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Zeta functions in Physics: a comprehensive review

Emilio Elizalde

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Zero point energy

QFT vacuum to vacuum transition: $\langle 0|H|0\rangle$

Spectrum, normal ordering (harm oscill):

$$H = \left(n + \frac{1}{2}\right) \lambda_n \, a_n \, a_n^\dagger$$

$$\langle 0|H|0\rangle = \frac{\hbar\, c}{2} \sum_n \lambda_n = \frac{1}{2} \, \text{tr} \; H$$

gives ∞ physical meaning?

Regularization $+$ Renormalization (cut-off, dim, ζ)

Even then: Has the final value real sense ?

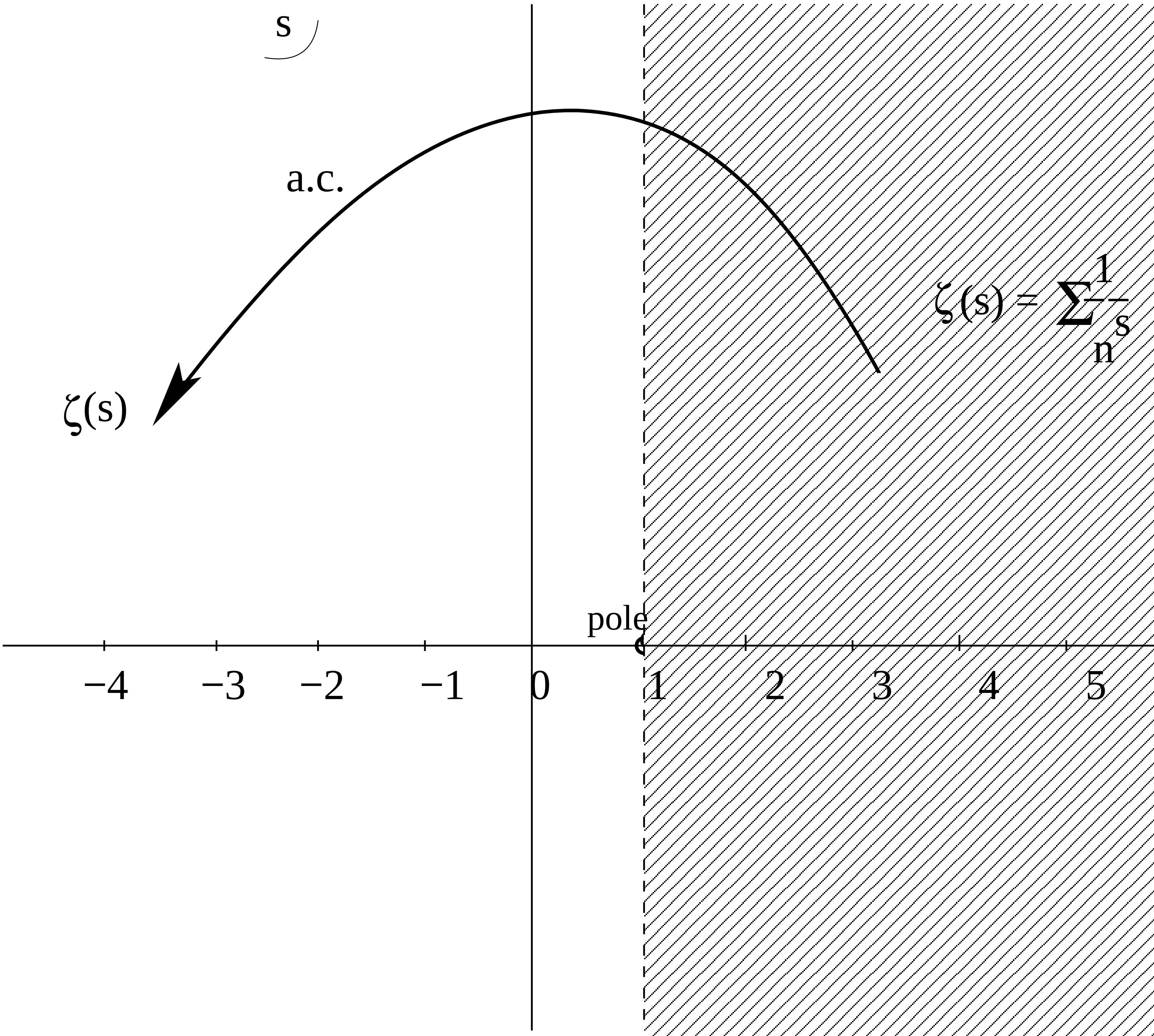
Effects of the Quantum Vacuum

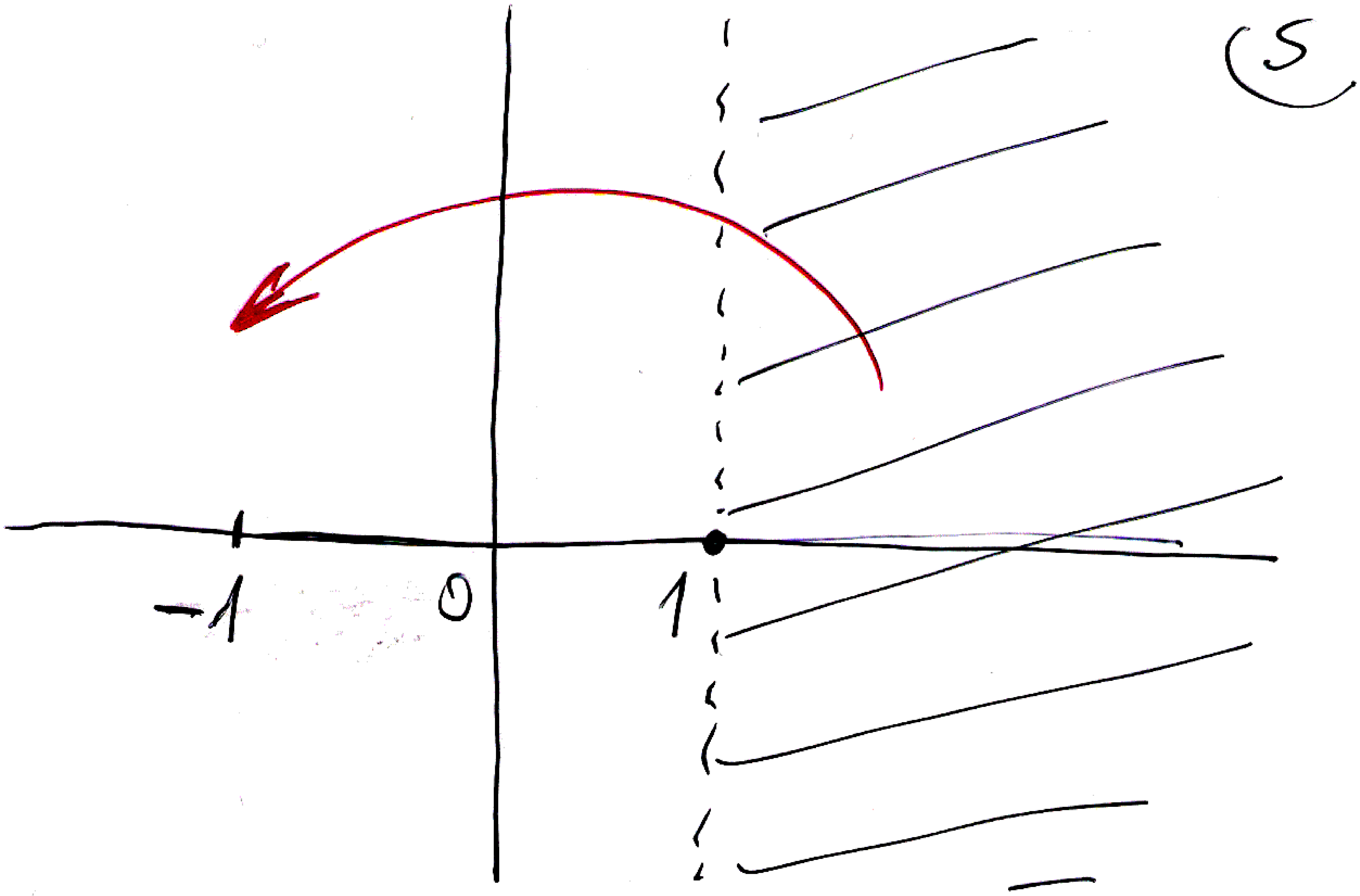
a) Negligible: Sonoluminiscence, Schwinger ~10^{-5}

b) Important: Wetting He3 – alcali ~30%

c) Incredibly big: Cosmological constant ~10^{120}

Riemann Zeta Function





$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty$$

$$\zeta(0) = -\frac{1}{2} \quad \text{or} \quad 1 + 1 + 1 + \dots \underset{\zeta}{=} -\frac{1}{2}$$

$$\zeta(-1) = -\frac{1}{12} \quad \text{or} \quad 1 + 2 + 3 + \dots \underset{\zeta}{=} -\frac{1}{12}$$

⋮

F Yndurain, A Slavnov
"As everybody knows ..."

Operator Zeta F's in MΦ: Origins

- The Riemann zeta function $\zeta(s)$ is a function of a complex variable, s. To define it, one starts with the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

  which converges for all complex values of s with real Re $s > 1$, and then defines $\zeta(s)$ as the analytic continuation, to the whole complex $s-$plane, of the function given, Re $s > 1$, by the sum of the preceding series.

  Leonhard Euler already considered the above series in 1740, but for positive integer values of s, and later Chebyshev extended the definition to Re $s > 1$.

- Godfrey H Hardy and John E Littlewood, "Contributions to the Theory of the Riemann Zeta-Function and the Theory of the Distribution of Primes", Acta Math 41, 119 (1916)

  Did much of the earlier work, by establishing the convergence and equivalence of series regularized with the heat kernel and zeta function regularization methods

  G H Hardy, Divergent Series (Clarendon Press, Oxford, 1949)

  Srinivasa I Ramanujan had found for himself the functional equation of the zeta function

