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Langlands Duality and Invariant Differential Operators

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Abstract

Langlands duality is one of the most influential topics in mathematical research. It has many different appearances and influential subtopics. Yet there is a topic that until now seems unrelated to the Langlands program. That is the topic of invariant differential operators. That is strange since both items are deeply rooted in Harish-Chandra's representation theory of semisimple Lie groups. In this paper we start building the bridge between the two programs. We first give a short review of our method of constructing invariant differential operators. A cornerstone in our program is induction of representations from parabolic subgroups P=MAN of semisimple Lie groups. The connection to the Langlands program is through the subgroup M which other authors use in the context of the Langlands program.

Next we consider the group $SL(2n,\mathbb{R})$ which is

currently prominently used via Langlands duality. In that case we have $M = SL(n,\mathbb{R}) \times SL(n,\mathbb{R})$. We classify the induced representations implementing $P = MAN$. We find out and classify the reducible cases. Using our procedure we classify the invariant differential operators in this case.

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Introduction

In the last 50 years Langlands duality is one of the most influential topics in mathematical research. It has many different appearances and influential subtopics, cf. an incomplete list in loc. cit. Note that some papers are written by authors who have created influential topics themselves. The last fact stresses the omnipresence of the Langlands program.

Yet, the concept of invariant differential operators has not been related to the Langlands program in literature. That is strange since both items are deeply rooted in Harish-Chandra's representation theory of semisimple Lie groups. Here we start building a bridge between the two programs.

Our attempt is based on our approach to the construction of *invariant differential operators*

- for an exposition we refer to [VKD1] which is based also on many papers, see loc. cit. Our approach is deeply related to the Langlands general classification of representations of real semisimple groups G taking into account the refinement by Knapp-Zuckermann.
[R.P. Langlands, On the Classification of Irreducible Representations of Real Algebraic Groups, Mimeographed notes Princeton 1973; Published in: Math.Surveys Monogr. 31 (1989) 101-170; A.W. Knapp, G.J. Zuckerman, Springer Lecture Notes in Math. Vol. 587, pp. 138-159 (1977); *Ann. Math. 116*, 389-501 (1982).

One main ingredient in Langlands approach are the parabolic subgroups $P = MAN$, such that M is semisimple subgroup of our group G under study, A is abelian subgroup, N is nilpotent subgroup preserved by the action A. Altogether, there is a local (Bruhat) decomposition of G using a subgroup $G' = P\tilde{N}$, where

$\tilde{N}$ is a nilpotent subgroup of G isomorphic to N also preserved by the action A, so that G' is dense in G. According to the construction of Langlands-Knapp-Zuckermann every admissible irreducible representation of G may be obtained as a subrepresentation of representations of G induced by a representations of some P (some class is enough - see details below).

Our construction of intertwining differential operators is based on the fact that the structure of parabolic subgroups is related to various subgroups of the Weyl groups $W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})$, where $\mathcal{G}$ is the Lie algebra of G, $\mathcal{H}$ is the Cartan subalgebra of some MA. This is also related to various intertwining operators in the Langlands dual group.

Another aspect of the above is the Chevalley automorphism in the case of real groups which is an exhibition of the local Langlands correspondence over R. Another application to representation theory is using Heisenberg modules.

A full review with literature is in our paper loc. cit. (in "Mathematics").
We just mention some points that are reviewed there.
- Gauge theory aspects of the geometric Langlands programme.
- An exotic 'Chtoucas' application of the Langlands programme.
- Langlands duality extends to Poisson-Lie duality via cluster theory and to representations of W-algebras in the quantum framework.
- A proof of the global Langlands conjecture for GL(2) over a function field
- Two-parameter generalization of the geometric Langlands correspondence is proved for all

simply-laced Lie algebras. This is related to two-parameter quantum groups, e.g., and to 6d conformal supersymmetry [VKD1] ;

- Applications to integrability.

Further, the present talk is organized as follows. In the next section we give a synopsis of our approach. Then we apply this to the group $SL(2n,\mathbb{R})$, using the Langlands duality of the subgroup M used in the example. The cases $n = 2, 3, 4$ are exposed in separate subsections.

Preliminaries

We start by giving a synopsis of our program of canonical construction of invariant differential operators.

Let G be a semi-simple, non-compact Lie group, and K a maximal compact subgroup of G. Then, we have an *Iwasawa decomposition* $G = KA_0N_0$, where A_0 is an Abelian simply connected vector subgroup of G and N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Furthermore, let M_0 be the centralizer of A_0 in K. Then, the subgroup $P_0 = M_0A_0N_0$ is a *minimal parabolic subgroup* of G. A *parabolic subgroup* $P' = M'A'N'$ is any subgroup of G which contains a minimal parabolic subgroup.

Furthermore let $\mathcal{G}, \mathcal{K}, \mathcal{P}, \mathcal{M}, \mathcal{A}, \mathcal{N}$ denote the Lie algebras of G, K, P, M, A, N, resp.

Further, for simplicity, we restrict to *maximal parabolic subgroups* $P = MAN$, i.e., rank $A = 1$, resp., to *maximal parabolic subalgebras* $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ with dim $\mathcal{A} = 1$.

Let ν be a (non-unitary) character of A, $\nu \in \mathcal{A}^*$, parameterized by a real number d, called (for historical reasons) the *conformal weight* or energy.

Furthermore, let μ fix a discrete series representation D^μ of M on the Hilbert space V_μ, or the finite-dimensional (non-unitary) representation of M with the same Casimirs.

