Бранкофест

Београд, Србија

Nonlinearity, Nonlocality and Ultrametricity

Dragovich 80

Stuart-Landau oscillators: Variations on a theme

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This talk is based on (ongoing) work in collaboration with

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Rahul Ghosh (Shiv Nadar Institute of Excellence / IISER Berhampur)
Aryan Patel (IISER Berhampur)
Awadhesh Prasad (University of Delhi) and
Ram Ramaswamy (Indian Institute of Technology - Delhi)

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- Close to the onset of oscillation (bifurcation), the Stuart-Landau oscillator is a universal mathematical model
- It is related to the (complex) Landau-Ginzburg theory of transition

Stuart-Landau oscillator

A non-linear oscillator in two dimensions ($D = 2$) described by the equation

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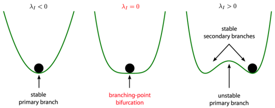

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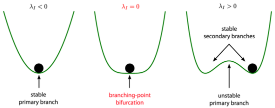

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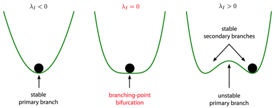

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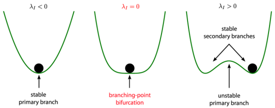

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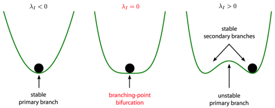

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The Stuart-Landau equations in components are

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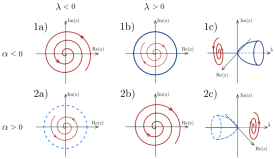

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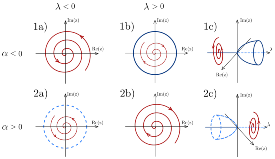

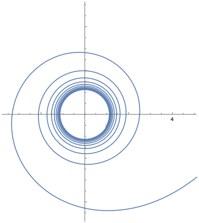

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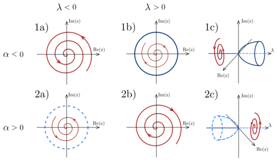
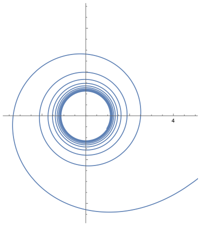
This is an example of (supercritical) Hopf bifurcation in which a pair of eigenvalues change sign from negative to positive values.





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$$\frac{d\mathfrak{q}}{dt} = (\mathfrak{m} - |\mathfrak{q}|^2)\, \mathfrak{q}, \qquad \begin{array}{l} \dot{q}_0 = -\vec{\omega} \cdot \mathbf{q} + (\varrho - |\mathfrak{q}|^2)q_0 \\ \dot{\vec{q}} = \vec{\omega} \times \vec{q} + \vec{\omega} q_0 + (\varrho - |\mathfrak{q}|^2)\vec{q} \end{array} , \qquad \begin{array}{l} |\dot{\mathfrak{q}}| = (\varrho - |\mathfrak{q}|^2)|\mathfrak{q}| \\ \mathfrak{U}^\dagger \dot{\mathfrak{U}} = \mathfrak{m} - \varrho \sim \vec{\omega} \end{array}$$

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- but one imaginary unit $\sqrt{-1} = \mathrm{i}$ (complex) to three $(\mathrm{i}, \mathrm{j}, \mathrm{k})$ (quaternions).

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The exercise is facilitated by the use of Clifford's geometric algebra.

Clifford algebra and imaginary units

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(Spoiler alert: Quaternions do not really generalise to $D = 4$, but to $D = 3$.)

Generalised Stuart-Landau equation in $\mathbb{R}^D$

Our generalisation: the coordinate vector $\mathbf{x}$ in $\mathbb{R}^D$ evolves as

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The generalised SL eqn multiplied by R.

$$\frac{d}{dt}\mathrm{R}\,\mathbf{x} = (\varrho - |\mathrm{R}\,\mathbf{x}|^2)\,\mathrm{R}\,\mathbf{x} + \mathrm{R}\,\boldsymbol{\mu}\,\mathrm{R}^T \cdot \mathrm{R}\,\mathbf{x}$$

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Diagonal blocks of the equations in $\mathbb{R}^D$

The equation for $r = |\mathbf{x}| = |\mathbf{y}|$ is decoupled

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where $\mathrm{a}_j^{(\infty)}$ is a constant that determines the radius of the asymptotic attracting circle on the z_j-plane.

