Isoperiodic deformations of meromorphic differentials on Riemann surfaces and applications to Mathematical Physics

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Dedicated to Prof. Branko Dragović and his 80th anniversary

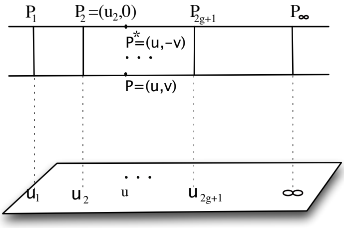
Based on joint work with Vasilisa Shramchenko:

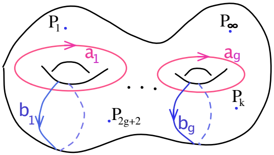
Publications

- VD, V. Shramchenko, *Algebro-geometric solutions of the Schlesinger systems and the Poncelet-type polygons in higher dimensions*, IMRN, (2018), 4229–4259.
- VD, V. Shramchenko, *Algebro-geometric approach to an Okamoto transformation, the Painlevé VI and Schlesinger equations*, Annales H. Poincaré, (2019), 1121-1148.
- VD, V. Shramchenko, *Deformations of the Zolotarev polynomials and Painlevé VI equations*, Lett. M. Phys. ('21).
- VD, V. Shramchenko, *Isoharmonic deformations and constrained Schlesinger systems*, arXiv: 2112:04110.
- VD, V. Shramchenko, *Isoperiodic deformations of Toda lattices and curves, $SU(N)$ Seiberg-Witten theory, and triangular Schlesinger systems*, in preparation

Hyperelliptic curves

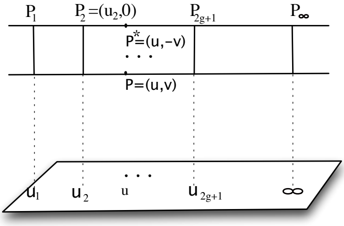
$$v^2 = (u - u_1)\cdots(u - u_{2g+1}), \qquad u : (u, v) \in \mathcal{C} \mapsto u.$$

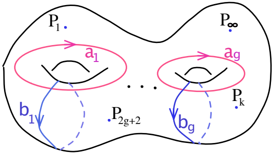


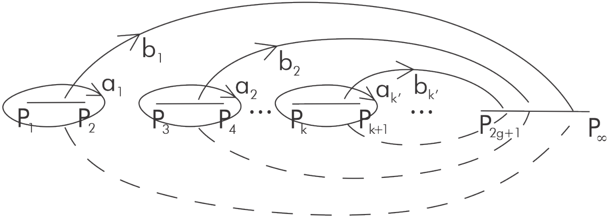


- Three bases in the space of holomorphic 1-forms:
  - $\phi, u\phi, \ldots, u^{g-1}\phi$ with $\boxed{\phi = \frac{du}{v}}$ $\Rightarrow$ matrix of a-periods is $\boxed{\mathbb{A}}$
  - $\omega = (\omega_1, \ldots, \omega_g)^t$ are holomorphic normalized 1-forms: $\oint_{a_j} \omega_i = \delta_{ij} \Rightarrow$ matrix of b-periods is $\boxed{\mathbb{B}}$
  - $v_1, \ldots, v_g$ with $v_i(Q_j) = \delta_{ij}$

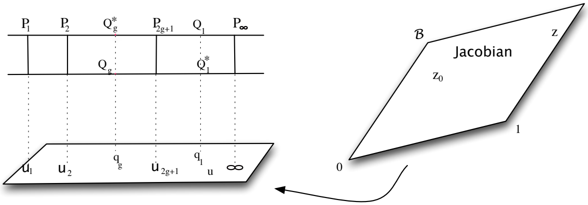
Choice of the canonical basis of cycles







A differential Ω on hyperelliptic curves



$$\Omega(P) = \sum_{j=1}^{g} \Omega_{Q_j Q_j^*}(P) - 4\pi\mathrm{i}\, c_2^t \omega(P)$$

