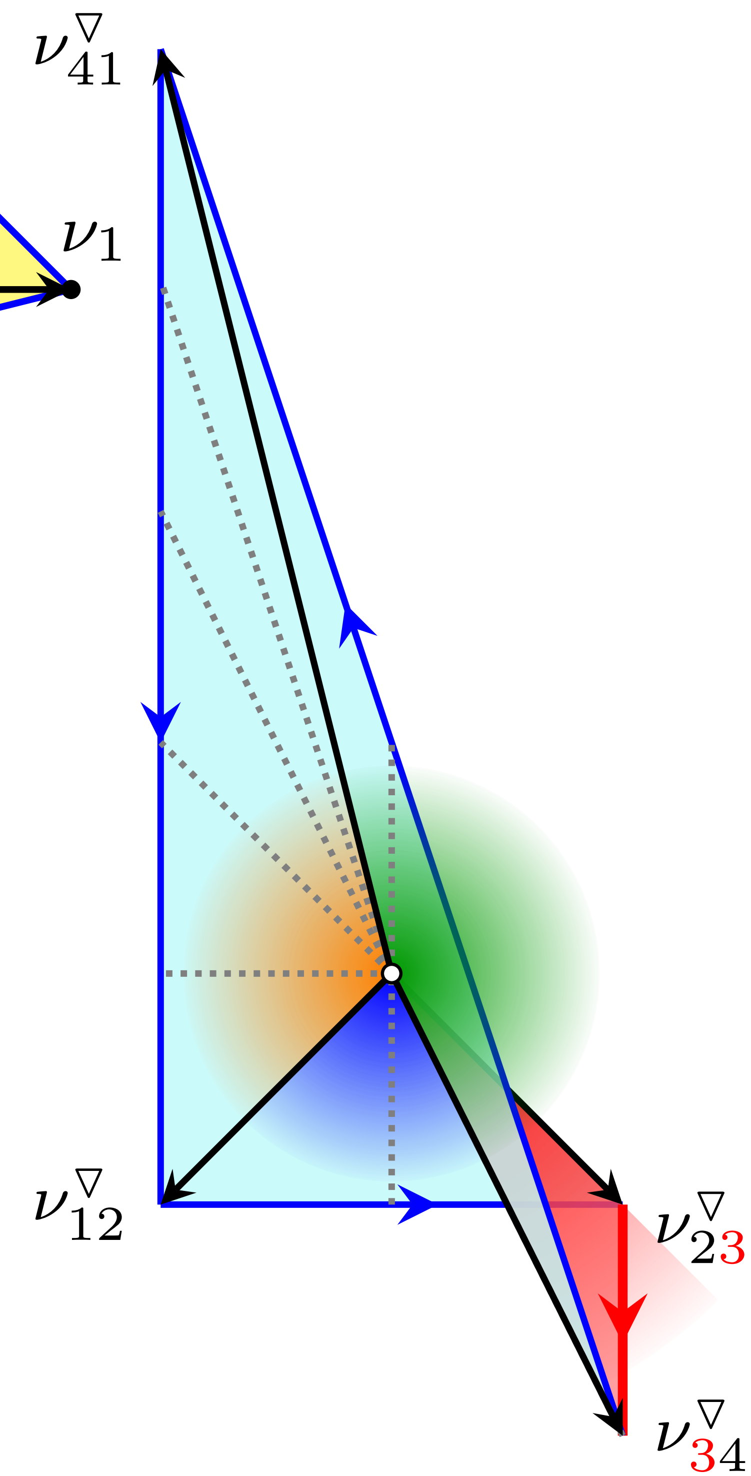
- Abstract the shape (for $m = 3$):