- Torsten Carleman, "Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes" (French), 8. Skand Mat-Kongr, 34-44 (1935)

  Zeta function encoding the eigenvalues of the Laplacian of a compact Riemannian manifold for the case of a compact region of the plane

- Robert T Seeley, “Complex powers of an elliptic operator. 1967 Singular Integrals" (Proc. Sympos. Pure Math., Chicago, Ill., 1966) pp. 288-307, Amer. Math. Soc., Providence, R.I.

  Extended this to elliptic pseudo-differential operators A on compact Riemannian manifolds. So for such operators one can define the determinant using zeta function regularization

- D B Ray, Isadore M Singer, “R-torsion and the Laplacian on Riemannian manifolds", Advances in Math 7, 145 (1971)

  Used this to define the determinant of a positive self-adjoint operator A (the Laplacian of a Riemannian manifold in their application) with eigenvalues $a_1, a_2, ...,$ and in this case the zeta function is formally the trace

$$\zeta_A(s) = \mathrm{Tr}\,(A)^{-s}$$

  the method defines the possibly divergent infinite product

$$\prod_{n=1}^{\infty} a_n = \exp[-\zeta_A{}'(0)]$$

- J. Stuart Dowker, Raymond Critchley

  "Effective Lagrangian and energy-momentum tensor in de Sitter space", Phys. Rev. D13, 3224 (1976)

  Abstract

  The effective Lagrangian and vacuum energy-momentum tensor $< T^{\mu\nu} >$ due to a scalar field in a de Sitter space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form $(\Box^2 + \xi R + m^2)\varphi = 0$. If $\xi = 1/6$ and $m = 0$, the renormalized $< T^{\mu\nu} >$ equals $g^{\mu\nu}(960\pi^2 a^4)^{-1}$, where a is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.

Effective Lagrangian and energy-momentum tensor in de Sitter space

J. S. Dowker and Raymond Critchley
Department of Theoretical Physics, The University, Manchester, 13, England
(Received 29 October 1975)

The effective Lagrangian and vacuum energy-momentum tensor $\langle T^{\mu\nu}\rangle$ due to a scalar field in a de Sitter-space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form $(\Box^2 + \xi R + m^2)\varphi = 0$. If $\xi = 1/6$ and $m = 0$, the renormalized $\langle T^{\mu\nu}\rangle$ equals $g^{\mu\nu}(960\pi^2 a^4)^{-1}$, where a is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.