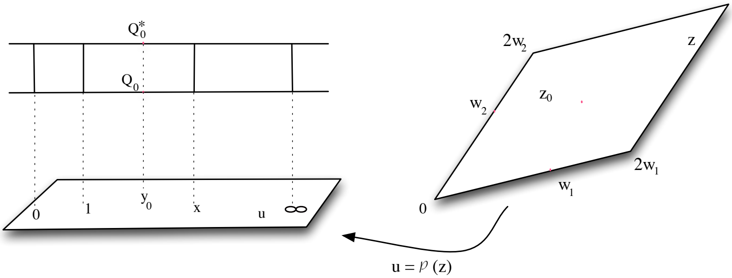
where $z_0 = c_1 + \mathbb{B}c_2$ and $\sum_{j=1}^{g} \mathcal{A}_\infty(Q_j) = z_0$; $c_1, c_2 \in \mathbb{C}^g$,

$$\oint_{a_k} \Omega = -4\pi\mathrm{i} c_{2k} \qquad \oint_{b_k} \Omega = 4\pi\mathrm{i} c_{1k}.$$

Jacobi inversion problem

- Let $z_0 \in Jac(\mathcal{L})$, $z_0 = c_1 + \mathbb{B}c_2$, and $\sum_{j=1}^{g} \mathcal{A}_\infty(Q_j) = z_0$.
- Let $q_j = u(Q_j)$
- $P_k = (u = u_k, v = 0)$- ramification points.
- Then
$$\boxed{\frac{\partial q_j}{\partial u_k} = -\frac{1}{4}\Omega(P_k)v_j(P_k)}$$
- where v_j are holomorphic differentials defined by
$$\boxed{v_j(Q_s) = \delta_{js}\,,} \qquad j, s = 1, \ldots, g$$

Genus one case and Painlevé-VI



- Modified $\wp$ satisfies: $(\wp'(z))^2 = \wp(z)\,(\wp(z) - 1)\,(\wp(z) - x)$.
- $z_0 := 2w_1 c_1 + 2w_2 c_2$.
- Picard's solution to $P_{VI}(0, 0, 0, \frac{1}{2})$:

$$y_0(x) = \wp(z_0(x)).$$

Okamoto transformations $\sim$ 1980

- a group of symmetries of PVI$(\alpha, \beta, \gamma, \delta)$.

- Example: Okamoto transformation from $P_{VI}(0,0,0,\frac{1}{2})$ to $P_{VI}(\frac{1}{8},-\frac{1}{8},\frac{1}{8},\frac{3}{8})$:

  y_0 - Picard's solution of $P_{VI}(0,0,0,\frac{1}{2})$

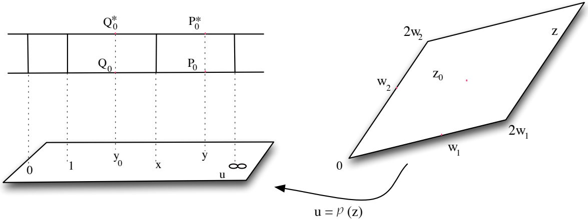
  y - Hitchin's solution of $P_{VI}(\frac{1}{8},-\frac{1}{8},\frac{1}{8},\frac{3}{8})$

$$y(x) = y_0 + \frac{y_0(y_0-1)(y_0-x)}{x(x-1)y_0' - y_0(y_0-1)}.$$

- Formula for y_0' :

$$\boxed{\frac{dy_0}{dx} = -\frac{1}{4}\Omega(P_x)\frac{\omega(P_x)}{\omega(Q_0)}}$$

Genus one case and Painlevé-VI



- Differential of the third kind on the elliptic curve $\mathcal{C}$:

$$\Omega(P) = \Omega_{Q_0,Q_0^*}(P) - 4\pi \mathrm{i} c_2 \omega(P).$$

- $\omega(P)$ -holomorphic normalized differential on $\mathcal{C}$
- Ω has two simple poles at Q_0 et Q_0^* which project to y_0, Picard's solution of $P_{VI}(0,0,0,\frac{1}{2})$.
- Ω has two simple zeros at P_0 et P_0^* which project to y, Hitchin's solution of $P_{VI}(\frac{1}{8},-\frac{1}{8},\frac{1}{8},\frac{3}{8})$.