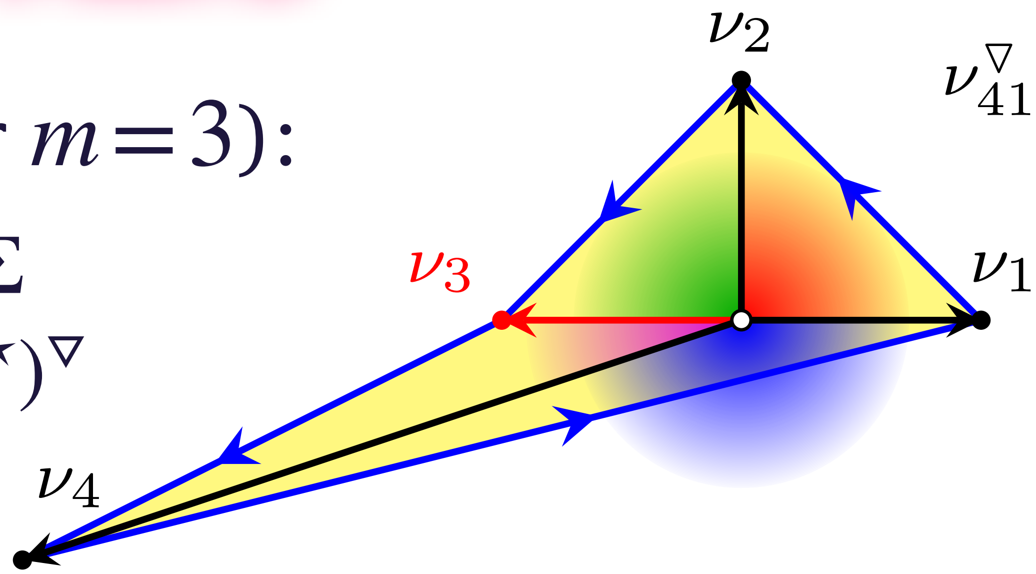


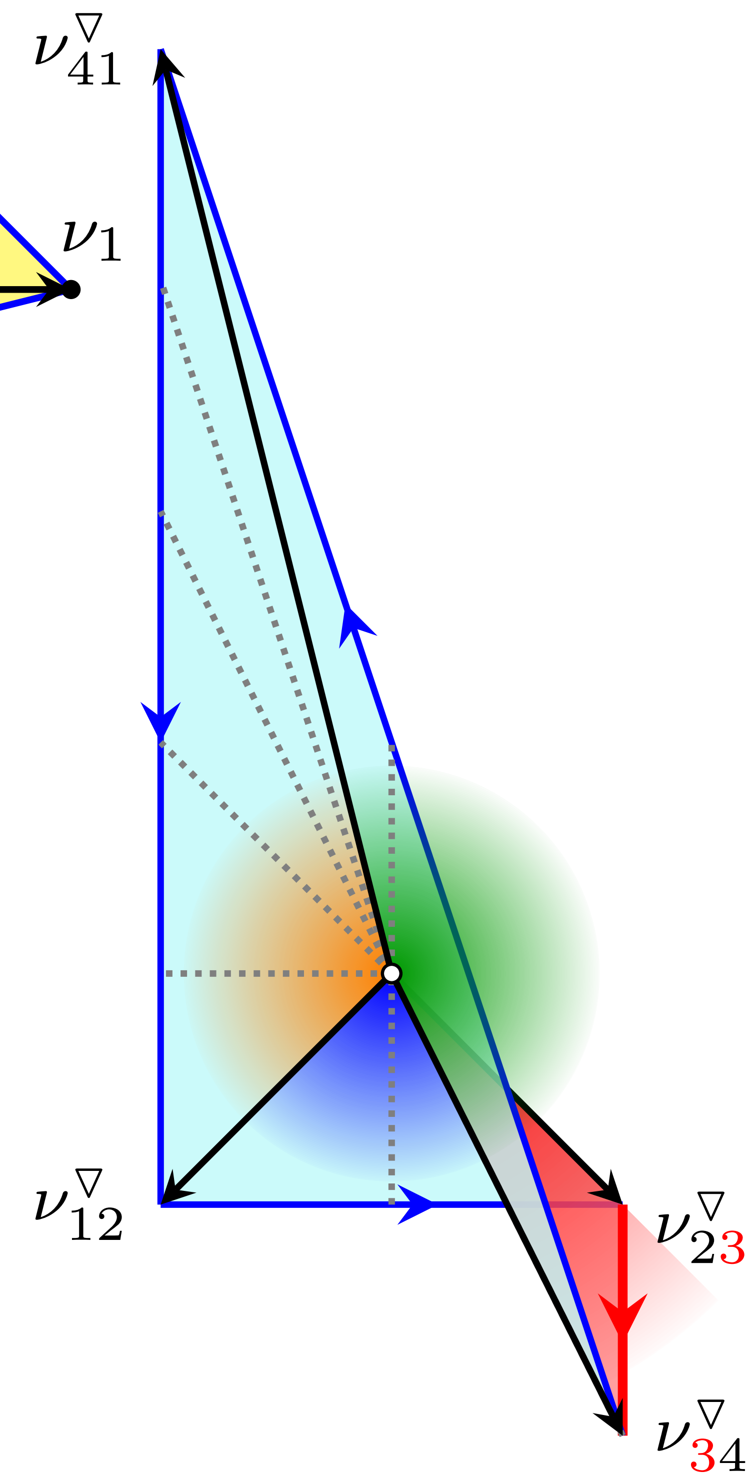
GLSM

→ Toric Geometry “Plan B”

$x_{n+j+1} := y_j$

- Abstract the shape (for $m = 3$):
  - Non-convexity in $\Delta^\star \succcurlyeq \Sigma$
    → flip-folding in $\Delta = (\Delta^\star)^\nabla$





GLSM

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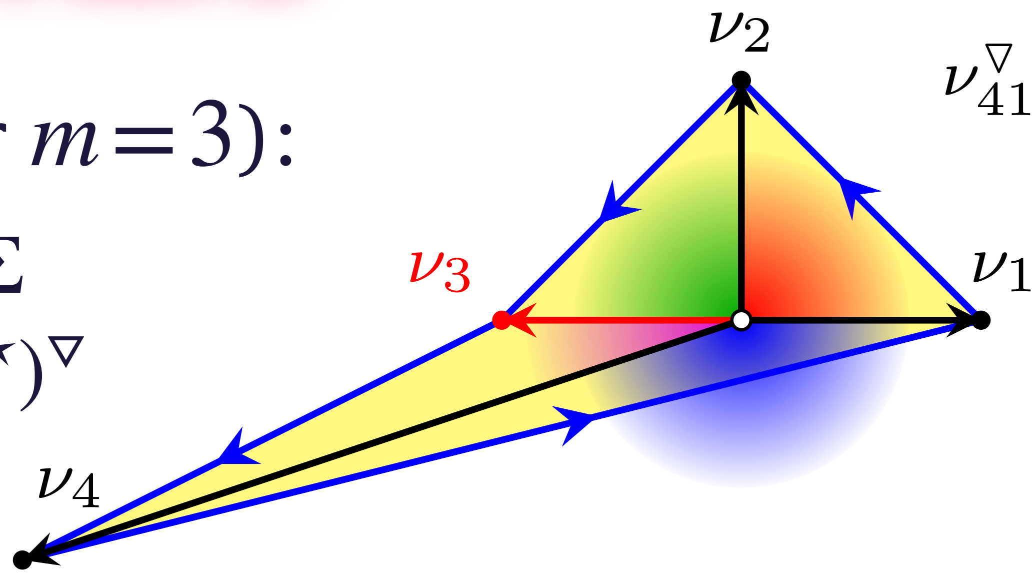