I. INTRODUCTION

In a previous paper[1] (to be referred to as I) the effective Lagrangian $\mathfrak{L}^{(1)}$ due to single-loop diagrams of a scalar particle in de Sitter space was computed. It was shown to be real and was evaluated as a principal-part integral. The regularization method used was the proper-time one due to Schwinger[2] and others. We now wish to consider the same problem but using different techniques. In particular, we wish to make contact with the work of Candelas and Raine,[3] who first discussed the same problem using dimensional regularization.

Some properties of the various regularizations as applied to the calculation of the vacuum expectation value of the energy-momentum tensor have been contrasted by DeWitt.[4] We wish to pursue some of these questions within the context of a definite situation.

II. GENERAL FORMULAS: REGULARIZATION METHODS

We use exactly the notation of I, which is more or less standard, and begin with the expression for $\mathfrak{L}^{(1)}$ in terms of the quantum-mechanical propagator, $K(x'', x', \tau)$,

$$\mathfrak{L}^{(1)}(x') = -\tfrac{1}{2} i \lim_{x''\to x'} \int_0^\infty d\tau\, \tau^{-1} K(x'', x', \tau) e^{-im^2\tau} + X(x') . \tag{1}$$

There are two points regarding this expression which need some further discussion. Firstly, if we adopt the proper-time regularization method so that the infinities appear only when the τ integration, which is the final operation, is performed, then we can take the coincidence limit, $x'' = x'$, through into the integrand. Further, since the regularized expression is continuous across the light cone, it does not matter how we let x'' approach x'. Secondly, the term X does not have to be a constant, but it should integrate to give a metric-independent contribution to the total action, $W^{(1)}$.

The Schwinger-DeWitt procedure is to derive an expression for K, either in closed form or as a general expansion to powers of τ, then to effect the coincidence limit in (1), and finally to perform the τ integration. This was the approach adopted in I. We proceed now to give a few more details.

We assume that we are working on a Riemannian space, $\mathfrak{M}$, of dimension d. The coincidence limit $K(x, x, \tau)$ can be expanded,[5]

$$K(x, x, \tau) = i(4\pi i\tau)^{-d/2} \sum_{n=0}^{\infty} a_n(x)(i\tau)^n , \tag{2}$$

where the a_n are scalars constructed from the curvature tensor on $\mathfrak{M}$ and whose functional form is independent of d. The manifold $\mathfrak{M}$ must not have boundaries, otherwise other terms appear in the expansion.

The expansion (2) is substituted into (1) to yield

$$\mathfrak{L}^{(1)}(x) = \tfrac{1}{2} i (4\pi)^{-d/2} \sum_n a_n(x) \int_0^\infty (i\tau)^{n-d/2-1} e^{-im^2\tau} d\tau . \tag{3}$$

The infinite terms are those for which $n \leq d/2$ (for d even) or $n \leq (d-1)/2$ (for d odd). For $d = 4$, e.g. space-time, there are three infinite terms. These terms are removed by renormalization; the details are given by DeWitt.[4]

Another popular regularization technique is dimensional regularization.[6] In this method the dimension, d, is considered to be complex and all expressions are defined in a region of the d plane where they converge. The infinities appear when an analytic continuation to $d = 4$ is performed to regain the physical quantities. This idea was originally developed for use in flat-space (i.e., Lorentz-invariant) situations for the momentum

- Stephen W Hawking, “Zeta function regularization of path integrals in curved spacetime", Commun Math Phys 55, 133 (1977)

  This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a generalized zeta function from the eigenvalues of the differential operator that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique agrees with dimensional regularization where one generalises to n dimensions by adding extra flat dims. The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation which describes diffusion over the four dimensional spacetime manifold in a fifth dimension of parameter time. Using the asymptotic expansion for the heat kernel, one can deduce the behaviour of the path integral under scale transformations of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This EM tensor has an anomalous trace.

Commun. math. Phys. 55, 133—148 (1977)

Communications in Mathematical Physics

Zeta Function Regularization of Path Integrals in Curved Spacetime

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Abstract. This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a generalized zeta function from the eigenvalues of the differential operator that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique agrees with dimensional regularization where one generalises to n dimensions by adding extra flat dimensions. The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation which describes diffusion over the four dimensional spacetime manifold in a fith dimension of parameter time. Using the asymptotic expansion for the heat kernel, one can deduce the behaviour of the path integral under scale transformations of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This energy momentum tensor has an anomalous trace.

1. Introduction

The purpose of this paper is to describe a technique for obtaining finite values to path integrals for fields (including the gravitational field) on a curved spacetime background or, equivalently, for evaluating the determinants of differential operators such as the four-dimensional Laplacian or D'Alembertian. One forms a gemeralised zeta function from the eigenvalues λ_n of the operator

$$\zeta(s) = \sum_n \lambda_n^{-s} . \tag{1.1}$$

In four dimensions this converges for $\mathrm{Re}(s) > 2$ and can be analytically extended to a meromorphic function with poles only at $s=2$ and $s=1$. It is regular at $s=0$. The derivative at $s=0$ is formally equal to $-\sum_n \log \lambda_n$. Thus one can define the determinant of the operator to be $\exp(-d\zeta/ds)|_{s=0}$.