Our tools

- The fundamental bi-differential $W(P,Q)$: defined by
  - $W(P,Q) = W(Q,P)$
  - $W(P,Q) = \frac{dz(P)dz(Q)}{(z(P)-z(Q))^2} + \textit{regular terms}$
  - $\oint_{a_j} W(P,Q) = 0$
- $$\Omega(P) = \sum_{j=1}^{g} \int_{Q_j^*}^{Q_j} W(P,\cdot) - 4\pi \mathrm{i}\, c_2^t \omega(P)$$
- Rauch variational formulas

$$\partial_{u_k} W(P,Q) = \frac{1}{2} W(P,P_k) W(P_k,Q);$$

$$\partial_{u_k} \omega_j(P) = \frac{1}{2} \omega_j(P_{u_k}) W(P,P_{u_k});$$

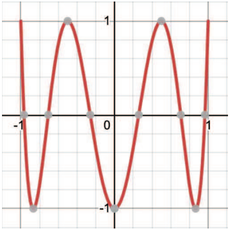
$$\partial_{u_k} \mathbb{B}_{ij} = \mathrm{i}\pi \omega_i(P_k) \omega_j(P_k).$$

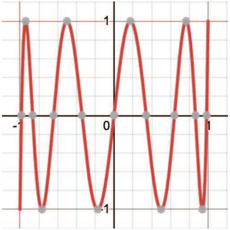
Chebyshev polynomials

$T_n(x),\ n = 0, 1, 2, \dots$ $\boxed{T_n(x) = \cos n\phi, \quad x = \cos\phi,}$

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

$$T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$$



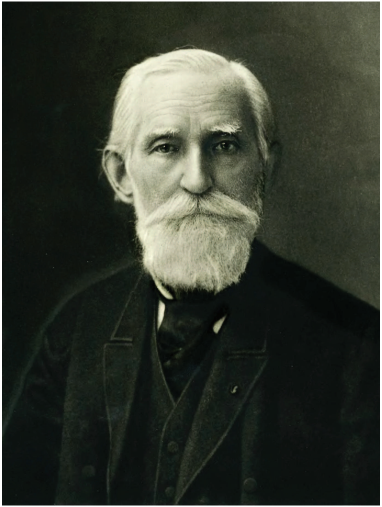


- The Chebyshev polynomials are solutions to the Pell equation

$$\boxed{T_n^2(x) - (x-1)(x+1)Q_{n-1}^2(x) = 1}$$

- $2^{1-n}T_n$ is the monic polynomial of degree n which minimizes the uniform norm on the interval $[-1, 1]$

Pafnuty Lvovich Chebyshev, 1821-1894



Zolotarev polynomials ($d = 2$)

Problem: find monic polynomial of degree n minimizing the uniform norm over the union of two (or more) intervals. Denote the solution by $\hat{P}_n$ and its norm by L_n.

The polynomial $\hat{p}_n$ is the solution of the Pell equation on $[\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4]$

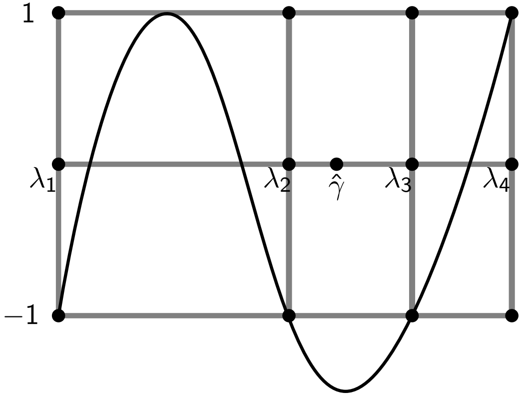
$$1 = \hat{p}_n(\lambda)^2 - \prod_{j=1}^{4} (\lambda - \lambda_j) Q_{n-2}^2(\lambda),$$

if and only if:

(i) $\hat{p}_n = \hat{P}_n / \pm L_n$

(ii) the set $[\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4]$ is the maximal subset of $\mathbf{R}$ for which $\hat{P}_n$ is the minimal polynomial in the above sense.