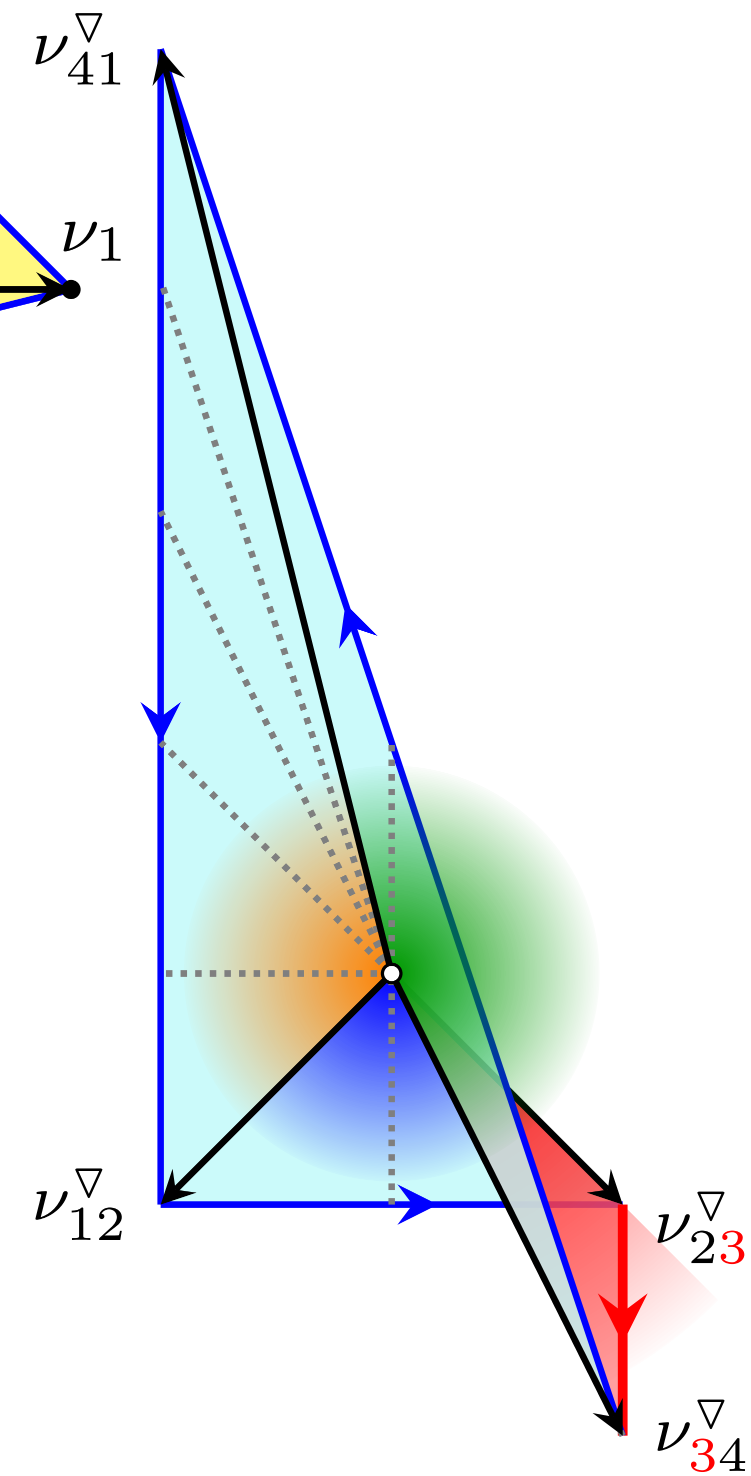

GLSM

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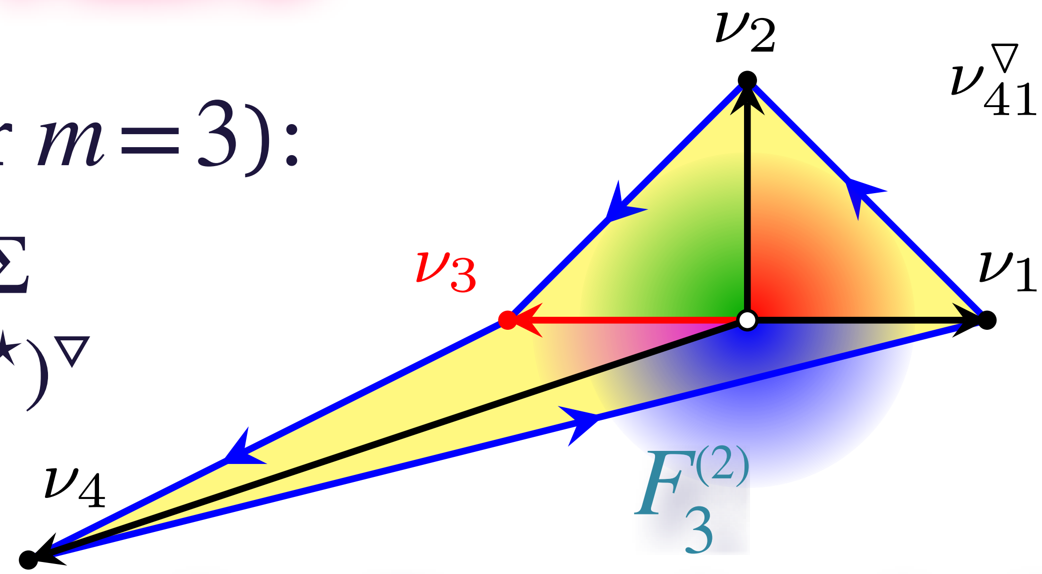
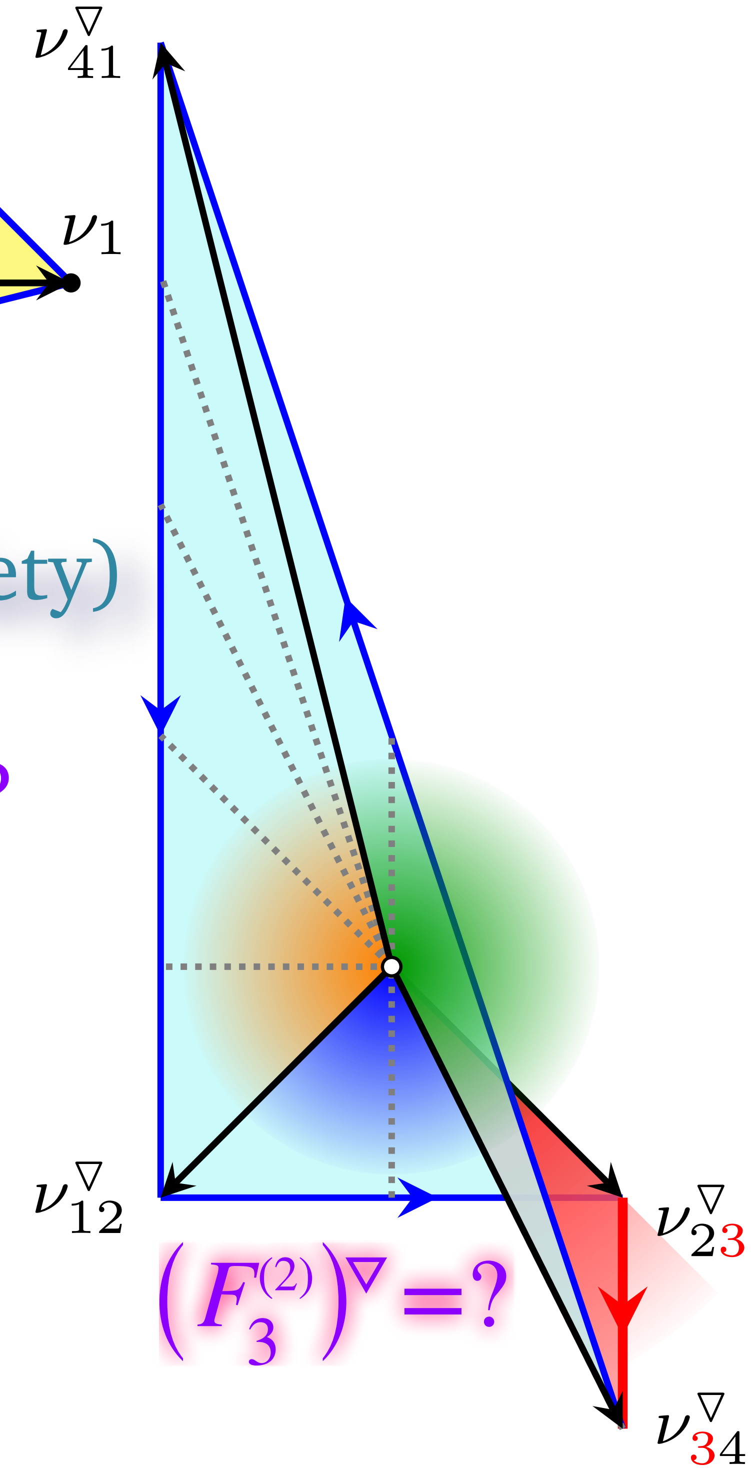
- On what sort of space is ${}^\mathsf{T}\!f(y)$ defined?



(non-Fano toric variety)