Basic strategies

- Jacobi's identity for the $\theta-$function

$$\theta_3(z,\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad q := e^{i\pi\tau},\ \tau \in \mathbb{C}$$

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{z^2/i\pi\tau}\, \theta_3\left(\frac{z}{\tau}\Big|\frac{-1}{\tau}\right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}} \cos(2\pi n z), \qquad z,t \in \mathbb{C},\ \text{Re}\,t > 0$$

- Higher dimensions: Poisson summ formula (Riemann)

$$\sum_{\vec{n}\in\mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^p} \widetilde{f}(\vec{m})$$

$\widetilde{f}$ Fourier transform

[Gelbart + Miller, BAMS '03, Iwaniec, Morgan, ICM '06]

- Truncated sums $\longrightarrow$ asymptotic series

ζ: EXPLICIT CALCULATIONS

Epstein zeta functions (quadratic)

$$\zeta_E = \sum_{\vec{n} \in \mathbb{Z}^d} Q(\vec{n})^{-s}$$

Q quadratic form

Barnes zeta functions (linear)

$$\zeta_B = \sum_{\vec{n} \in \mathbb{N}^d} L(\vec{n})^{-s}$$

L affine form (coeff's $\in \mathbb{Q}^+$)

Extensions:

$\zeta_E \rightarrow Q + L$ affine

$\rightarrow \sum_{\vec{n} \in \mathbb{N}^d}$ (truncation)

$\zeta_B \rightarrow \zeta_B'(0)$ (new formulas)

$\rightarrow \sum'_{\vec{n} \in \mathbb{Z}^d}$ (by analyt. cont.)

Ex

ζ-REGULARIZ: SPECTRUM KNOWN IMPLICITLY

- Example of the ball:
  - Operator
    $$(-\Delta + m^2)$$
    on the D-dim ball $B^D = \{x \in R^D; |x| \le R\}$ with Dirichlet, Neumann or Robin BC
  - The zeta function
    $$\zeta(s) = \sum_k \lambda_k^{-s}$$
  - Eigenvalues implicitly obtained from
    $$(-\Delta + m^2)\phi_k(x) = \lambda_k \phi_k(x) \quad + \quad BC$$
  - In spherical coordinates:
    $$\phi_{l,m,n}(r,\Omega) = r^{1-\frac{D}{2}} J_{l+\frac{D-2}{2}}(w_{l,n} r) Y_{l+\frac{D}{2}}(\Omega)$$
    $J_{l+(D-2)/2}$ Bessel functions
    $Y_{l+D/2}$ hyperspherical harmonics
  - Eigenvalues $w_{l,n}$ (>0) determined through BC
    $$J_{l+\frac{D-2}{2}}(w_{l,n} R) = 0, \qquad \text{for Dirichlet BC}$$

$$\frac{u}{R} J_{l+\frac{D-2}{2}}(w_{l,n}R) + w_{l,n} J'_{l+\frac{D-2}{2}}(w_{l,n}r)\,|_{r=R} = 0, \text{ for Robin BC}$$

- Now, $\lambda_{l,n} = w_{l,n}^2 + m^2$

$$\zeta(s) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} d_l(D)(w_{l,n}^2 + m^2)^{-s}$$

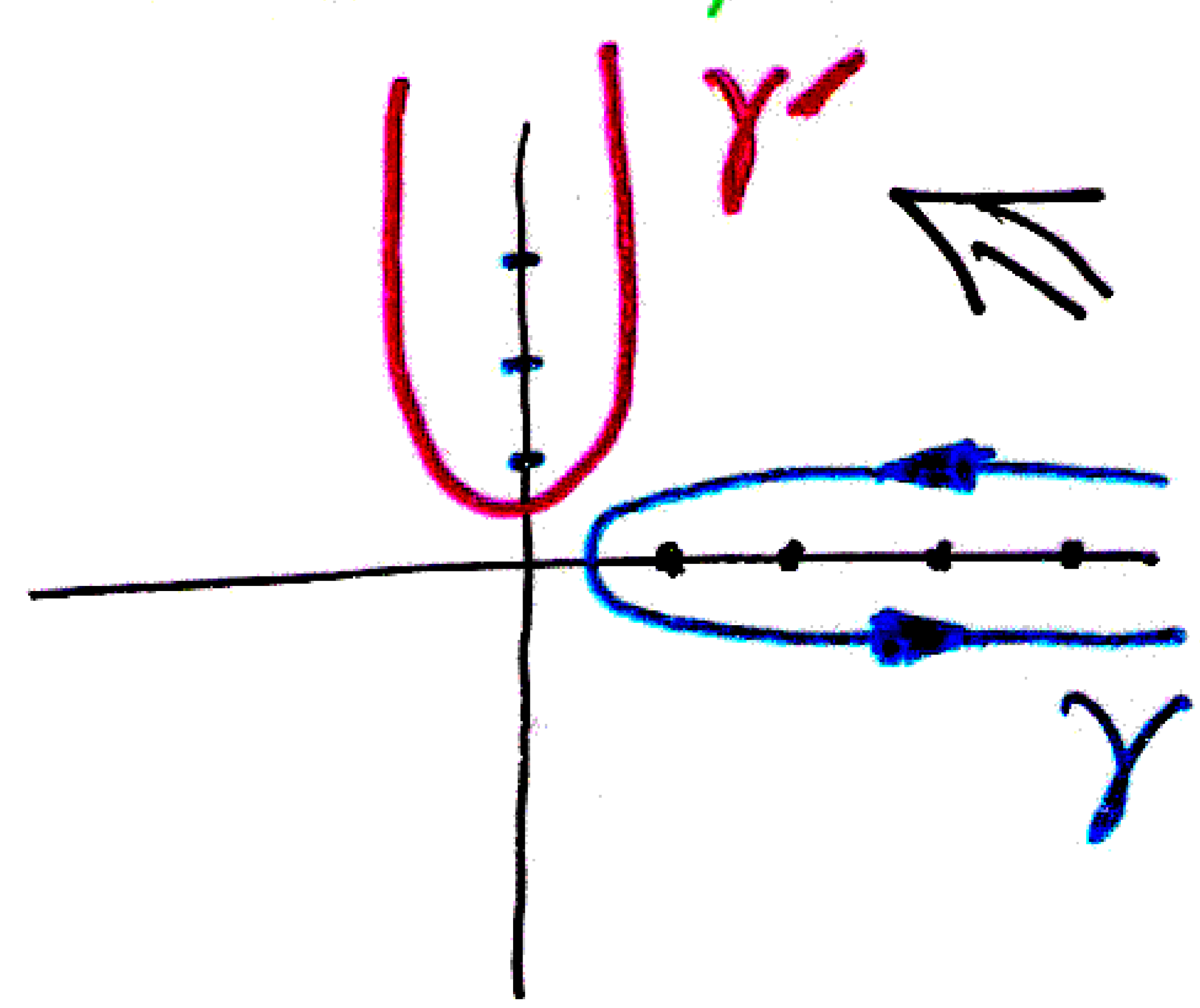
$w_{l,n}$ (> 0) is defined as the n-th root of the l-th equation, $d_l(D) = (2l + D - 2)\frac{(l+D-3)!}{l!\,(D-2)!}$