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G. (These are called *generalized principal series representations* (or *limits thereof*) in [Knapp].) Their spaces of functions are:

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) | \mathcal{F}(gman) =$$

$$= e^{-\nu(H)} \cdot D^\mu(m^{-1})\, \mathcal{F}(g)\}$$

where $a = \exp(H) \in A'$, $H \in \mathcal{A}'$, $m \in M'$, $n \in N'$. The representation action is the *left regular action*:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g'), \quad g, g' \in G. \qquad (1)$$

An important ingredient in our considerations are the *highest/lowest-weight representations* of $\mathcal{G}^{\mathbb{C}}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}^{\mathbb{C}}$, where $\Lambda \in (\mathcal{H}^{\mathbb{C}})^*$, $\mathcal{H}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$ and the weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [VKD1].

Actually, since our ERs may be induced from finite-dimensional representations of $\mathcal{M}$ (or their limits) the Verma modules are always reducible.

Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest-weight vector v_0 is taken by the (finite-dimensional) space $V_\mu \, v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators, only the reducibility with regard to non-compact roots is essential.

Another main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets*. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the *vertices* of which correspond to the reducible ERs and the *lines (arrows)* between the vertices correspond to intertwining operators. The explicit parameterization of the multiplets and of their ERs is important in understanding of the situation. The notion of multiplets was introduced in 1985 and applied to representations of $SO_o(p,q)$ and $SU(2,2)$, resp., induced from their minimal parabolic subalgebras. Then it was applied to the conformal superalgebra, to quantum groups, to infinite-dimensional (super)algebras, see later volumes of [VKD1].

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consist of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbb{C}}$, $m \in \mathbb{N}$, such that the *BGG Verma module reducibility condition* (for highest-weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) = m, \quad \beta^\vee \equiv 2\beta/(\beta,\beta) \qquad (2)$$

where ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. When the above holds, then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$

for GVM and β non-compact) is embedded in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\ v_0$, and $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbb{C}}$ generated by the negative root generators. More explicitly [**?**], $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, V_\mu\, v_0$ for GVMs). Then, there exists [VKD1] **intertwining differential operator**

$$\mathcal{D}_{m,\beta} : \mathcal{C}_{\chi(\Lambda)} \longrightarrow \mathcal{C}_{\chi(\Lambda - m\beta)} \qquad (3)$$

given explicitly by:

$$\mathcal{D}_{m,\beta} = \mathcal{P}_{m,\beta}(\widehat{\mathcal{G}^-}) \qquad (4)$$

where $\widehat{\mathcal{G}^-}$ denotes the *right action* on the functions $\mathcal{F}$.

Main results

Restricted Weyl groups and related notions

In our exposition below, we shall use the so-called Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \alpha_i^\vee), \quad i = 1, \ldots, n, \tag{5}$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$.

We shall use also the so-called *Harish–Chandra parameters*:

$$m_\beta \equiv (\Lambda + \rho, \beta^\vee) \ , \tag{6}$$

where β is any positive root of $\mathcal{G}^{\mathbb{C}}$. These parameters are redundant, since they are expressed in terms of the Dynkin labels; however, some statements are best formulated in their terms. (Clearly, both the Dynkin labels and Harish–Chandra parameters have their origin in the BGG reducibility condition (2).)

Next, we recall the action of the Weyl group on highest weights:

$$w_\beta(\Lambda) \doteq \Lambda - (\Lambda + \rho, \beta^\vee)\beta \qquad (7)$$

and thus,

$$w_\beta(\Lambda) = \Lambda - m_\beta \beta \qquad (8)$$

and the shifted weight in (3) results by the action of the Weyl group as in (8).

Next we mention the important notion of *restricted Weyl group*. We first need the so-called restricted roots.

Let $\Delta_{\mathcal{A}'}$ be the *restricted root system* of $(\mathcal{G}, \mathcal{A}')$:

$$\Delta_{\mathcal{A}'} \doteq \{\lambda \in \mathcal{A}'^* \mid \lambda \neq 0,\ \mathcal{G}^\lambda_{\mathcal{A}'} \neq 0\} , \\ \mathcal{G}^\lambda_{\mathcal{A}'} \doteq \{X \in \mathcal{G} \mid [Y, X] = \lambda(Y)X \ , \ \ \forall Y \in \mathcal{A}'\} \qquad (9)$$

The elements of $\Delta_{\mathcal{A}'}$ are called $\mathcal{A}'$-*restricted roots*.

[The terminology comes from the fact that things may be arranged so that these roots are obtained as restriction to $\mathcal{A}'$ of some roots of the root system Δ of the pair $(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})$.]

For $\lambda \in \Delta_{\mathcal{A}'}$, $\mathcal{G}^{\lambda}_{\mathcal{A}'}$ are called *$\mathcal{A}'$-restricted root spaces*, $\dim_R \mathcal{G}^{\lambda}_{\mathcal{A}'} \geq 1$.

Next we introduce some ordering (e.g., the lexicographic one) in $\Delta_{\mathcal{A}'}$. Accordingly the latter is split into positive and negative restricted roots: $\Delta_{\mathcal{A}'} = \Delta^{+}_{\mathcal{A}'} \cup \Delta^{-}_{\mathcal{A}'}$.

Furthermore, we introduce the simple restricted root system $\Delta^{R}_{\mathcal{A}'}$, which is the simple root system of the restricted roots. Next we introduce the *restricted Weyl reflections*: for each root $\lambda \in \Delta^{+}_{\mathcal{A}'}$ we define a reflection s_λ in $\mathcal{A}'^*$:

$$s_\lambda(\mu) \ \equiv \ \mu - 2\frac{(\lambda,\mu)}{(\lambda,\lambda)}\lambda \ , \quad \mu \in \mathcal{A}'^* \qquad (10)$$

Clearly, $s_\lambda(\lambda) = -\lambda$, $s^2_\lambda = \mathrm{id}_{\mathcal{A}'^*}$.