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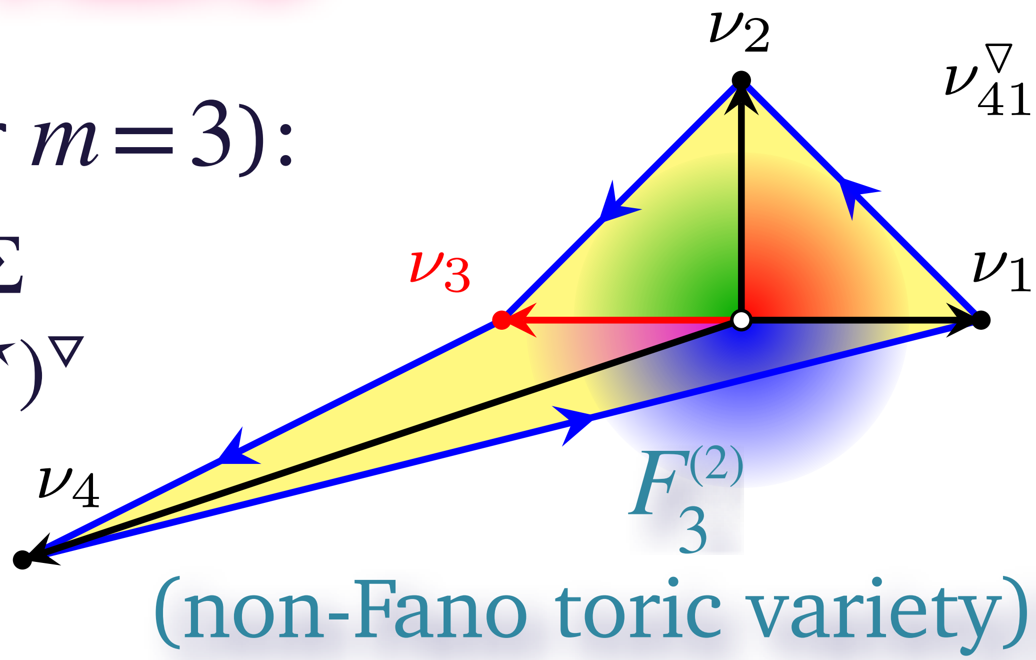
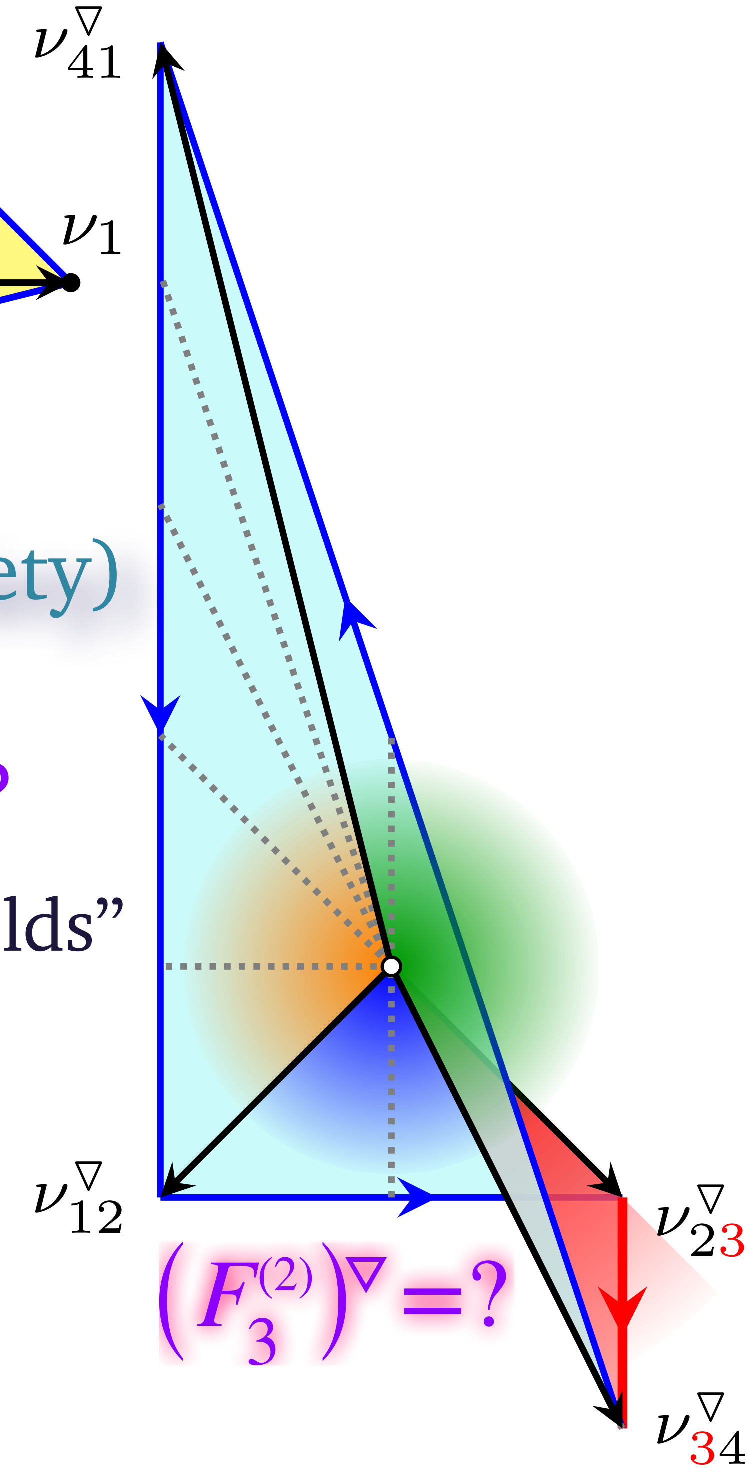


- On what sort of space is ${}^{\mathsf{T}}f(y)$ defined?
  - Flip-folded Δ's span "multifans" $\xleftarrow{\text{~~1-1~~}}$ "torus manifolds"



GLSM

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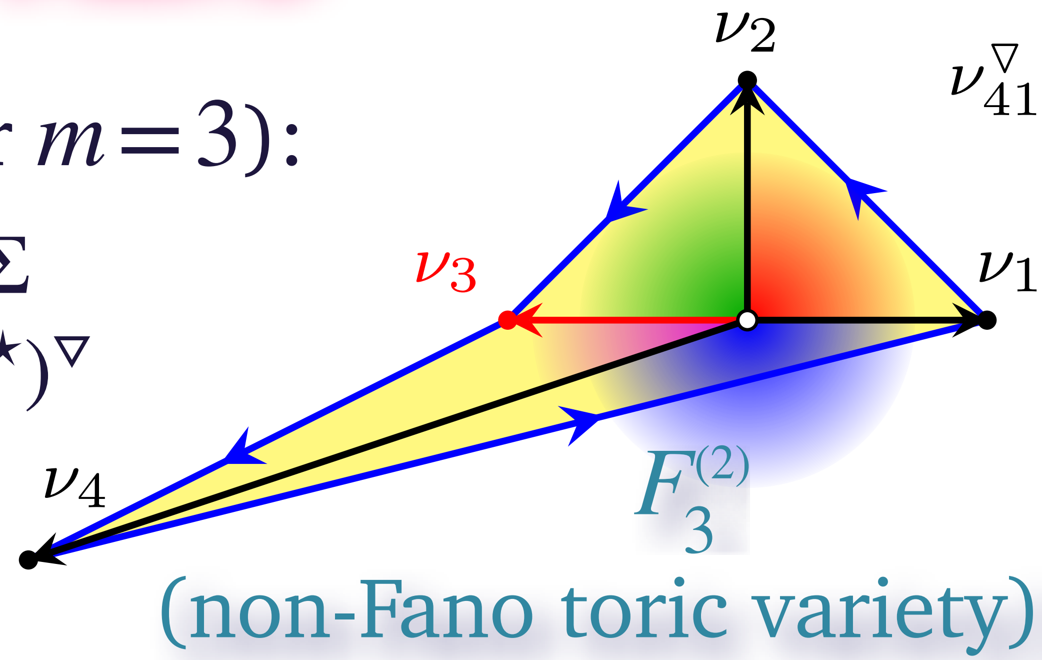