- **Procedure:**
  - Contour integral on the complex plane

$$\zeta(s) = \sum_{l=0}^{\infty} d_l(D) \int_\gamma \frac{dk}{2\pi i}\, (k^2+m^2)^{-s} \frac{\partial}{\partial k} \ln \Phi_{l+\frac{D-2}{2}}(kR)$$

γ runs counterclockwise and must enclose all the solutions [Ginzburg, Van Kampen, EE + I. Brevik]

- **Obtained:** [with Bordag, Kirsten, Leseduarte, Vassilievich,...]
  - Zeta functions
  - Determinants
  - Seeley [heat-kernel] coefficients



Existence of ζ_A for A a ΨDO

1. A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}^+$
2. A acts on the space of smooth sections of
3. E, n-dim vector bundle over
4. M closed n-dim manifold

(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr}\, A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re}\, s > \frac{n}{m} := s_0$$

$\{\lambda_j\}$ ordered spect of A, $s_0 = \dim M / \text{ord}\, A$ abscissa of converg of $\zeta_A(s)$

(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of A, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg}\, \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec}\, A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

(c) The definition of $\zeta_A(s)$ depends on the position of the cut L_θ

(d) The only possible singularities of $\zeta_A(s)$ are poles at

$$s_j = (n - j)/m, \qquad j = 0, 1, 2, \ldots, n-1, n+1, \ldots$$

Definition of Determinant

H ΨDO operator $\{\varphi_i, \lambda_i\}$ spectral decomposition

$\prod_{i\in I} \lambda_i$?! $\ln \prod_{i\in I} \lambda_i = \sum_{i\in I} \ln \lambda_i$

Riemann zeta func: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \; Re\, s > 1$ (& analytic cont)

Def nition: zeta function of H $\zeta_H(s) = \sum_{i\in I} \lambda_i^{-s} = \text{tr}\; H^{-s}$

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1}\, \text{tr}\; e^{-tH}$, Re$s > s_0$

Derivative: $\zeta_H'(0) = -\sum_{i\in I} \ln \lambda_i$

Determinant: Ray & Singer, '67

$\det_\zeta H = \exp\left[-\zeta_H'(0)\right]$

Weierstrass def: subtract leading behavior of λ_i in i, as $i \to \infty$, until series $\sum_{i\in I} \ln \lambda_i$ converges $\Longrightarrow$ non-local counterterms !!

C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...

Properties

- The definition of the determinant $\det_\zeta A$ only depends on the homotopy class of the cut

- A zeta function (and corresponding determinant) with the same meromorphic structure in the complex s-plane and extending the ordinary definition to operators ofcomplex order $m \in \mathbb{C}\backslash\mathbb{Z}$ (they do not admit spectral cuts), has been obtained [Kontsevich and Vishik]

- Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = {\sum}'_{\lambda \in \text{Spec } A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \, \text{Res}_{s=s_j} \zeta_A(s), \; s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[\text{PP } \zeta_A(-k) + \psi(k+1) \, \text{Res}_{s=-k} \zeta_A(s) \right], \quad s_j = -k, \; k \in \mathbb{N}$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N}\backslash\{0\}$$

$$\text{PP } \phi := \lim_{s \to p} \left[\phi(s) - \frac{\text{Res}_{s=p} \phi(s)}{s-p} \right]$$

“ Hi, Emilio. This is a question I have been trying to solve for years. With a bit of luck you could maybe provide me with a hint or two.

- Imagine I’ve got a functional integral and I perform a point transformation (doesn’t involve derivatives). Its Jacobian is a kind of functional determinant, but of a non-elliptic operator (it is simply infinite times multiplication by a function.) Did anybody study this seriously?
- I do know, from at least one paper I did with Luis AG, that in some cases (T duality) one is bound to define something like
- det f(x) ~ det [f(x) O] / det O where O es an elliptic operator (e.g. the Laplacian)
- This is what Schwarz and Tseytlin did in order to obtain the dilaton transformation
- And LAG and I did also proceed in a basically similar way
- As I know, Konsevitch, too, uses a related method involving the multiplicative anomaly

Tell me what you know about, please. Thanks so much.- Hugs, Enrique “

Multipl or N-Comm Anomaly, or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

$$\det{}_\zeta(AB) \neq \det{}_\zeta A \, \det{}_\zeta B$$

$$\det_\zeta(AB) \overset{?}{=} \det_\zeta A \, \det_\zeta B$$

$$\log \det_\zeta = \mathrm{tr}_\zeta \log \quad , \det_\zeta = e^{\mathrm{tr}_\zeta \log}$$

$$\det_\zeta(AB) \overset{1}{=} e^{\mathrm{tr}_\zeta \log(AB)} \overset{2}{=} e^{\mathrm{tr}_\zeta(\log A + \log B)}$$

$$\overset{3}{=} e^{\mathrm{tr}_\zeta \log A + \mathrm{tr}_\zeta \log B} =$$

$$\overset{4}{=} e^{\mathrm{tr}_\zeta \log A} \, e^{\mathrm{tr}_\zeta \log B} =$$

$$\overset{5}{=} \det_\zeta A \cdot \det_\zeta B$$

$[A, B] = 0$ assumed!