The above reflections generate the $\mathcal{A}'$-*restricted Weyl group* $W(\mathcal{G}, \mathcal{A}')$.

The above may be applied to the case when instead of some $\mathcal{A}'$ we use an arbitrary subalgebra $\mathcal{H}'$ of $\mathcal{H}$.

The case of $SL(2n,\mathbb{R})$

In this talk we treat the case of $G = SL(2n,\mathbb{R})$, $\mathcal{G} = sl(2n,\mathbb{R})$. We restrict to maximal parabolic subalgebra

$$\begin{aligned} \mathcal{P} &= \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \qquad (11) \\ \mathcal{M} &= sl(n,\mathbb{R}) \oplus sl(n,\mathbb{R})\,, \ \dim \mathcal{A} = 1, \ \dim \mathcal{N} = n^2 \end{aligned}$$

In the context of relative Langlands duality this case was studied as the subcase of hyperspherical dual pairs. There the relation to physics appeared as arithmetic analog of the electric-magnetic duality of boundary conditions in four-dimensional supersymmetric Yang-Mills theory. This aspect will be recovered for $n = 2$ below.

In this section we start with $G = SL(m,\mathbb{R})$, the group of invertible $m \times m$ matrices with real elements and determinant 1. Then $\mathcal{G} =$

$sl(m,\mathbb{R})$ and the Cartan involution is given explicitly by: $\tilde{h}X = -\ {}^tX$, where ${}^tX$ is the transpose of $X \in \mathcal{G}$. Thus, $\mathcal{K} \cong so(m)$, and is spanned by matrices (r.l.s. stands for real linear span):

$$\mathcal{K} = \text{r.l.s.}\{X_{ij} \equiv e_{ij} - e_{ji}\ , \quad 1 \le i < j \le m\}\ , \tag{12}$$

where e_{ij} are the standard matrices with only nonzero entry (=1) on the i-th row and j-th column, $(e_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}$. (Note that $\mathcal{G}$ does not have discrete series representations if $m > 2$.)

Further, the complementary space $\mathcal{P}$ is given by:

$$\mathcal{P} = \text{r.l.s.}\{Y_{ij} \equiv e_{ij} + e_{ji}\ , \quad 1 \le i < j \le m\ , \tag{13}$$

$$H_j \equiv e_{jj} - e_{j+1,j+1}\ , \quad 1 \le j \le m-1\}\ , \tag{14}$$

The split rank is $r = m - 1$, and from (13) it is obvious that in this setting one has:

$$\mathcal{H} = \text{r.l.s.}\{H_j\ , \quad 1 \le j \le n-1 = r\}\ . \tag{15}$$

The simple root vectors are given explicitly by:

$$X_j^+ \doteq e_{j,j+1} \ , \quad X_j^- \doteq e_{j+1,j} \ , \quad 1 \leq j \leq m-1 \ . \qquad (16)$$

Note that matters are arranged so that

$$[X_j^+, X_j^-] = H_j \ , \quad [H_j, X_j^\pm] = \pm 2 X_j^\pm \ , \qquad (17)$$

and further we shall denote by $sl(2,\mathbb{R})_j$ the $sl(2,\mathbb{R})$ subalgebra of $\mathcal{G}$ spanned by $X_j^\pm, H_j$.

In our case of consideration $m = 2n$ we have

$$\mathcal{M} \ = \ sl(n) \oplus sl(n) \qquad (18)$$

and we use representations of $\mathcal{M}$ indexed as follows:

$$\hat{\mathcal{M}} = (m_1, \ldots, m_{n-1} \ ; \ m_{n+1}, \ldots, m_{2n-1}) \qquad (19)$$

When all m_j are natural numbers $\hat{\mathcal{M}}$ indexes the unitary finite-dimensional irreps of $\mathcal{M}$.

sl(4)

In the case of $sl(4)$ the parabolic $\mathcal{M}$ factor is:

$$\mathcal{M}_4 \;=\; sl(2) \oplus sl(2) \tag{20}$$

the representations being indexed by the numbers m_1, m_3

Relatedly the representations of $\mathcal{G}$ are indexed by:

$$\chi_4 \;=\; [m_1, m_2, m_3] \tag{21}$$

It is well-known that when all m_j are natural numbers then χ_4 exhausts the finite-dimensional representations of $\mathcal{G}$. Each representation χ_4 is part of 24-member multiplet naturally corresponding to the 24 elements of the Weyl group of $sl(4)$. When we consider induction from $\mathcal{M}_4$ then we have six-member multiplets (sextets) parametrized as follows:

$$\begin{aligned}
\chi^- &= \{\, m_1\,, m_2\,, m_3\,\}, \qquad (22)\\
\chi'^- &= \{\, m_{12}\,, -m_2\,, m_{23}\}, \ \Lambda'^- = \Lambda^- - m_2\alpha_2\\
\chi''^- &= \{\, m_2\,, -m_{12}\,, m_{13}\}, \ \Lambda''^- = \Lambda'^- - m_1\alpha_{12}\\
\chi''^+ &= \{\, m_{13}\,, -m_{23}\,, m_2\}, \ \Lambda''^+ = \Lambda'^- - m_3\alpha_{23}\\
\chi'^+ &= \{\, m_{23}\,, -m_{13}\,, m_{12}\}, \ \Lambda'^+ = \Lambda''^- - m_3\alpha_{23} =\\
&= \Lambda''^+ - m_1\alpha_{12}\\
\chi^+ &= \{\, m_3\,, -m_{13}\,, m_1\}, \ \Lambda^+ = \Lambda'^+ - m_2\alpha_2
\end{aligned}$$

where $m_{12} \equiv m_1 + m_2$, $m_{23} \equiv m_2 + m_3$, $m_{13} \equiv m_1{+}m_2{+}m_3$. Note that the $\pm$ pairs are related by Knapp-Stein integral intertwining operators $G^\pm$ so that the operators G^+ act from χ^- to χ^+, while G^- act from χ^+ to χ^-, etc.