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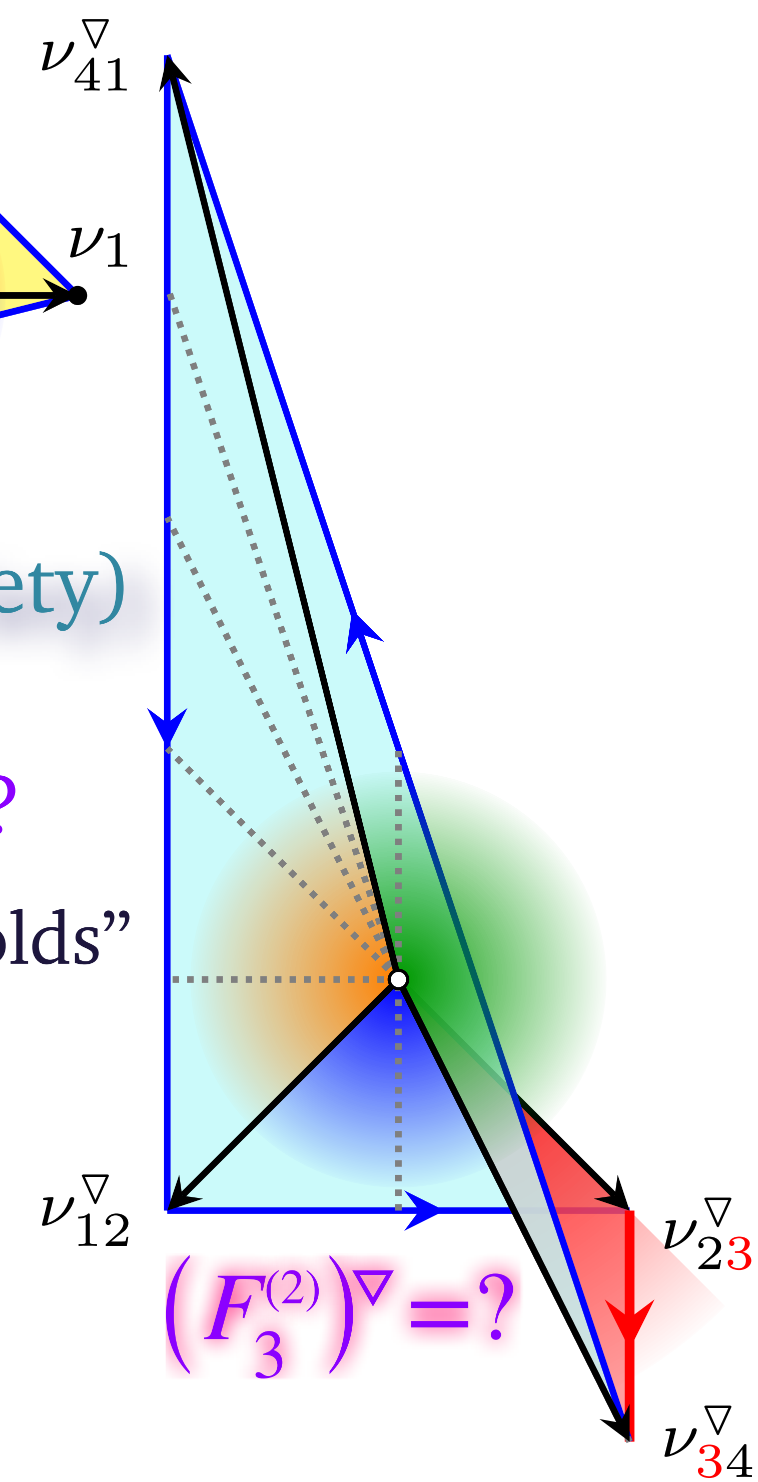

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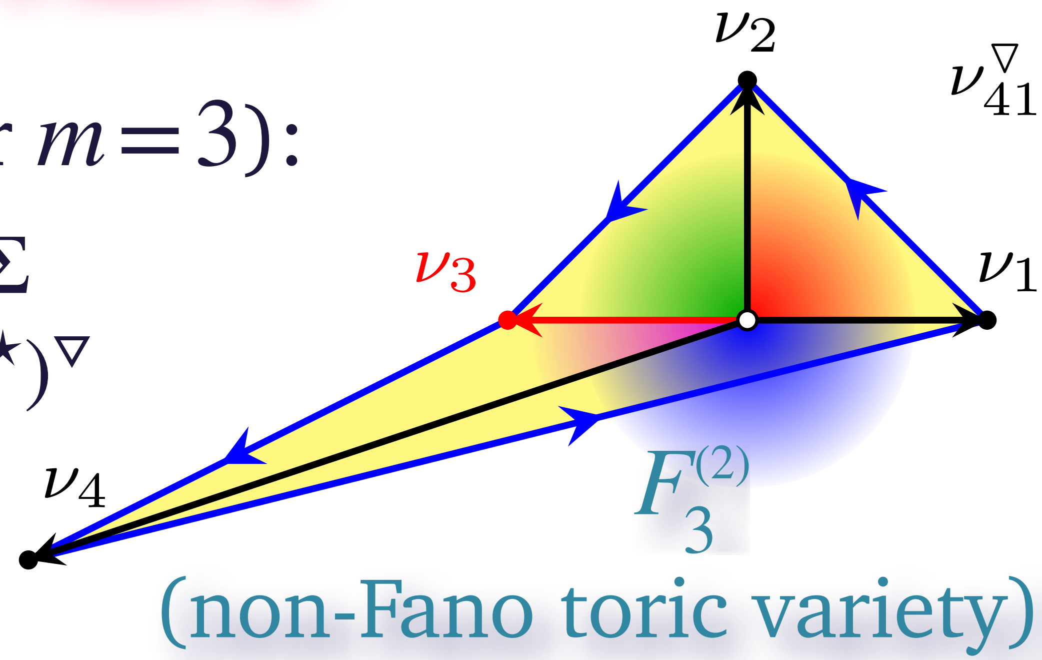

- On what sort of space is ${}^{\mathsf{T}}\!f(y)$ defined?
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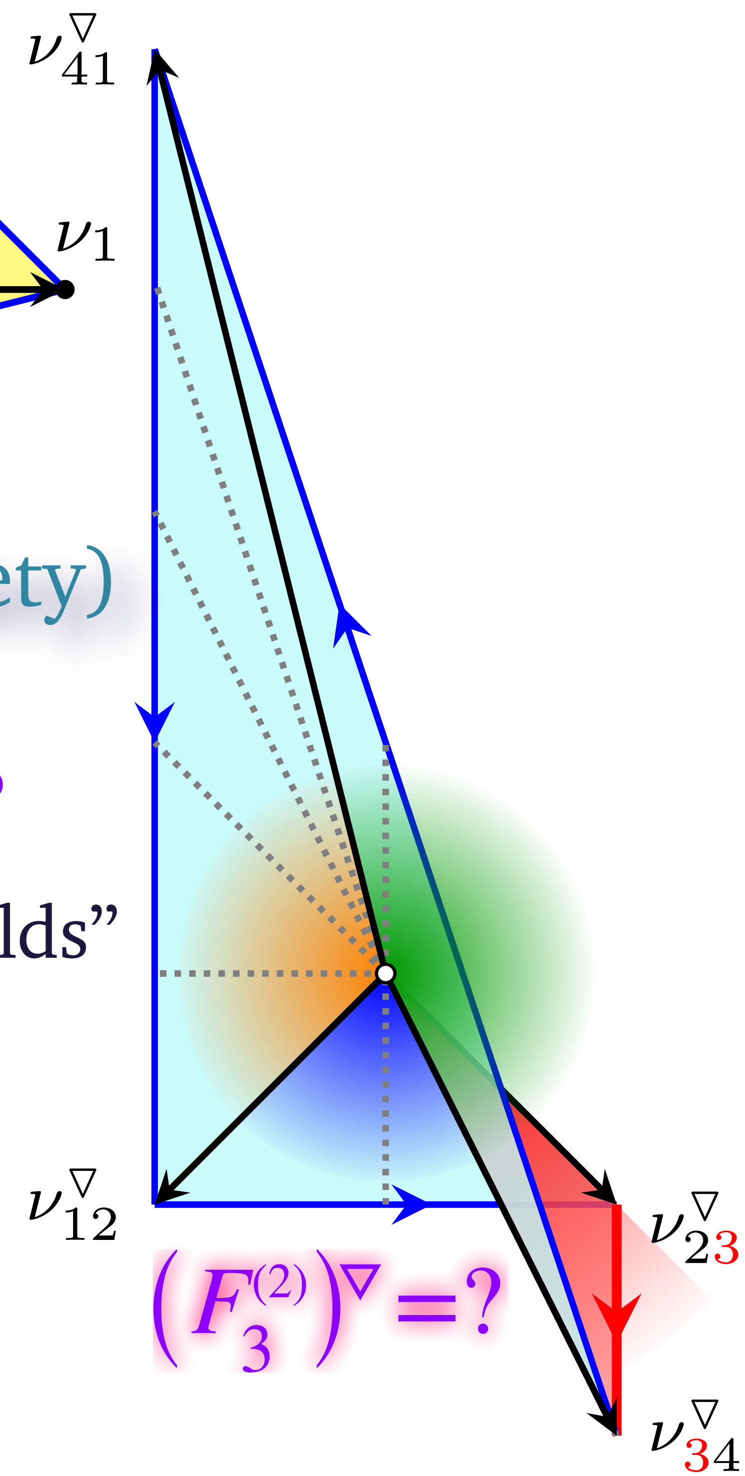