Zolotarev polynomials



The differential Ω on hyperelliptic curves

- The curves

$$v^2 = \prod_{j=1}^{2g+1} (u - u_j)$$

- Define $z_0 = \hat{c}_1 + \mathbb{B}\hat{c}_2$ with $\hat{c}_1, \hat{c}_2 \in \mathbb{C}^g$ constant vectors

- Jacobi inversion:
  - Usual way (generic) $\sum_{j=1}^{g} \mathcal{A}_\infty(Q_j) = z_0$ and

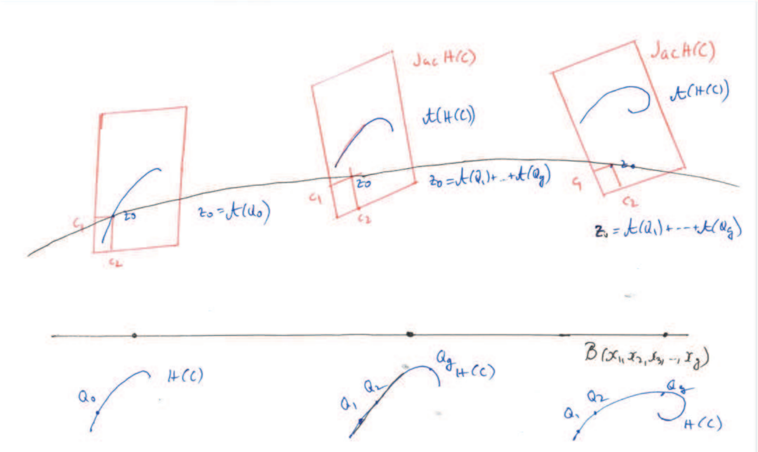
$$\boxed{\Omega(P) = \sum_{j=1}^{g} \Omega_{Q_j Q_j^*}(P) - 4\pi\mathrm{i}\, \hat{c}_2^t \omega(P)}$$

  - Unusual way (constrained) $\mathcal{A}_\infty(Q_0) = z_0$ and

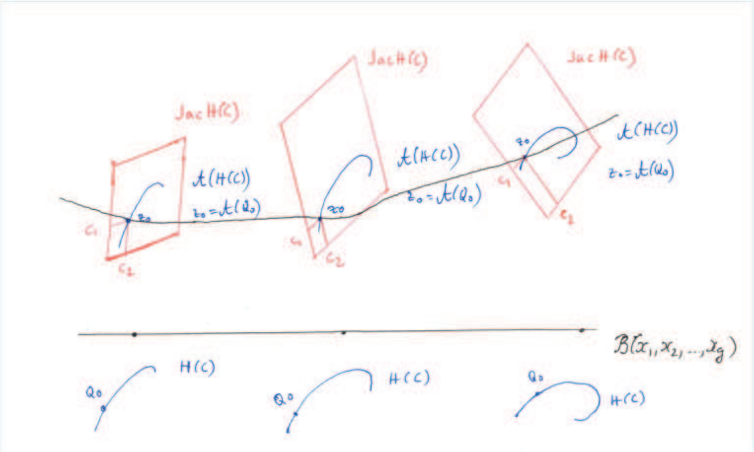
$$\boxed{\Omega(P) = \Omega_{Q_0 Q_0^*}(P) - 4\pi\mathrm{i}\, \hat{c}_2^t \omega(P)}$$

Here $\omega = (\omega_1, \ldots, \omega_g)^t$ are holom. normalized differentials.