Thus, the *Knapp-Stein duality* is a manifestation of the *Langlands duality*.

We recall that the number N_M of ERs in a multiplet corresponding to induction from a parabolic given by [VKD1]:

$$N_M \;=\; \frac{|W(\mathcal{G},\mathcal{H})|}{|W(\mathcal{M},\mathcal{H}_m)|} \qquad (23)$$

which in our case ($\mathcal{M} = \mathcal{M}_4$) gives:

$$N_M = \frac{24}{4} = 6. \qquad (24)$$

what we have obtained.

An alternative parametrization stressing the duality is given as follows:

$$\begin{aligned}
\chi^{\pm} &= \{ (m_1 ; m_3)^{\pm} ;\ c = \pm (m_2 + \tfrac{1}{2}(m_1 + m_3)) \} \\
\chi'^{\pm} &= \{ (m_{12} , m_{23})^{\pm} ;\ c = \pm \tfrac{1}{2}(m_1 + m_3)\}, \\
\chi''^{\pm} &= \{ (m_2 , m_{13})^{\pm} ;\ c = \pm \tfrac{1}{2}(m_1 - m_3)\},
\end{aligned}$$

where $(p;q)^+ = (q;p)$, $(p;q)^- = (p;q)$,

The irreducible subrepresentations $\mathcal{E}$ of χ^- are finite-dimensional, exhausting all finite-dimensional (non-unitary) representations of $sl(4)$, and of all real forms.

Note also that the dimensions of the $\pm$ inducing pair of $\mathcal{M}$ are the same, namely, $m_1 m_3$ for $\chi^{\pm}$, $m_{12} m_{23}$ for $\chi'^{\pm}$, $m_2 m_{13}$ for $\chi''^{\pm}$.

Finally, we use the simplest case $m_1 = m_2 = m_3 = 1$ to exhibit the electro-magnetic duality which has transparent physical meaning for the conformal real form $su(2,2)$. The multiplet is depicted on Fig. 1. (Complete treatment may be found in [VKD1].)

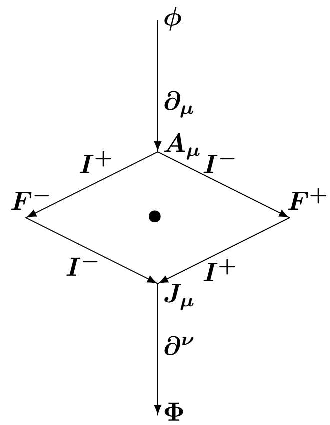

Fig. 1. Simplest case of conformal invariant differential operators
$F^\pm$ depict the duality decomposition of the electromagnetic field $F_{\mu\nu}$.
A_μ, resp., J_μ, is the electromagnetic potential, resp., the current.
$I^\pm$ depict the differential operators which come from the equation $\partial_{[\mu A_\nu]=F_{\mu\nu}}$.
Knapp-Stein operators relate cases symmetric w.r.t. the central black dot.

Multiplets containing the finite-dimensional subrepresentations are called *main multiplets*. The other multiplets are called *reduced multiplets*. These contain inducing finite-dimensional representations of $\mathcal{M}$.

In the case at hand there are three such cases so that each reduced multiplet is a doublet (containing two ERs). Explicitly the three cases are:

$$\begin{aligned}\chi_1^\pm &= \{ (m_2\,, m_{23})^\pm\,; \\ &\qquad c \;=\; \pm\,\tfrac{1}{2} m_3\}, \qquad\qquad (25a)\\ &\qquad \Lambda_1^+ = \Lambda_1^- - m_3\alpha_{23}\\ \chi_2^\pm &= \{ (m_1\,; m_3)^\pm\,; \\ &\qquad c^\pm \;=\; \pm\,\tfrac{1}{2}(m_1 + m_3)\}, \qquad (25b)\\ \chi_3^\pm &= \{ (m_{12}\,, m_2)^\pm\,; \\ &\qquad c \;=\; \pm\,\tfrac{1}{2} m_1\}, \qquad\qquad (25c)\\ &\qquad \Lambda_3^+ = \Lambda_3^- - m_1\alpha_{12}\end{aligned}$$

Note that here the invariant operators are deformations of the Knapp-Stein integral operators from the sextet picture. Thus, those from

χ^+ to χ^- are still integral operators, while those from χ^- to χ^+ are differential operators via degeneration of the Knapp-Stein integral operators. Yet in the first and third case these are differential operators inherited from the sextets, only the operators in (25b) from χ_2^- to χ_2^+ are obtained due to genuine degeneration of the Knapp-Stein integral operators. This is the standard degeneration of the two-point function-kernel which at the reducibility points is a generalized function with regularization turning it into delta-function (cf. Gelfand et al (Vol 5)). Finally, we add that in the case $m_1 = m_3 = n$ the operators (25b) become a degree of the d'Alembert operator:

$$\mathcal{D}_{n,n} \ = \ const \, \Box^{c^+} \ = \ const \, \Box^{n} \qquad (26)$$

sl(6)