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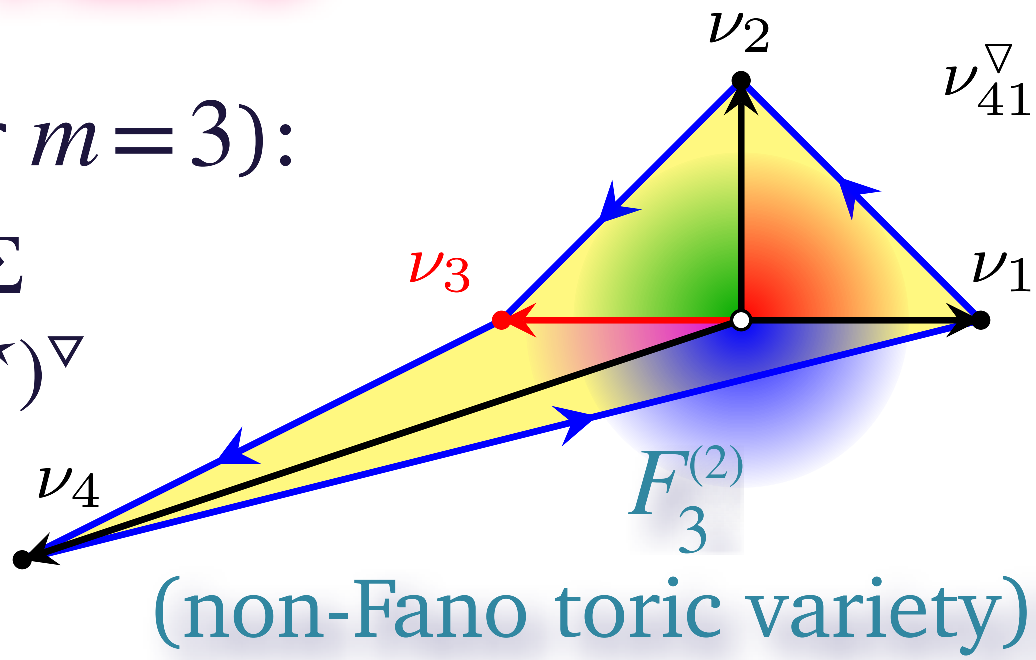

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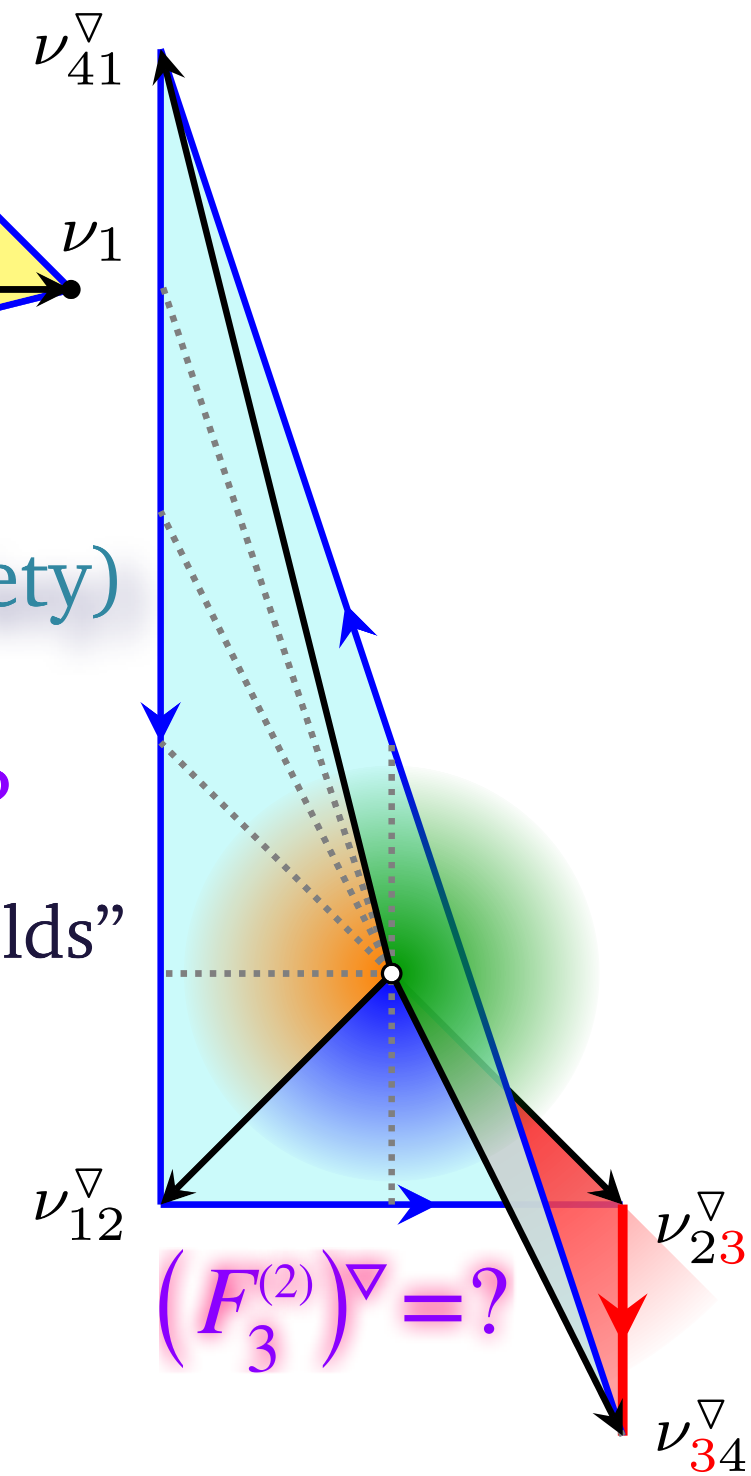

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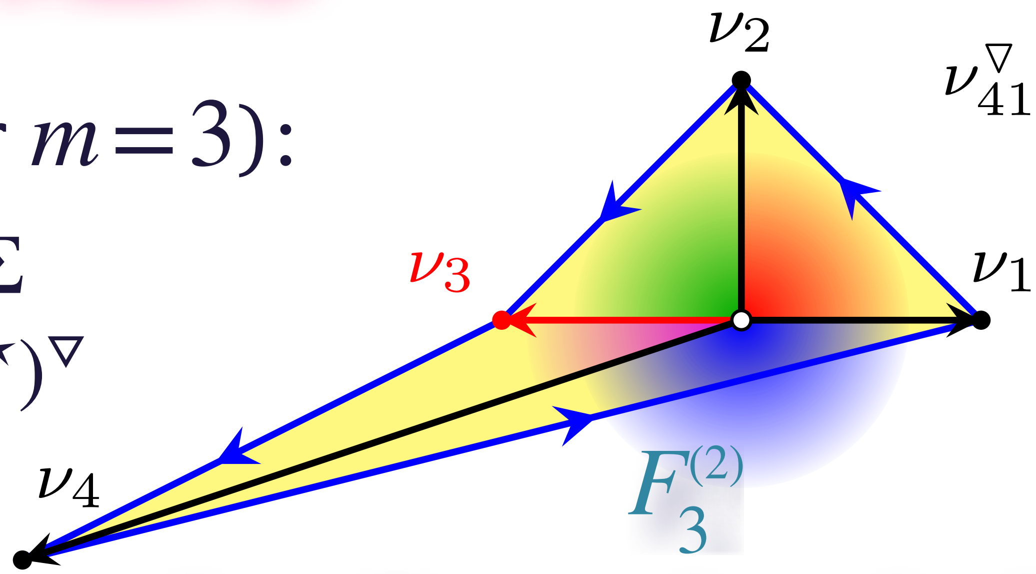