Jacobi inversion problem

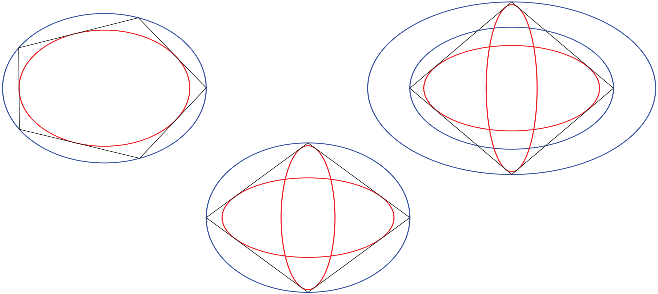


Constrained Jacobi inversion problem



Usual and Constrained Jacobi inversion: billiard perspective

Billiard ordered games, see VD, M. Radnović, JMPA 2006



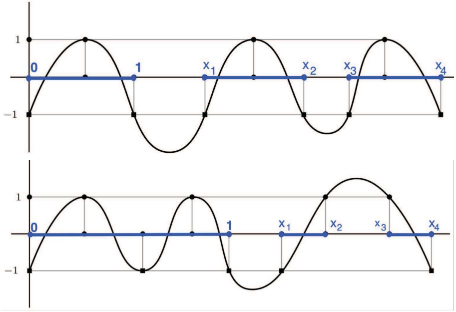
Generalized Chebyshev polynomials

Let

$$\mu^2 = x(x-1)\prod_{j=1}^{2g}(x - x_j)$$

The generalized Chebyshev polynomials satisfy the Pell equation

$$P_n^2(x) - \mu^2(x) Q_{n-g-1}^2(x) = 1\,.$$



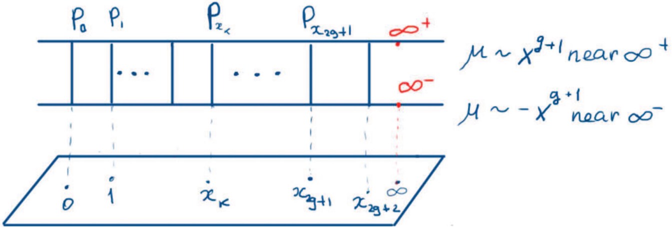
Given $x_1, \ldots, x_{2g}$, the existence of a solution $P_n,\ Q_{n-g-1}$ to the Pell equation is not guaranteed.

Pell equations and points of a finite order

Let $\mu,\ P_n,\ Q_{n-g-1}$ be such that the Pell equation holds:

$$P_n^2(x) - \mu^2(x) Q_{n-g-1}^2(x) = 1$$

and $\mu^2 = x(x-1)\prod_{j=1}^{2g}(x - x_j)$ defines a hyperelliptic compact surface $\mathcal{L}$



Then the point ∞^+ is of order n that is: $\boxed{n\mathcal{A}_{\infty^-}(\infty^+) \equiv 0}$

Pell equations and points of a finite order - proof

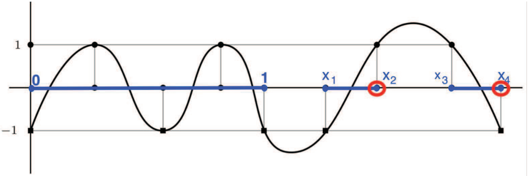
- We have
  - $P_n^2(x) - \mu^2(x) Q_{n-g-1}^2(x) = 1$
  - and $\mu^2 = x(x-1)\prod_{j=1}^{2g}(x - x_j)$.
- Define $s(P) := P_n(x) + \mu Q_{n-g-1}(x)$.
- Then $\boxed{s(P^*) = \frac{1}{s(P)}}$
- because

$$s(P^*)s(P) = (P_n - \mu Q_{n-g-1})(P_n + \mu Q_{n-g-1}) = 1.$$

- $s(P)$ has a pole of order n at $P \sim \infty^+$ (where $\mu \sim x^2$)
- therefore $s(P)$ has a zero of order n at $P \sim \infty^-$ ($\mu \sim -x^2$)
- Thus $n\mathcal{A}_{\infty^-}(\infty^+) \equiv 0$. $\square$
- The converse is also true: if ∞^+ is a point of order n, then there is a solution to the Pell equation for the μ in question.

Dynamics of Chebyshev polynomials

- $\mu^2 = x(x-1)\prod_{j=1}^{2g}(x-x_j)$
- The Pell equation: $P_n^2(x) - \mu^2(x)Q_{n-g-1}^2(x) = 1\,.$
- Positions of x_2, x_4 are determined by x_1, x_3 and the Pell eq.



- Möbius transformation:

$$u(x) = \frac{x(1-x_{2g})}{x-x_{2g}}$$

- Denote

$$\boxed{y_0 := u(\infty) = 1 - x_{2g}\,,\quad \hat{x}_j := u(x_{2j-1})\,,\quad u_j := u(x_{2j})}$$

- $Q_0 =$ "$u(\infty^+)$" is of order $2n$ on the curve $\forall \hat{x}_j \in \mathbb{R}\setminus\{0,1\}$

$$\mathcal{C}\ :\quad v^2 = u(u-1)\prod_{j=1}^{g}(u-\hat{x}_j)\prod_{j=1}^{g-1}(u-u_j)$$