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These are determined from the initial condition and are asymptotically conserved charges. The limit cycle is a torus

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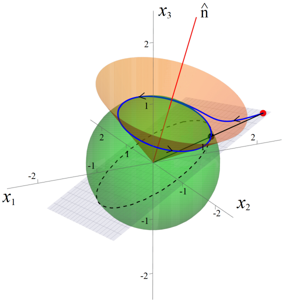
$D = 3$

The limit cycle is the blue circle of radius $\sqrt{\varrho\,(1 - (\frac{\mathbf{x}_0}{|\mathbf{x}_0|}\cdot\hat{\mathbf{n}})^2)}$. (where $\mathbf{x}_0$ is the initial value).



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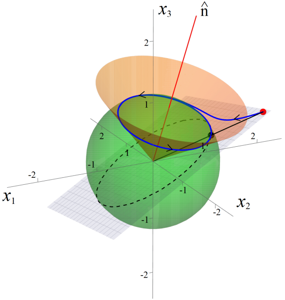

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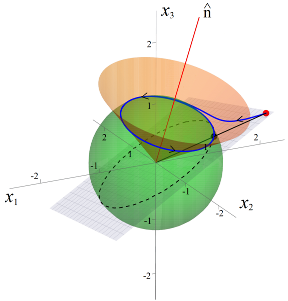

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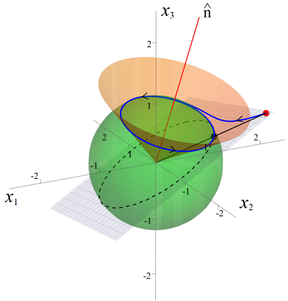

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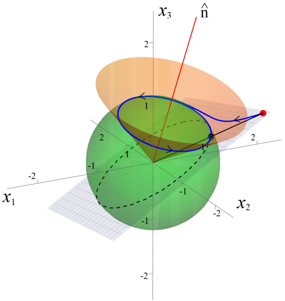

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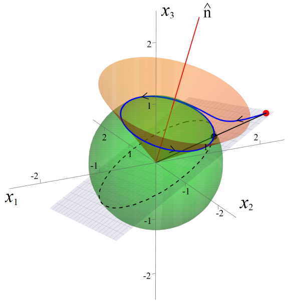

The limit cycle is inscribed in the projection of the great circle defined by the intersection of the initial position-velocity (two-)plane ($\mathbf{x}\wedge\dot{\mathbf{x}}$) and the attractor $\mathbb{S}^3 : r^2 = \varrho$.

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The limit cycle is two-torus T^2, which is a product of two circles on the $z_1 = y_1 + \mathrm{i}\, y_2$ and $z_2 = y_3 + \mathrm{i}\, y_4$ planes.

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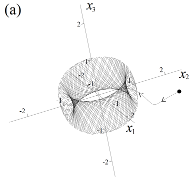

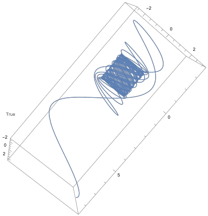

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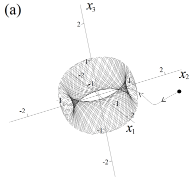
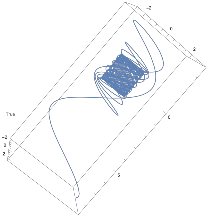
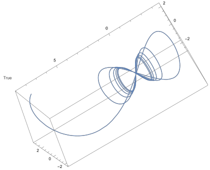
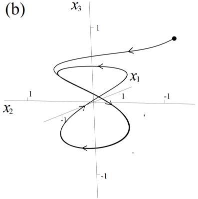
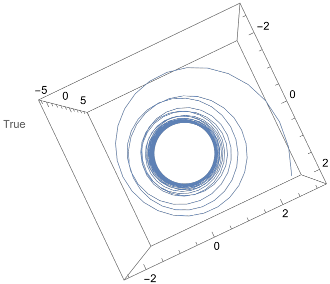








4D redux



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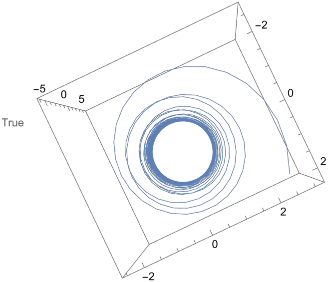

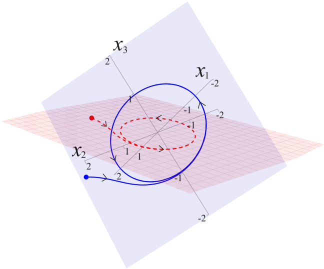

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The 4D case may be truncated by imposing three relations among the six μ_{ij} to get a quaternionic SL oscillator.