Here we take up the case $sl(6)$ with parabolic $\mathcal{M}$ factor

$$\mathcal{M}_5 = sl(3,\mathbb{R}) \oplus sl(3,\mathbb{R}) = \mathcal{M}_{2L} \oplus \mathcal{M}_{2R} \quad (27)$$

We start with elementary representations of $sl(6,\mathbb{R})$ indexed by five numbers:

$$\chi = \{m_1\,, m_2\,, m_3\,, m_4\,, m_5\}\,, \qquad (28)$$

so that m_1, m_2 index the representations of $\mathcal{M}_{2L}$, m_4, m_5 index the representations of $\mathcal{M}_{2R}$, while m_3 indexes the representations of the dilatation subalgebra $\mathcal{A}$.

When all m_j are positive integers we use formula (23) so we have a multiplet of 20 members since:

$$N_M = \frac{|W(\mathcal{G},\mathcal{H})|}{|W(\mathcal{M}_5,\mathcal{H}_5)|} = \frac{6!}{(3!)^2} = 20. \quad (29)$$

Their signatures are:

$$
\begin{aligned}
\chi_1 &= \{m_1, m_2, m_3, m_4, m_5\}, \qquad (30)\\
\chi_2 &= \{m_1, m_{23}, -m_3, m_{34}, m_5\},\\
\chi_3 &= \{m_{12}, m_3, -m_{23}, m_{24}, m_5\},\\
\chi_4 &= \{m_1, m_{24}, -m_{34}, m_3, m_{45}\},\\
\chi_5 &= \{m_2, m_3, -m_{13}, m_{14}, m_5\},\\
\chi_6 &= \{m_{12}, m_{34}, -m_{24}, m_{23}, m_{45}\},\\
\chi_7 &= \{m_1, m_{25}, -m_{35}, m_3, m_4\},\\
\chi_8 &= \{m_2, m_{34}, -m_{14}, m_{13}, m_{45}\},\\
\chi_9 &= \{m_{13}, m_4, -m_{24}, m_2, m_{35}\},\\
\chi_{10} &= \{m_{12}, m_{35}, -m_{25}, m_{23}, m_4\},\\
\chi_{11} &= \{m_{23}, m_4, -m_{14}, m_{12}, m_{35}\},\\
\chi_{12} &= \{m_2, m_{35}, -m_{15}, m_{13}, m_4\},\\
\chi_{13} &= \{m_{13}, m_{45}, -m_{25}, m_2, m_{34}\},\\
\chi_{14} &= \{m_3, m_4, -m_{14}, m_1, m_{25}\},\\
\chi_{15} &= \{m_{23}, m_{45}, -m_{15}, m_{12}, m_{34}\},\\
\chi_{16} &= \{m_{14}, m_5, -m_{25}, m_2, m_3\},\\
\chi_{17} &= \{m_3, m_{45}, -m_{15}, m_1, m_{24}\},\\
\chi_{18} &= \{m_{24}, m_5, -m_{15}, m_{12}, m_3\},\\
\chi_{19} &= \{m_{34}, m_5, -m_{15}, m_1, m_{23}\},\\
\chi_{20} &= \{m_4, m_5, -m_{15}, m_1, m_2\}
\end{aligned}
$$

The Proof is constructive. We start with the representation χ_1, then by our procedure we find the embedded representation χ_2. Then from the latter we find the embedded representations χ_3 and χ_4. We proceed to the last case χ_{20} which is reducible only by the Knapp-Stein operator intertwining it with its Langlands dual χ_1. $\diamondsuit$

The full picture of embeddings is seen on Fig. 2.

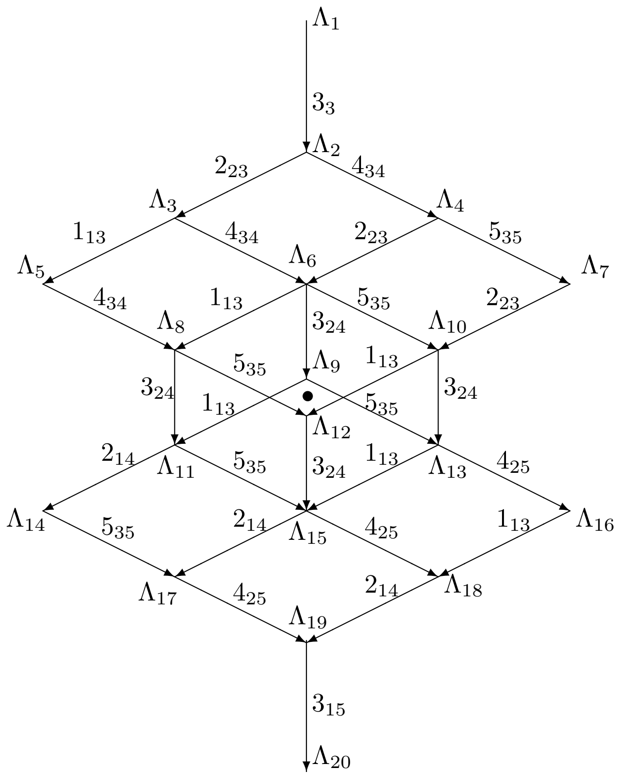

Fig. 2. Main multiplets for $sl(6,\mathbb{R})$

We quickly observe that the representations χ_n and χ_{21-n} are Langlands duals related by Knapp-Stein operators. More explicitly, this duality is given by the following presentation of the same multiplet:

$$\begin{aligned}
\chi_1^{\pm} &= \{\,(m_1\,,m_2\,;\,m_4\,,m_5)^{\pm}\,; \\
&\quad c=\pm(m_3+\tfrac{1}{2}m_{12,45})\,\}, \\
\chi_2^{\pm} &= \{\,(m_1\,,m_{23}\,;\,m_{34}\,,m_5)^{\pm}\,;\,c=\pm\tfrac{1}{2}m_{12,45}\,\}, \\
\chi_3^{\pm} &= \{\,(m_{12}\,,m_3\,;\,m_{24}\,,m_5)^{\pm}\,;\,c=\pm\tfrac{1}{2}m_{1,45}\,\}, \\
\chi_4^{\pm} &= \{\,(m_1\,,m_{24}\,;\,m_3\,,m_{45})^{\pm}\,;\,c=\pm\tfrac{1}{2}m_{12,5}\,\}, \\
\chi_5^{\pm} &= \{\,(m_2\,,m_3\,;\,m_{14}\,,m_5)^{\pm}\,;\,c=\pm\tfrac{1}{2}(m_{45}-m_1)\,\} \\
\chi_6^{\pm} &= \{\,(m_{12}\,,m_{34}\,;\,m_{23}\,,m_{45})^{\pm}\,; \\
&\quad c=\pm\tfrac{1}{2}(m_1+m_5)\,\}, \\
\chi_7^{\pm} &= \{\,(m_1\,,m_{25}\,;\,m_3\,,m_4)^{\pm}\,;\,c=\pm\tfrac{1}{2}(m_{12}-m_5)\,\} \\
\chi_8^{\pm} &= \{\,(m_2\,,m_{34}\,;\,m_{13}\,,m_{45})^{\pm}\,;\,c=\pm\tfrac{1}{2}(m_5-m_1) \\
\chi_9^{\pm} &= \{\,(m_{13}\,,m_4\,;\,m_2\,,m_{35})^{\pm}\,;\,c=\pm\tfrac{1}{2}(m_1+m_5)\,\} \\
\chi_{10}^{\pm} &= \{\,(m_{12}\,,m_{35}\,;\,m_{23}\,,m_4)^{\pm}\,;\,c=\pm\tfrac{1}{2}(m_1-m_5)
\end{aligned}$$

where $(p,q;r,s)^+ \equiv (r,s;p,q)$, $(p,q;r,s)^- \equiv (p,q;r,s)$, and the inducing number of the dilatation subalgebra $\mathcal{A}$ is replaced by the conformal factor c. Clearly, $\chi_n^- = \chi_n$, $\chi_n^+ = \chi_{21-n}$ for $1 \leq n \leq 10$.

Reduced multiplets

Here we just list the reduced multiplets which contain finite-dimensional irreps of the inducing $\mathcal{M}$.

$$\begin{array}{lll}
1 & \chi_3'^{\pm} = & \{ (m_2\,, m_3\,;\, m_{24}\,, m_5)^{\pm}\,; \\
 & & c = \pm\frac{1}{2} m_{45} \}, \\
 & \chi_6'^{\pm} = & \{ (m_2\,, m_{34}\,;\, m_{23}\,, m_{45})^{\pm}\,;\, c = \pm\frac{1}{2} m_5 \}, \\
 & \chi_9'^{\pm} = & \{ (m_{23}\,, m_4\,;\, m_2\,, m_{35})^{\pm}\,;\, c = \pm\frac{1}{2} m_5 \}, \\
13 & \chi_6''^{\pm} = & \{ (m_2\,, m_4\,;\, m_2\,, m_{45})^{\pm}\,;\, c = \pm\frac{1}{2} m_5 \}, \\
14 & \chi_3''^{\pm} = & \{ (m_2\,, m_3\,;\, m_{23}\,, m_5)^{\pm}\,;\, c = \pm\frac{1}{2} m_5 \}, \\
15 & \chi_9''^{\pm} = & \{ (m_{23}\,, m_4\,;\, m_2\,, m_{34})^{\pm}\,;\, c = 0 \}, \quad (31a) \\
135 & \chi_6 = & \{ (m_2\,, m_4\,;\, m_2\,, m_4)\,;\, c = 0 \}, \qquad (31b)
\end{array}$$

$$\begin{array}{lll}
2 & \chi_2'^{\pm} = & \{ (m_1\,, m_3\,;\, m_{34}\,, m_5)^{\pm}\,;\, c = \pm\frac{1}{2} m_{1,45} \}, \\
 & \chi_4'^{\pm} = & \{ (m_1\,, m_{34}\,;\, m_3\,, m_{45})^{\pm}\,;\, c = \pm\frac{1}{2} m_{1,5} \}, \\
 & \chi_7'^{\pm} = & \{ (m_1\,, m_{35}\,;\, m_3\,, m_4)^{\pm}\,;\, c = \pm\frac{1}{2}(m_1 - m_5 \\
24 & \chi_2''^{\pm} = & \{ (m_1\,, m_3\,;\, m_3\,, m_5)^{\pm}\,;\, c = \pm\frac{1}{2} m_{1,5} \}, \\
25 & \chi_4''^{\pm} = & \{ (m_1\,, m_{34}\,;\, m_3\,, m_4)^{\pm}\,;\, c = \pm\frac{1}{2} m_1 \},
\end{array}$$

$$
\begin{aligned}
3 \quad \chi'^{\pm}_1 &= \{ (m_1 , m_2 ; m_4 , m_5)^{\pm} ; c = \pm\tfrac{1}{2} m_{12,45} \}, \\
\chi'^{\pm}_6 &= \{ (m_{12} , m_4 ; m_2 , m_{45})^{\pm} ; c = \pm\tfrac{1}{2} m_{1,5} \}, \\
\chi'^{\pm}_8 &= \{ (m_2 , m_4 ; m_{12} , m_{45})^{\pm} ; c = \pm\tfrac{1}{2}(m_5 - m_1
\end{aligned}
$$

Note that the numbers on the left indicate which representation numbers are missing in the displayed signatures.