Chebyshev variation of a hyperelliptic surface, $g \geq 2$

We have the hyperelliptic surface of the curve

$$\mathcal{C} \ : \quad v^2 = u(u-1)\prod_{j=1}^{g}(u-x_j)\prod_{j=1}^{g-1}(u-u_j)$$

where

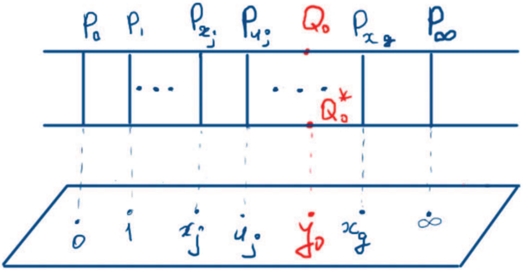
- $1 < x_j < u_j < x_{j+1} < \infty$
- $u_j = u_j(x_1, \ldots, x_g)$
- $\omega = (\omega_1, \ldots, \omega_g)^t$ a vector of holom. norm. differentials
- $\exists Q_0 \in \mathcal{C}$ such that

$$n\int_{Q_0^*}^{Q_0} \omega = k_1 + \mathbb{B}k_2 \quad \text{with} \quad k_1,\ k_2 \in \mathbb{Z}^g \quad \text{for} \quad x_j \in \mathbb{R}\setminus\{0,1\}$$

- That is $\exists Q_0 \in \mathcal{C}$ such that

$$\boxed{2\int_{P_\infty}^{Q_0} \omega = \int_{Q_0^*}^{Q_0} \omega = \hat{c}_1 + \mathbb{B}\hat{c}_2 \quad \text{with constant} \ \ \hat{c}_1,\ \hat{c}_2 \in \mathbb{Q}^g \quad \forall x_j \in \mathbb{R}}$$

Chebyshev variation of a hyperelliptic surface, $g \geq 2$



The set of branch points $B := \{0, 1, x_1, \ldots, x_g, u_1, \ldots, u_{g-1}\}$.

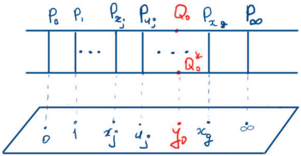
- $x_1, \ldots, x_g$ vary independently in $\mathbb{R} \setminus \{0, 1\}$
- $u_1, \ldots, u_{g-1} \in \mathbb{R}$ are functions of $x_1, \ldots, x_g$

$$\int_{P_\infty}^{Q_0} \omega = \hat{c}_1 + \mathbb{B}\hat{c}_2 =: z_0 \quad \text{with constant } \hat{c}_1, \ \hat{c}_2 \in \mathbb{Q}^g \quad \forall x_i \in \mathbb{R} \setminus \{0, 1\}$$

- Define

$$\Omega(P) = \Omega_{Q_0 Q_0^*}(P) - 4\pi \mathrm{i}\, \hat{c}_2^t \omega(P)$$

Chebyshev variation of a hyperelliptic surface, $g \geq 2$



- Define a basis of holomorphic differentials $v_1, \dots, v_g$:

$$\boxed{v_i(P_{u_j}) = \delta_{ij}, \quad v_i(Q_0) = \delta_{ig}} \quad \text{with } 1 \leq i \leq g, \; 1 \leq j \leq g-1.$$

- Or, explicitly with $\varphi(P){=}du/\sqrt{\prod_{a_j \in B}(u-a_j)}$

$$v_i(P) = \frac{\varphi(P)\prod_{\alpha=1,\alpha\neq i}^{g-1}(u-u_\alpha)(u-y_0)}{\varphi(P_{u_i})\prod_{\alpha,\alpha\neq i}(u_i-u_\alpha)(u_i-y_0)}, \qquad i=1,\cdots,g-1$$

$$v_g(P) = \frac{\varphi(P)\prod_{\alpha=1}^{g-1}(u-u_\alpha)}{\varphi(Q_0)\prod_{\alpha=1}^{g-1}(y_0-u_\alpha)}.$$

- Then

$$\boxed{\frac{\partial u_m}{\partial x_i} = -\frac{\Omega(P_{x_i})}{\Omega(P_{u_m})} v_m(P_{x_i}), \qquad \frac{\partial y_0}{\partial x_i} = -\frac{1}{4}\Omega(P_{x_i}) v_g(P_{x_i}).}$$

Example: genus 1

Theorem (V.D, V. Shramchenko 2025)