Which step is wrong?

tr_ζ is <u>no</u> trace at all

$$\mathrm{tr}_\zeta (A_1 + A_2) \neq \mathrm{tr}_\zeta A_1 + \mathrm{tr}_\zeta A_2$$

recall

$$\mathrm{tr}_\zeta A = \zeta_A(s=-1) = \sum_n \lambda_n^{-s} \Big|_{s=-1}$$

Multipl or N-Comm Anomaly, or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

$$\det_\zeta(AB) \neq \det_\zeta A \det_\zeta B$$

- The multiplicative (or noncommutative) anomaly (defect) is def ned as

$$\delta(A,B) = \ln\left[\frac{\det_\zeta(AB)}{\det_\zeta A \ \det_\zeta B}\right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0)$$

- Wodzicki formula

$$\delta(A,B) = \frac{\text{res}\left\{[\ln \sigma(A,B)]^2\right\}}{2 \text{ ord } A \text{ ord } B \,(\text{ord } A + \text{ord } B)}$$

where $\sigma(A,B) = A^{\text{ord } B} B^{-\text{ord } A}$

The Dixmier Trace

- In order to write down an action in operator language one needs a functional that replaces integration
- For the Yang-Mills theory this is the Dixmier trace
- It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N), \qquad \mu_0 \geq \mu_1 \geq \cdots$$

- Definition of the Dixmier trace of T:

$$\text{Dtr}\, T = \lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]

- The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ [Connes]

$$\text{Dtr}\, T = \lim_{s\to 1^+} (s-1)\zeta_{T^{-1}}(s)$$

The Wodzicki Residue

- The Wodzicki (or noncommutative) residue is the only extension of the Dixmier trace to ΨDOs which are not in $\mathcal{L}^{(1,\infty)}$
- Only trace one can define in the algebra of ΨDOs (up to multipl const)
- Definition: res $A = 2\,\text{Res}_{s=0}\,\text{tr}(A\Delta^{-s})$, Δ Laplacian
- Satisfies the trace condition: res $(AB) =$ res (BA)
- Important!: it can be expressed as an integral (local form)

$$\text{res } A = \int_{S^*M} \text{tr } a_{-n}(x,\xi)\, d\xi$$

with $S^*M \subset T^*M$ the co-sphere bundle on M (some authors put a coefficient in front of the integral: Adler-Manin residue)

- If dim $M = n = -$ ord A (M compact Riemann, A elliptic, $n \in \mathbb{N}$) it coincides with the Dixmier trace, and $\text{Res}_{s=1}\zeta_A(s) = \frac{1}{n}\,\text{res } A^{-1}$
- The Wodzicki residue makes sense for ΨDOs of arbitrary order. Even if the symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

Consequences of the Multipl Anomaly

- In the path integral formulation

$$\int [d\Phi] \; \exp\left\{ - \int d^D x \left[\Phi^\dagger(x) (\quad) \Phi(x) + \cdots \right] \right\}$$

Gaussian integration: $\longrightarrow \quad \det\,(\quad)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} A & \\ & B \end{pmatrix}$$

$\det(AB)$ or $\det A \cdot \det B$?

- In a situation where a superselection rule exists, AB has no sense (much less its determinant): $\Longrightarrow \det A \cdot \det B$

- But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: $\Longrightarrow \det(AB)$

Google Scholar: "*zeta regularization*" (about 25,100 results) 07/03/2025

Sorted by relevance

Zeta function **regularization** of path integrals in curved spacetime
SW Hawking - Communications in Mathematical Physics, 1977 - Springer
… One therefore has to adopt some **regularization** procedure. The technique that will be used in this paper will be called the **zeta** function method. One forms a generalized **zeta** function …
Cited by 1858

Zeta regularization techniques with applications
E Elizalde ea - 1994 – World Sci.
… the authors on different aspects of **zeta** functions and related topics. … with the different aspects of **zeta**-function **regularization** (analytic … Virtually all types of **zeta** functions are dealt with …
Cited by 1147

Effective Lagrangian and energy-momentum tensor in de Sitter space
JS Dowker, R Critchley - Physical Review D, 1976 - APS
… Firstly, if we adopt the proper-time **regularization** method so that the infinities appear only when … We now turn to another **regularization** method the **zeta**-function method. We start from the …
Cited by 995

Ten physical applications of spectral **zeta** functions
E Elizalde - 2012 – Springer-Nature
… formulation of a fully-fledged **zeta regularization** method, it is … the first time, a **zeta** function **regularization** method for quantum … that a well-defined and clear **regularization** prescription for a …
Cited by 831

EE 12/22 most cited works on "zeta regularization" (includ. 2nd & 4th)

Path integrals and the indefiniteness of the gravitational action
GW Gibbons, SW Hawking, MJ Perry - Nuclear Physics B, 1978 - Elsevier
… We perform a **zeta** function **regularization** of the one-loop term for gravity and obtain a non-trivial scaling behavior in cases in which the background metric has non-zero curvature …
Cited by 829

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K Kirsten - AIP Conference Proceedings, 1999 - pubs.aip.org
… Of course, there are various possible **regularization** procedures; let us mention only Pauli-… **regularization** and **zeta** function **regularization**. Here, we will use **zeta** function **regularization**
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A Voros - Communications in Mathematical Physics, 1987 - Springer
… sequence, as defined by **zeta regularization**, can be simply … a factorization of the Selberg **zeta** function into two functional … its connection to the Selberg **zeta** function and evaluates the …
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One-loop f (R) gravity in de Sitter universe
G Cognola, E Elizalde, S Nojiri… - Journal of Cosmology …, 2005 - iopscience.iop.org
… have been regularized by means of **zeta**-function **regularization** (see, for example [12,13]). However, we should recall that within the **zeta**-function **regularization** it is no longer true that …
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Quantum gravity: the new synthesis
BS DeWitt - General relativity, 1979 - inis.iaea.org
… **zeta** function, **regularization** and renormalization, conformal invariance and the trace anomaly, conformally flat spacetimes); the full quantum theory (the gauge group, the Feynman …
Cited by 462

Regularization, renormalization, and covariant geodesic point separation
SM Christensen - Physical Review D, 1978 - APS
… We will see that the breaking of conformal invariance in this **regularization** technique leads to … **zeta**-function, and point-separation **regularization** methods beginning from the same initial …
Cited by 459

The path-integral approach to quantum gravity
SW Hawking - General relativity, 1979 - inis.iaea.org
… approximation; **zeta** function **regularization**; the background fields (some positive-definite metrics which are solutions of the Einstein equations in vacuum or with a Λ-term); gravitational …
Cited by 408

