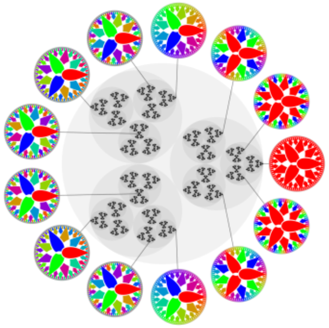
# Adelic Dynamics and DBI Lagrangians

**NONLINEARITY, NONLOCALITY AND ULTRAMETRICITY**

**International Conference on the Occasion of**

**Branko Dragovich 80th Birthday**

**26 - 30.05.2025, Belgrade, Serbia**

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# Some background

- It is a real pleasure and a privilege to be in position to give a lecture at the Conference on occasion of 80th Birthday of my supervisor (ментор) Branko Dragovich (Бранко Драговић)!
- Branko ``appeared`` as our professor on the subject ``Theoretical Physics / Electrodynamics`` in the second semester of the 3rd year of studying Physics in Niš, likely, in January 1986.
- In March1989 I defended my ``Diploma work`` on ``Friedmann Cosmological Models``.
- In June 1995, it was ``Master thesis`` on ``Adelic (time-dependent) Harmonic Oscillator …``, while in November 1999 ``The PhD thesis``- ``On p-adic and Adelic Quantum Mechanics`` was defended.
- **All these steps and works were supervised by Branko!**
- Between 1995 and 2003 we have published around 15 papers (some of them in collaboration with Lj. Nesic).
- This talk is a possibility to make a short review on this joint work, and the new results, with a different motivation but presented in a ``good old framework``!

# Plan

- IN SHORT – AS GOOD AS POSSIBLE …
- AQM, it was but …
- *p*-Adic and Adelic Quantum Mechanics
- *p*-Adic and Adelic Path Integrals – Quadratic Systems
- Tachyons-From Field Theory to Classical Analogue – DBI and Sen approach
- Tachyons and Inflation

- Primordial BHs …
- Beyond?

# Motivation …

- The main task of quantum cosmology is to describe the evolution of the universe in a very early stage.
- Since quantum cosmology is related to the Planck scale phenomena it is logical to consider various geometries (in particular nonarchimedean, noncommutative …)
- Despite some evident problems such as a non-sufficiently long period of inflation, tachyon-driven scenarios, both real and *p*-adic, remain highly interesting for study.
- But this time, it is a bit different …

# Adelic Quantum Theory

- Reasons to use *p*-adic numbers and adeles in quantum physics:
- The field of rational numbers **Q**, which contains all observational and experimental numerical data, is a dense subfield not only in **R** but also in the fields of *p*-adic numbers $Q_p$.
- There is an analysis within and over $\boldsymbol{Q_p}$ like that one related to **R.**
- General mathematical methods and fundamental physical laws should be invariant [I.V. Volovich, (1987), Vladimirov, Volovich, (1994)] under an interchange of the number fields **R** and $\boldsymbol{Q_p}$ .
- There is a quantum gravity uncertainty ($\Delta x \geq l_0 = \sqrt{\hbar G / c^3}$), when measures distances around the Planck length , which restricts priority of Archimedean geometry based on the real numbers and gives rise to employment of non-Archimedean geometry.
- It seems to be quite reasonable to extend standard Feynman's path integral method to non-Archimedean spaces.

# Adelic Quantum Theory –
## *p*-ADIC FUNCTIONS AND INTEGRATION

- There are primary two kinds of analyses on **Q***p* : **Q***p* → **Q***p* (class.) and **Q***p* → **C** (quant.).
- Usual complex valued functions of p-adic variable, which are employed in mathematical physics, are :
- an additive character $\chi_p(x) = \exp 2\pi i \{x\}_p,$
- locally constant functions with compact support

$$\Omega(|x|_p) = \begin{cases} 1 & |x|_p \leq 1 \\ 0 & |x|_p > 1 \end{cases}$$

# Adelic Quantum Theory –

## *p*-ADIC FUNCTIONS AND INTEGRATION

- There is well defined Haar measure and integration. Important integrals are

$$\int_{Q_p} \chi_p(ayx)dx = \delta_p(ay) = |a|_p^{-1}\,\delta_p(y),\ a \neq 0$$

$$\int_{Q_p} \chi_p(\alpha x^2 + \beta x)dx = \lambda_p(\alpha)\,|2\alpha|_p^{-1/2}\,\chi_p\left(-\frac{\beta^2}{4\alpha}\right),\ \alpha \neq 0$$