Let $\mathcal{T} \to X$ be a Toda family of the elliptic curves of equation

$$\mu^2 = \lambda(\lambda - 1)(\lambda - x)(\lambda - u),$$

parameterized by x. Then the position of the branch point u of the coverings, as a function of x satisfies the following equation

$$u'' = \frac{1}{2}\left(\frac{u}{x-1} - \frac{u-1}{x} + \frac{1}{u-x}\right) - \frac{u'}{2}\left(\frac{2}{x} + \frac{2}{x-1} + \frac{1}{u-x}\right)$$
$$+\frac{(u')^2}{2}\left(\frac{2}{u} + \frac{2}{u-1} + \frac{1}{x-u}\right) - \frac{(u')^3}{2}\left(\frac{x}{u-1} - \frac{x-1}{u} + \frac{1}{x-u}\right).$$

The Toda Lattice

A one-dimensional chain of particles with exponential interaction of immediate neighbours

$$\ddot{x}_n(t) = e^{x_{n+1}-x_n} - e^{x_n - x_{n-1}},$$

$x_n(t)$ is the position of the n-th particle at the moment t, $\dot{x}_n(t)$ denotes the derivative with respect to t. Denote: $v_n := \dot{x}_n$ and $c_n = \exp(x_n - x_{n-1})$, $c_n > 0$, Toda lattice equations can be rewritten in the form

$$\begin{aligned} \dot{v}_n &= c_{n+1} - c_n, \\ \dot{c}_n &= c_n(v_n - v_{n-1}). \end{aligned}$$

The Toda Lattice: Krichever 1981

The expressions for the solutions:

$$v_n = \frac{d}{dt} \ln \frac{\theta((n+1)U + tV + z_0)}{\theta(nU + tV + z_0)};$$

$$c_n = \frac{\theta((n+1)U + tV + z_0)\theta((n-1)U + tV + z_0)}{\theta^2(nU + tV + z_0)},$$

z_0 is an arbitrary vector; U is the period vector of the differential of the third kind $\Omega_{\infty^+,\infty_-}$. V is a linear combination of the b-periods of the normalized differentials of the second kind Ω_{∞^-} and Ω_{∞^+} having only a double pole at ∞^- and ∞^+, respectively, and no other singularities.

The Toda Lattice: Krichever 1981

Theorem (Krichever 1981)

A solution to the Toda lattice is periodic in n with a period N if and only if the solution (v_n, c_n) relative to a hyperelliptic curve and the vector $U/(2\pi\mathrm{i})$ is an N-division point of the lattice generating the Jacobian of Γ, that is

$$\frac{1}{2\pi\mathrm{i}} U = \int_{\infty^-}^{\infty^+} \omega = M_1 + \mathbb{B} M_2,$$

$M_1, M_2 \in \mathbb{Q}^g$ *are vectors, such that* $NM_1, NM_2 \in \mathbb{Z}^g$.

The Toda Lattice: Isoperiodic deformations

Theorem (V.D, V. Shramchenko 2025)

Consider an N-periodic in n solution to the Toda lattice constructed by the above formulas from the hyperelliptic surface $\Gamma_{\mathbf{x_0}}$ of genus g. For a value of $\mathbf{x_0}$ away from a set of measure zero, there exists a continuous g-parameter deformation of this solution which remains N-periodic in n. Moreover, any continuous deformation of this solution constructed from a family of curves $\Gamma_{\mathbf{x}}$, $\mathbf{x} \in \mathcal{X}$, obtained from $\Gamma_{\mathbf{x_0}}$ by varying $\mathbf{x} = (x_1, \ldots, x_g)$ and allowing $\mathbf{u} = (u_1, \ldots, u_g)$ to be functions of $\mathbf{x}$ and which remains N-periodic in n solution to the Toda lattice is obtained by the above formulas. In this case, the branch points $u_1, \ldots, u_g$ of the coverings as functions of $\mathbf{x}$ have the derivatives expressed by our equations and satisfy our system of the second order PDEs with rational coefficients.

Applications to the $SU(N)$ Seiberg-Witten theory

In a special case of the $SU(N)$ super-symmetric Yang-Mills theory, the so-called case without fundamental hypermultiplets, the main object is the family of curves

$$\mathcal{C}_T(x_1, \ldots, x_n) : \mu^2 = \mathcal{P}_N^2(z) - \hat{\Lambda}^2,$$

parametrized by n complex parameters $x_1, \ldots, x_n$, where $\hat{\Lambda}$ is a real constant; $\mathcal{P}_N(z)$ is a polynomial of degree N. The parameters, vacuum moduli of the Yang-Mills theory, can be chosen as a subset of the set of zeros of the varying polynomial $\mathcal{P}_N^2(z) - \hat{\Lambda}^2$. In general, these curves are non-singular hyperelliptic curves of genus $n = N - 1$, when all zeros of $\mathcal{P}_N^2(z) - \hat{\Lambda}^2$ are simple.