Trace anomaly of a conformally invariant quantum field in curved spacetime
RM Wald - Physical Review D, 1978 - APS
… , "" dimensional regularization, "and **zeta**-function **regularization**. "In addition, I have … Explicit calculations in four dimensions using dimensional **regularization**, **zeta**-function regul…
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Zeta functions and the Casimir energy
SK Blau, M Visser, A Wipf - Nuclear Physics B, 1988 - Elsevier
… **Zeta** functions on manifolds with boundary. As a **regularization** technique we shall use the **zeta**… Its relation to other methods (eg, dimensional **regularization**) has been discussed in the …
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Heat kernel coefficients of the Laplace operator on the *D*-dimensional ball
M Bordag, E Elizalde, K Kirsten - Journal of Mathematical Physics, 1996 - pubs.aip.org
… is the complex argument s of the **zeta** function of the Laplace … a representation of the associated **zeta** function in terms of a … an analytical representation of the **zeta** function —valid in the …
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Casimir energies for massive scalar fields in a spherical geometry
M Bordag, E Elizalde, K Kirsten, S Leseduarte - Physical Review D, 1997 - APS
… **zeta** function **regularization** was developed see also 24. In this approach, a knowledge of the **zeta** … Recently, a detailed description of how to obtain the **zeta** function for a massive scalar …
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E Elizalde, S Nojiri, SD Odintsov, S Ogushi - Physical Review D, 2003 - APS
… We show that **zeta-regularization** techniques at its full power 21 can be used in order to calculate the bulk effective potential in such braneworlds, in a quite general setting. One …
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Casimir energy for a massive fermionic quantum field with a spherical boundary
E Elizalde, M Bordag, K Kirsten - Journal of Physics A …, 1998 - iopscience.iop.org
… In this approach, a knowledge of the **zeta** function of the … Recently, a detailed description of how to obtain the **zeta** … needed in the subsequent study of the **zeta** function of the problem we …
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E Elizalde, A Romeo - American Journal of Physics, 1991 - ui.adsabs.harvard.edu
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Zeta-function **regularization**, the multiplicative anomaly and the Wodzicki residue
E Elizalde, L Vanzo, S Zerbini - Communications in mathematical physics, 1998 - Springer
… This may be quite ambiguous, since one has to employ necessarily a **regularization** procedure. In fact, it turns out that the **zeta**-function regularized determinants do not satisfy the above …
Cited by 147

Expressions for the **zeta**-function regularized Casimir energy
E Elizalde, A Romeo - Journal of mathematical physics, 1989 - pubs.aip.org
… The method used here, which rests upon the properties of the Riemann **zeta** function, is actually different from the procedure of direct **zeta**-function **regularization**. By this we mean that …
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Multidimensional extension of the generalized Chowla–Selberg formula
E Elizalde - Communications in mathematical physics, 1998 - Springer
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Cited by 129

Uses of **zeta regularization** in QFT with boundary conditions: a cosmo-topological Casimir effect
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… **Zeta regularization** has proven to be a powerful and reliable tool for the **regularization** of the vacuum energy density in ideal situations. With the Hadamard complement, it has been …
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EE 12/22 most cited works on "zeta regularization" (includ. 2nd & 4th)

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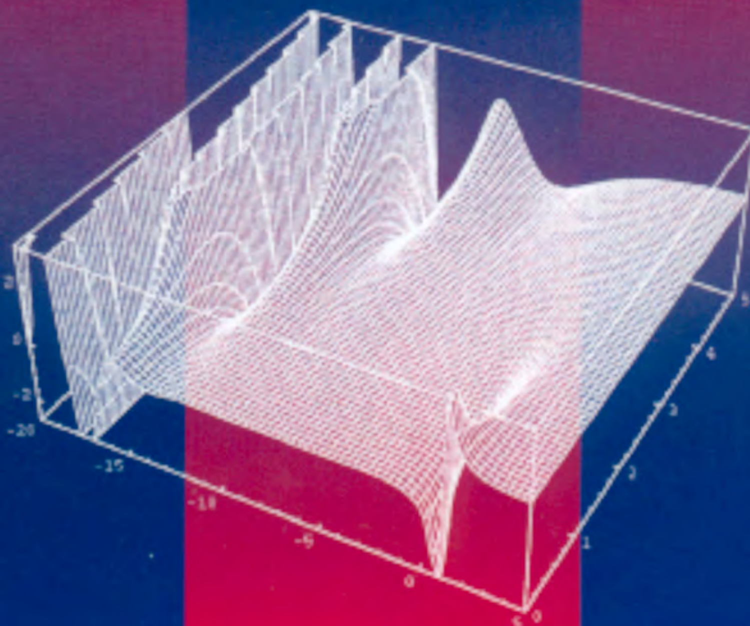