(non-Fano toric variety)

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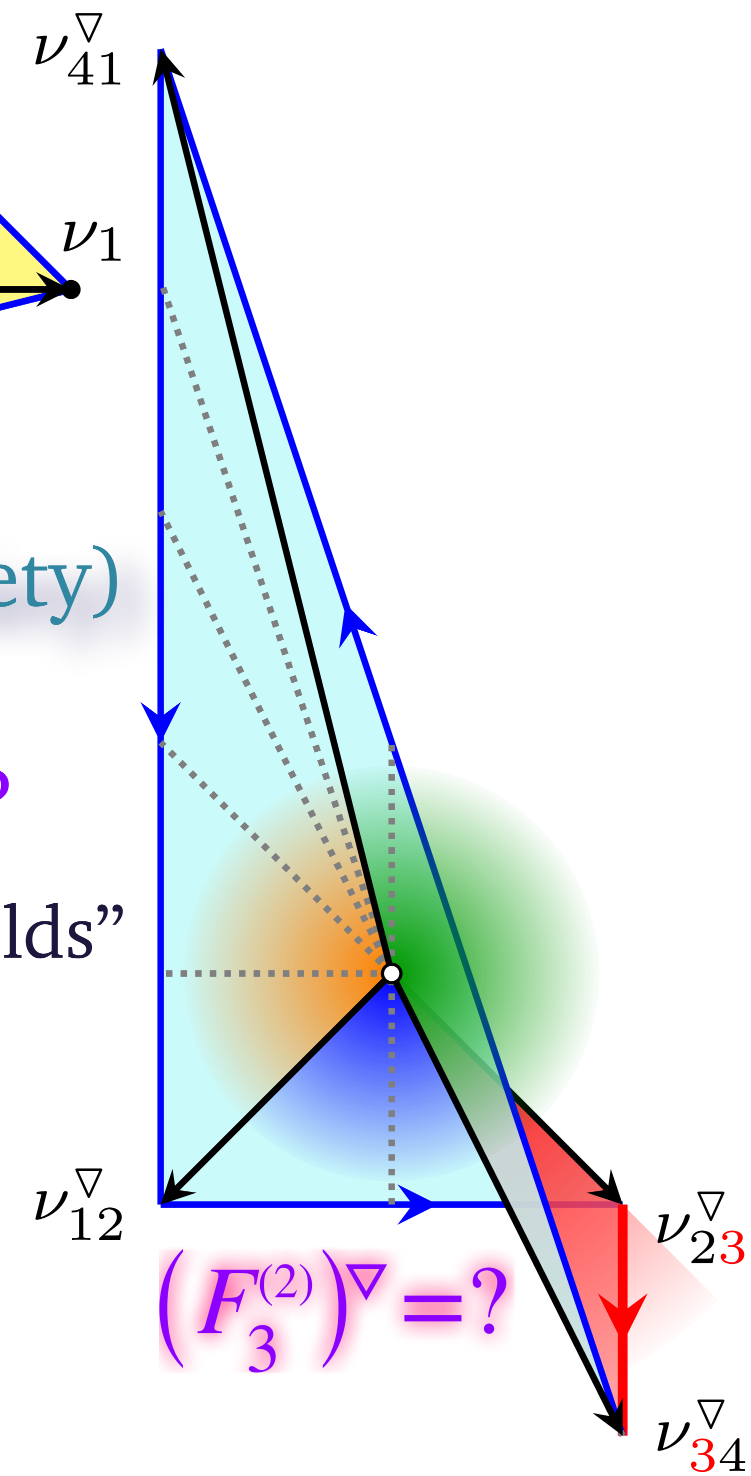

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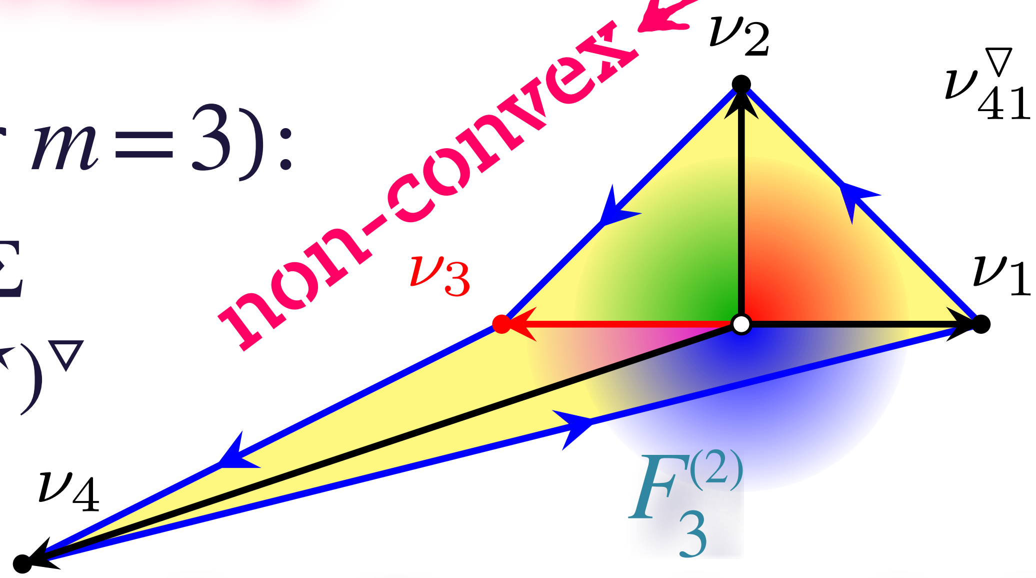