Applications to the $SU(N)$ Seiberg-Witten theory

For certain values of the parameters, the curves may become singular, as some of the zeros of $\mathcal{P}_N^2(z) - \hat{\Lambda}^2$ merge to form a double zero. Such singularities occur exactly when some of the particles in the Yang-Mills theory become massless. We will call such situations *singular regimes* and denote by $N - g - 1$ the number of massless particles, that is of double zeros of $\mathcal{P}_N^2(z) - \hat{\Lambda}^2$. The desingularization of the singular curve $\mathcal{C}_T$ is a hyperelliptic curve $\Gamma_{\mathbf{x}}$ of genus g defined by the equation

$$\Gamma_{:} w^2 = \Delta_{2g+2}(z).$$

where $\mathbf{x} = (x_1, \ldots, x_g)$ is a subset of zeros of Δ_{2g+2}. The $N - g - 1$ massless particles correspond to the zeros of $\mathcal{Q}_{N-g-1}$. If all zeros of $\Delta_{2g+2}(z)$ are real, zeros of $\mathcal{Q}_{N-g-1}$ are the *internal* critical points of the generalized Chebyshev polynomial $\mathcal{P}_N$, that is critical points lying inside the intervals of the support of $\mathcal{P}_N$.

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Theorem (V. D, V. Shramchenko 2025)

Consider the vacuum moduli parameters $\mathbf{x_0} = (x_1^0, \ldots, x_g^0)$ of a singular regime of the Yang-Mills theory with $N - g - 1$ massless particles. Let Δ_{2g+2}^0 be given with distinct $x_j^0, u_j^0 \in \mathbb{C}$. For generic moduli parameters $\mathbf{x_0}$ lying away from some set of measure zero, there exists a local continuous deformation of this singular regime which fixes the number of massless particles in the theory. This deformation is constructed from a family of curves $\Gamma_{\mathbf{x}}$ with $\mathbf{x}$ varying in some neighbourhood of $\mathbf{x_0}$ where the branch points $u_1, \ldots, u_g$ as functions of $= (x_1, \ldots, x_g)$ have the derivatives expressed by our initial conditions and satisfy our system second order PDEs with rational coefficients.

Applications to the $SU(N)$ Seiberg-Witten theory

Seiberg and Witten (1994) considered the curve

$$Y^2 = X^3 + 2uX^2 + \Lambda^4 X.$$

By a Möbius transformations, it can be brought to:

$$\mathcal{C}_T(u) : \mu^2 = (z^2 - \frac{1}{2}u)^2 - \Lambda^4.$$

This is an elliptic curve for $u \neq \pm 2\Lambda^2$. The polynomial $\mathcal{P}_2$ is

$$\mathcal{P}_2(z) = z^2 - \frac{u}{2}.$$

Applications to the $SU(N)$ Seiberg-Witten theory

Consider the singular case $u = 2\Lambda^2$. The curve becomes $\mu^2 = (z^2 - \Lambda^2)^2 - \Lambda^4$, which is a singular cubic

$$\mathcal{C}_T(2\Lambda^2) : \mu^2 = z^2(z^2 - 2\Lambda^2).$$

With (z, μ) mapping to (z, w) by $w = \mu/z$, we get a nonsingular rational curve:

$$\Gamma : w^2 = z^2 - 2\Lambda^2.$$

$\mathcal{P}_2(z) = z^2 - \Lambda^2$ is the monic Chebyshev polynomial of degree 2 on $[-\Lambda\sqrt{2}, \Lambda\sqrt{2}]$. It satisfies Pell's equation $\mathcal{P}_2(z)^2 - w^2 z^2 = \Lambda^4$, where $\Delta_2(z) = w^2 = z^2 - 2\Lambda^2$ and $\mathcal{Q}_1(z) = z$. The internal critical point of $\mathcal{P}_2$ is $z = 0$, the zero of $\mathcal{Q}_1$, corresponding to one massless particle which arises in this singular case.
The set of the vacuum moduli parameters is empty $(g = d - 1 = 0)$: there are no nontrivial deformations in this case.