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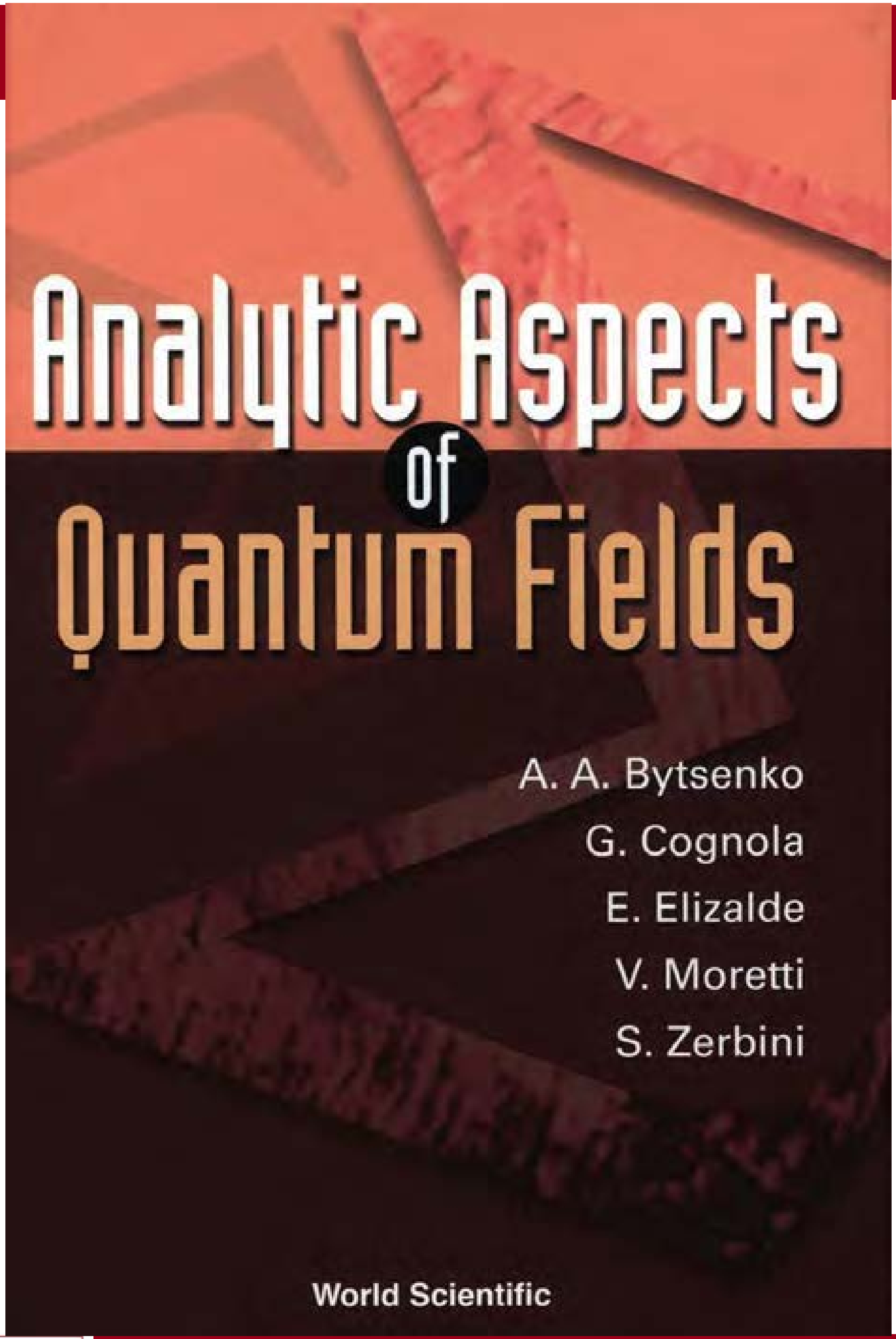
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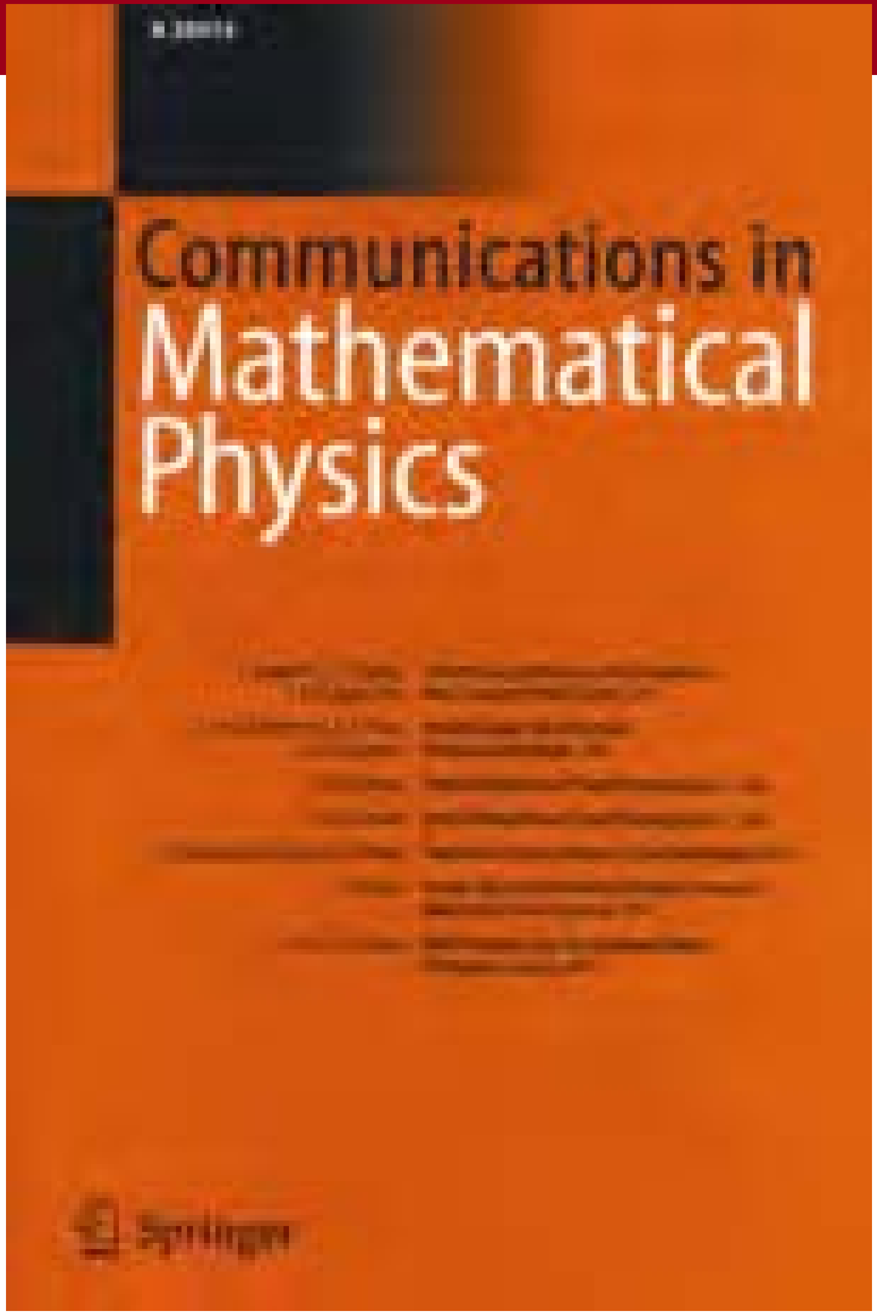
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Happy 80th Birthday!

Branko

Best Wishes for the Future