Further, note that the $\pm$ pairs are Knapp-Stein pairs, except the case (31a) where the operator is just a flip of the finite-dimensional inducing irreps. Note also that the case (31b) is a singlet.

Note that we do not display reduced multiplets with missing labels m_4 and m_5 since due to duality they are equivalent to multiplets with missing labels m_2, m_1, resp.

The case sl(8)

Here we consider the case $sl(8)$ with parabolic $\mathcal{M}$ factor

$$\mathcal{M}_7 = sl(4,\mathbb{R})\oplus sl(4,\mathbb{R}) = \mathcal{M}_{2L}\oplus\mathcal{M}_{2R} \quad (32)$$

Analogously to the previously considered cases the representations of $sl(8,\mathbb{R})$ are indexed by seven numbers:

$$\chi = \{m_1\,, m_2\,, m_3\,, m_4\,, m_5\,, m_6,\,, m_7\}\,, \quad (33)$$

so that m_1, m_2, m_3 index the representations of $\mathcal{M}_{2L}$, m_5, m_6, m_7 index the representations of $\mathcal{M}_{2R}$, and m_4 indexes the representations of the dilatation subalgebra $\mathcal{A}$.

When all m_j are positive integers we again use the formula (23) so we have a multiplet of 70 members since:

$$N_M = \frac{|W(\mathcal{G},\mathcal{H})|}{|W(\mathcal{M}_7,\mathcal{H}_7)|} = \frac{8!}{(4!)^2} = 70. \quad (34)$$

Their signatures are:

$$
\begin{aligned}
\chi_1^\pm &= \{\,(m_1\,,m_2\,,m_3\,,m_4\,,m_5\,,m_6\,,m_7)^\pm, \\
c^\pm &= \pm\,(m_4+\tfrac{1}{2}m_{13,57})\,\}, \\
\chi_2^\pm &= \{\,(m_1\,,m_2\,,m_{34}\,,-m_4\,,m_{45}\,,m_6\,,m_7)^\pm\,, \\
c^\pm &= \pm\,(\tfrac{1}{2}m_{17})\} \\
\chi_3^\pm &= \{\,(m_1\,,m_{23}\,,m_4\,,-m_{34}\,,m_{35}\,,m_6\,,m_7)^\pm \\
c^\pm &= \pm\,(\tfrac{1}{2}m_{12,57})\,\} \\
\chi_4^\pm &= \{\,(m_1\,,m_2\,,m_{35}\,,-m_{45}\,,m_4\,,m_{56}\,,m_7)^\pm\,, \\
c^\pm &= \pm\,(\tfrac{1}{2}m_{13,67})\,\} \\
\chi_5^\pm &= \{\,(m_{12}\,,m_3\,,m_4\,,-m_{24}\,,m_{25}\,,m_6\,,m_7)^\pm\,, \\
c^\pm &= \pm\,(\tfrac{1}{2}m_{1,57})\,\} \\
\chi_6^\pm &= \{\,(m_1\,,m_{23}\,,m_{45}\,,-m_{35}\,,m_{34}\,,m_{56}\,,m_7)^\pm\,, \\
c^\pm &= \pm\,(\tfrac{1}{2}m_{12,67})\,\} \\
\chi_7^\pm &= \{\,(m_1\,,m_2\,,m_{36}\,,-m_{46}\,,m_4\,,m_5\,,m_{67})^\pm\,, \\
c^\pm &= \pm\,(\tfrac{1}{2}m_{13,7})\,\} \\
\chi_8^\pm &= \{\,(m_2\,,m_3\,,m_4\,,-m_{14}\,(m_{15}\,,m_{56}\,,m_7)^\pm \\
c^\pm &= \pm\,(\tfrac{1}{2}m_{-1,57})\,\} \\
\chi_9^\pm &= \{\,(m_{12}\,,m_3\,,m_{45}\,,-m_{25}\,(m_{24}\,,m_{56}\,,m_7)^\pm\,, \\
c^\pm &= \pm\,(\tfrac{1}{2}m_{1,67})\,\} \\
\chi_{10}^\pm &= \{\,(m_1\,,m_{24}\,,m_5\,,-m_{35}\,(m_3\,,m_{46}\,,m_7)^\pm\,, \\
c^\pm &= \pm\,(\tfrac{1}{2}m_{12,67})\,\} \\
\chi_{11}^\pm &= \{\,(m_1\,,m_{23}\,,m_{46}\,,-m_{36}\,(m_{34}\,,m_5\,,m_{67})^\pm\,, \\
c^\pm &= \pm\,(\tfrac{1}{2}m_{12,7})\,\}
\end{aligned}
\tag{35}
$$