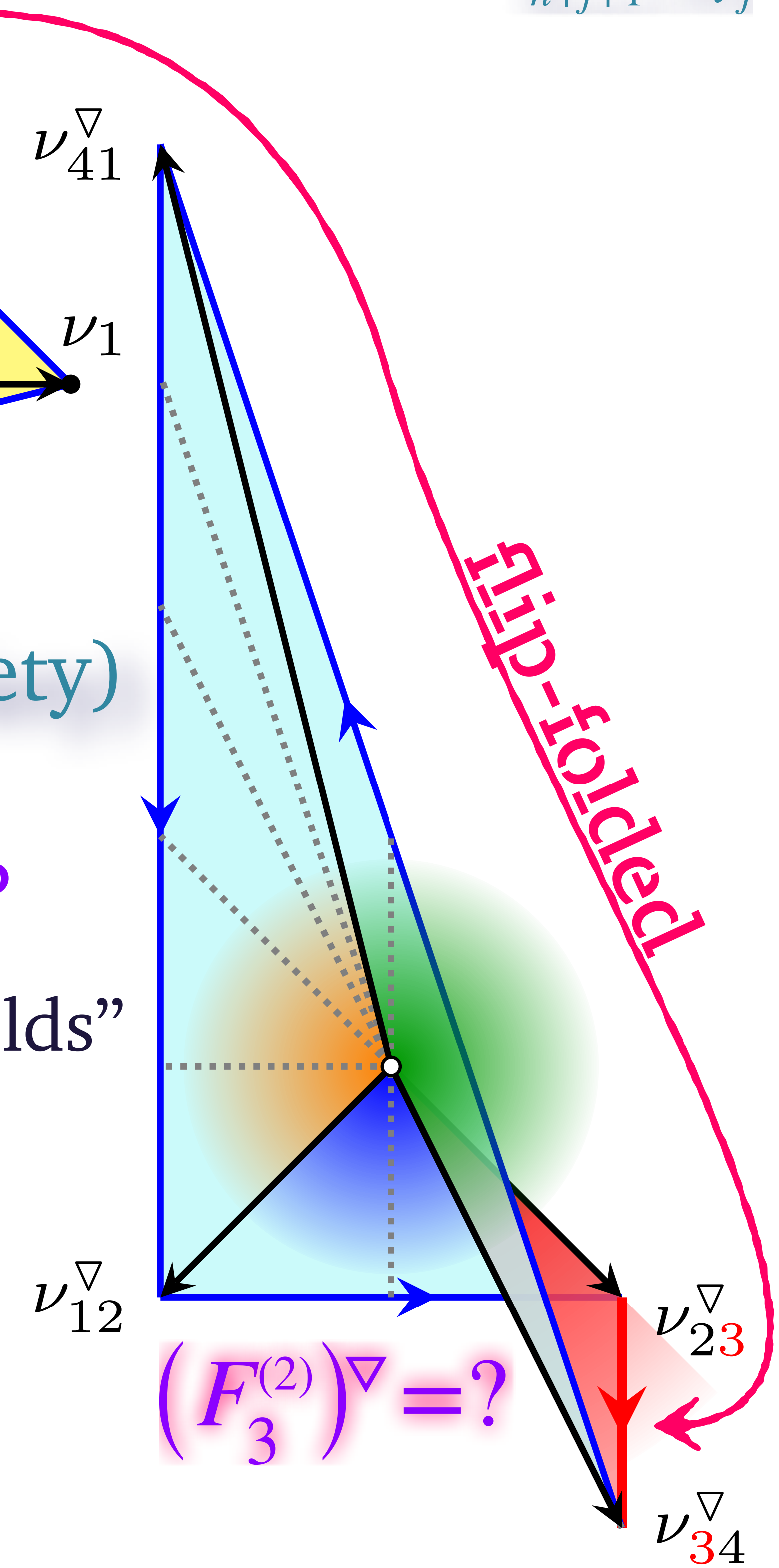

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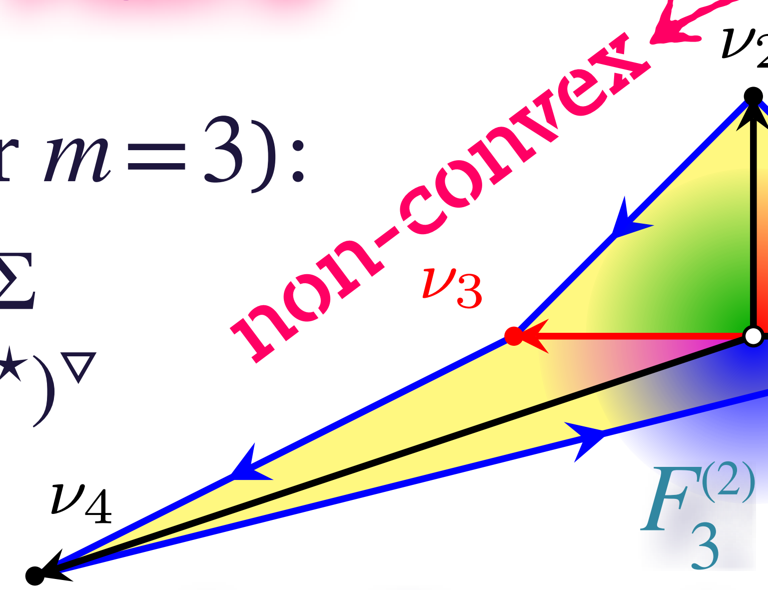

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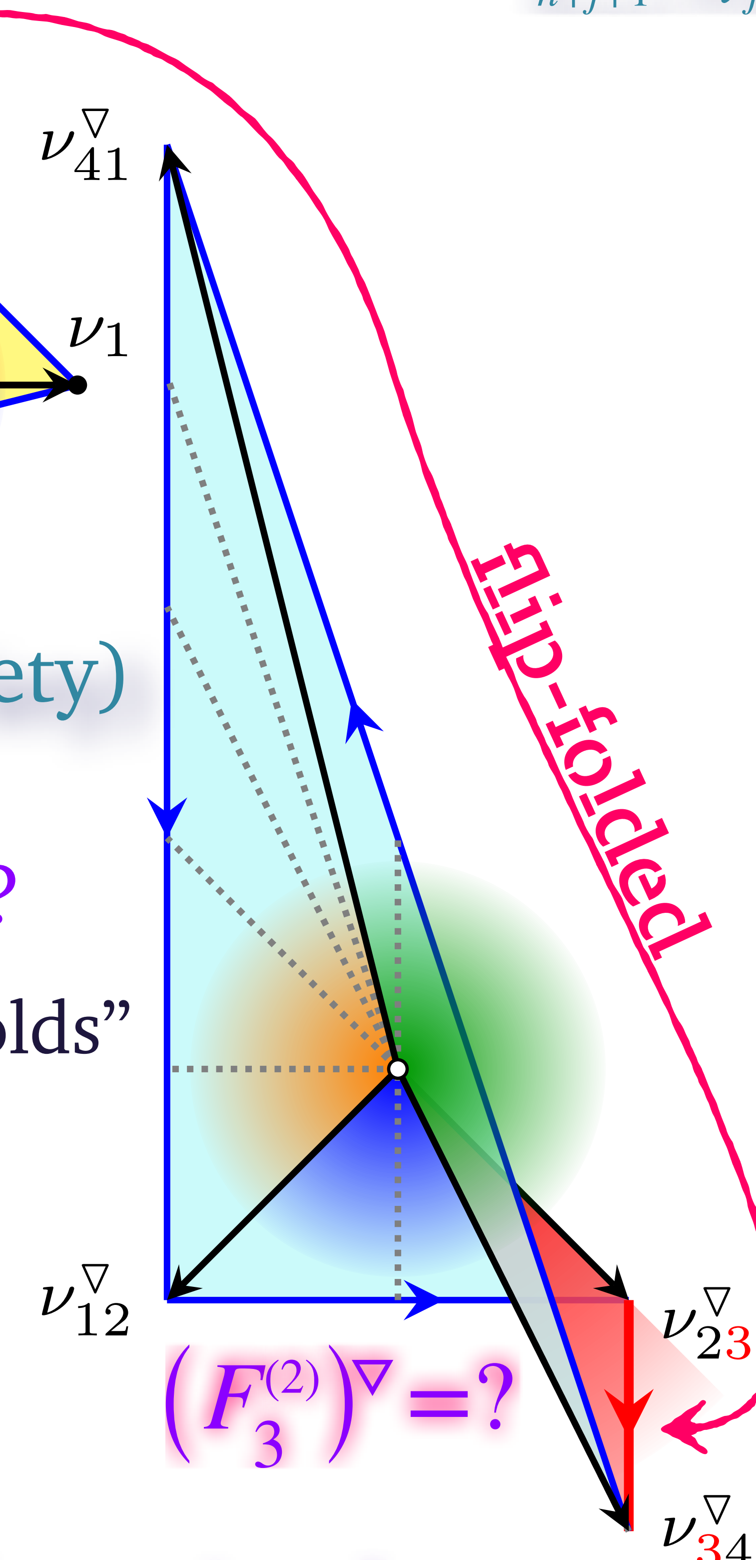

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(for now, see arXiv:2502.08002)

Thank You!

Tristan Hübsch

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