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The instantaneous position and velocity defines the bivector-plane $\mathrm{P}_2(t) \sim \mathbf{x} \wedge \dot{\mathbf{x}}$ through the origin. The unit bivector evolves as

$$\frac{d}{dt}\boldsymbol{\ell} = \frac{1}{|\mathbf{x}|\ |\boldsymbol{\mu}\cdot\mathbf{x}|} \sum_{i,j,k} x_i (\mathbb{M}^2)_{jk} x_k \,(\mathbf{e}_i \wedge \mathbf{e}_j)$$

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Consistent reduction to a G_2 symmetric system possible in $D = 7$.

Coupled oscillators: symmetry preserving coupling

Consider two coupled SL oscillators:

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A closed system of equations **if** $\boldsymbol{\mu} = \boldsymbol{\nu}$.

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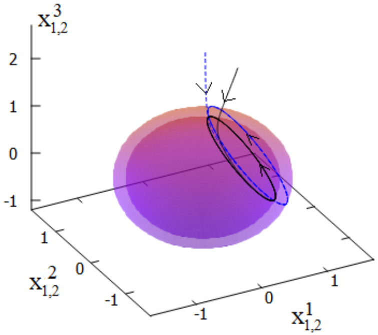
Since $\alpha_* = 0$, the two oscillators are completely synchronized.

Two identical oscillators in synchronisation



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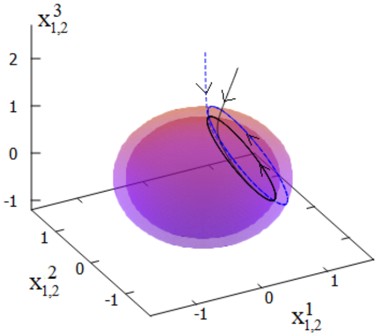

N coupled oscillators (with symmetry preserving couplings) with parameters $\mu_1, \mu_2, \cdots, \mu_N$ which can all be brought to the normal form by the same rotation (different 'eigenvalues', i.e., frequencies, but the same 'eigenvectors') can also be treated analytically to a large extent.

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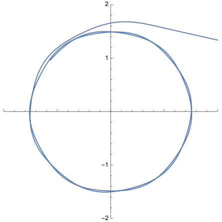

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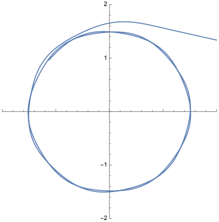

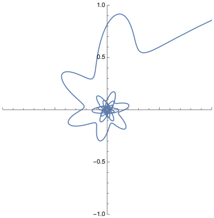

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Discretise time $t \in \mathbb{R} \to n \in \mathbb{Z}$

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This would make sense for $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} = |\mathbf{x}|\,\mathcal{U}$, where $|\mathcal{U}| = 1$, so $\mathcal{U}$ is a unit element.

Towards an ultrametric nonlinear oscillator

A p-adic dynamical variable $x \in \mathbb{Q}_p$ has a polar decomposition: $x = |x|_p U(x)$, with $U(x) \in \mathbb{Z}_p$, the p-adic units.

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The proposed dynamics is described by (other variations possible):

$$\begin{aligned} |x_{n+1}|_p &= |(\mu - g(|x_n|_p))\, x_n| \\ &= |x_n|_p \times \max\left(|\mu|_p, |x_n|_p\right) \\ U(x_{n+1}) &= U(\mu)\, U(x_n) \end{aligned}$$

The limit cycle is the unit circle $|x|_p = 1$. Therefore, it reduces to dynamics on the finite fields $(\mathbb{Z}/p\mathbb{Z})^\times$ (coefficients of the *p*-adic Laurent expansion)—this has been discussed earlier by [Nambu], [Meurice], [Freund-Olson], $\cdots$. Additionally, [Meurice] considered '*p*-adic time' using the fact that $\mathbb{Z}_p \subset \mathbb{Z}$ is dense.
The dynamical equations can easily be modified to include *nonlocality*.

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THANK YOU