$$\chi_{12}^{\pm} = \{ (m_1 , m_2 , m_{37} , -m_{47} , m_4 , m_5 , m_6)^{\pm} \ ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{13,-7}) \}$$
$$\chi_{13}^{\pm} = \{ (m_2 , m_3 , m_{45} , -m_{15} (m_{14} , m_{56} , m_7)^{\pm} \ ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{-1,67}) \}$$
$$\chi_{14}^{\pm} = \{ (m_{12} , m_{34} , m_5 , -m_{25} (m_{23} , m_{46} , m_7)^{\pm} \ ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{1,67}) \}$$
$$\chi_{15}^{\pm} = \{ (m_{12} , m_3 , m_{46} , -m_{26} (m_{24} , m_5 , m_{67})^{\pm} \ ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{1,7}) \}$$
$$\chi_{16}^{\pm} = \{ (m_1 , m_{24} , m_{56} , -m_{36} (m_3 , m_{45} , m_{67})^{\pm} \ ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{12,7}) \}$$
$$\chi_{17}^{\pm} = \{ (m_1 , m_{23} , m_{47} , -m_{37} , m_{34} , m_5 , m_6)^{\pm} \ ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{12,-7}) \}$$
$$\chi_{18}^{\pm} = \{ (m_2 , m_3 , m_{46} , -m_{16} , m_{14} , m_5 , m_{67})^{\pm} \ ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{-1,7}) \}$$
$$\chi_{19}^{\pm} = \{ (m_2 , m_{34} , m_5 , -m_{15} , m_{13} , m_{46} , m_7)^{\pm} \ ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{-1,67}) \}$$
$$\chi_{20}^{\pm} = \{ (m_{13} , m_4 , m_5 , -m_{25} , m_2 , m_{36} , m_7)^{\pm} \ ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{1,67}) \}$$
$$\chi_{21}^{\pm} = \{ (m_{12} , m_{34} , m_{56} , -m_{26} , m_{23} , m_{45} , m_{67})^{\pm}$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{1,7}) \}$$
$$\chi_{22}^{\pm} = \{ (m_1 , m_{25} , m_6 , -m_{36} , m_3 , m_4 , m_{57})^{\pm} \ ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{12,7}) \}$$

$$\chi^{\pm}_{23} = \{ (m_1, m_{24}, m_{57}, -m_{37}, m_3, m_{45}, m_6)^{\pm} ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{12,-7}) \}$$
$$\chi^{\pm}_{24} = \{ (m_{12}, m_3, m_{47}, -m_{27}, m_{24}, m_5, m_6)^{\pm} ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{1,-7}) \}$$
$$\chi^{\pm}_{25} = \{ (m_2, m_3, m_{47}, -m_{17}, m_{14}, m_5, m_6)^{\pm} ,$$
$$c^{\pm} = \mp (\tfrac{1}{2} m_{1,7}) \}$$
$$\chi^{\pm}_{26} = \{ (m_2, m_{34}, m_{56}, -m_{16}, m_{13}, m_{45}, m_{67})^{\pm} ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{-1,7}) \}$$
$$\chi^{\pm}_{27} = \{ (m_{23}, m_4, m_5, -m_{15}, m_{12}, m_{36}, m_7)^{\pm} ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{-1,67}) \}$$
$$\chi^{\pm}_{28} = \{ (m_{13}, m_4, m_{56}, -m_{26}, m_2, m_{35}, m_{67})^{\pm} ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{1,7}) \}$$
$$\chi^{\pm}_{29} = \{ (m_{12}, m_{35}, m_6, -m_{26}, m_{23}, m_4, m_{57})^{\pm} ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{1,7}) \}$$
$$\chi^{\pm}_{30} = \{ (m_1, m_{25}, m_{67}, -m_{37}, m_3, m_4, m_{56})^{\pm} ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{12,-7}) \}$$
$$\chi^{\pm}_{31} = \{ (m_{12}, m_{34}, m_{57}, -m_{27}, m_{23}, m_{45}, m_6)^{\pm} ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{1,-7}) \}$$
$$\chi^{\pm}_{32} = \{ (m_2, m_{34}, m_{57}, -m_{17}, m_{13}, m_{45}, m_6)^{\pm} ,$$
$$c^{\pm} = \mp (\tfrac{1}{2} m_{1,7}) \}$$
$$\chi^{\pm}_{33} = \{ (m_2, m_{35}, m_6, -m_{16}, m_{13}, m_4, m_{57})^{\pm} ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{-1,7}) \}$$

$$\chi^{\pm}_{34} = \{ (m_{23}, m_4, m_{56}, -m_{16}, m_{12}, m_{35}, m_{67})^{\pm} ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{-1,7}) \}$$
$$\chi^{\pm}_{35} = \{ (m_3, m_4, m_5, -m_{15}, m_1, m_{26}, m_7)^{\pm} ,$$
$$c^{\pm} = \pm (\tfrac{1}{2} m_{-12,67}) \}$$

where $(p,q,u;r,s,v)^+ \equiv (r,s,v;p,q,u)$, $(p,q,u;r,s,v)^- \equiv (p,q,u;r,s,v)$.

The Proof is constructive. We start with the representation χ^-_1, then by our procedure we find the embedded representation χ^-_2. Then from the latter we find the embedded representations χ^-_3 and χ^-_4. We proceed to the last case χ^+_1 which is reducible only by the Knapp-Stein operator intertwining it with its Langlands dual χ^-_1. $\diamondsuit$

Conclusion

On the example of the group $SL(2n,\mathbb{R})$ we started building a bridge between the Langlands program and our approach to construction and classification of invariant differential operators. We have obtained full new results in the cases of $sl(6,\mathbb{R})$ and $sl(8,\mathbb{R})$.

Our paper opens the perspective of applications to many other groups, in particular, the group $SL(2n+1,\mathbb{R})$ which looks similar but has different families of intertwining differential operators - this work is already in progress.

Thank you for your attention!