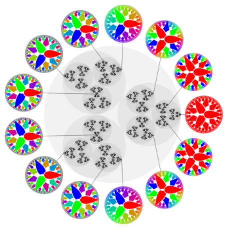
- Real analogues of integrals

$$\int_{Q_\infty} \chi_\infty(ayx)dx = \delta_\infty(ay) = |a|_\infty^{-1}\,\delta_\infty(y),\ a \neq 0$$

$$\int_{Q_\infty} \chi_\infty(\alpha x^2 + \beta x)dx = \lambda_\infty(\alpha)\,|2\alpha|_\infty^{-1/2}\,\chi_\infty\left(-\frac{\beta^2}{4\alpha}\right),\ \alpha \neq 0$$

$$Q_\infty \equiv R, \quad \chi_\infty(x) = \exp(-2\pi i x)$$

# Adelic Quantum Theory –
## Dynamics of a *p*-adic quantum model



- **Dynamics of *p*-adic quantum model**
- *p*-adic quantum mechanics is given by a triple

$$(L_2(Q_p), W_p(z_p), U_p(t_p))$$

- Adelic evolution operator is defined by

$$U(t)\psi(x) = \int_A K_t(x,y)\psi(y)dy = \prod_{v=\infty,2,3,\dots,p,\dots} \int_{Q_v} K_t^v(x_v,y_v)\psi^{(v)}(y_v)dy_v$$

- The eigenvalue problem

$$U(t)\psi_\alpha(x) = \chi(E_\alpha t)\psi_\alpha(x)$$

# Adelic Quantum Theory – path integrals

- The main problem in our approach is computation of *p*-adic transition amplitude in Feynman's PI method

$$K_p(x'',t'';x',t') = \int_{(x',t')}^{(x'',t'')} \chi_p\left(-\frac{1}{h}\int_{t'}^{t''} L(\dot{q},q,t)\right) Dq$$

- Exact general expression ($\bar{S}$-classical action)

$$K_p(x'',t'';x',t') = \lambda_p\left(-\frac{1}{2h}\frac{\partial^2 \bar{S}}{\partial x'\partial x''}\right) \times \left|\frac{1}{h}\frac{\partial^2 \bar{S}}{\partial x'\partial x''}\right|_p^{1/2} \chi_p\left(-\bar{S}(x'',t'';x',t')\right)$$

$$K_p(x'',y'',z'',t'';x',y',z',t') = \lambda_p\left(\det\begin{pmatrix} -\frac{1}{2}\frac{\partial^2 \bar{S}}{\partial x'\partial x''} & -\frac{1}{2}\frac{\partial^2 \bar{S}}{\partial x'\partial y''} & -\frac{1}{2}\frac{\partial^2 \bar{S}}{\partial x'\partial z''} \\ -\frac{1}{2}\frac{\partial^2 \bar{S}}{\partial y'\partial x''} & -\frac{1}{2}\frac{\partial^2 \bar{S}}{\partial y'\partial y''} & -\frac{1}{2}\frac{\partial^2 \bar{S}}{\partial y'\partial z''} \\ -\frac{1}{2}\frac{\partial^2 \bar{S}}{\partial z'\partial x''} & -\frac{1}{2}\frac{\partial^2 \bar{S}}{\partial z'\partial y''} & -\frac{1}{2}\frac{\partial^2 \bar{S}}{\partial z'\partial z''} \end{pmatrix}\right)$$

$$\times \left|\det\begin{pmatrix} -\frac{\partial^2 \bar{S}}{\partial x'\partial x''} & -\frac{\partial^2 \bar{S}}{\partial x'\partial y''} & -\frac{\partial^2 \bar{S}}{\partial x'\partial z''} \\ -\frac{\partial^2 \bar{S}}{\partial y'\partial x''} & -\frac{\partial^2 \bar{S}}{\partial y'\partial y''} & -\frac{\partial^2 \bar{S}}{\partial y'\partial z''} \\ -\frac{\partial^2 \bar{S}}{\partial z'\partial x''} & -\frac{\partial^2 \bar{S}}{\partial z'\partial y''} & -\frac{\partial^2 \bar{S}}{\partial z'\partial z''} \end{pmatrix}\right|_p^{1/2} \chi_p\left(-\bar{S}(x'',y'',z'',t'';x',y',z't')\right)$$

# Adelic Quantum Theory

- Adelic quantum mechanics [Dragovich (1994), G. Dj. and Dragovich (1997, 2000), G. Dj, Dragovich and Lj. Nesic (1999)].
- Adelic quantum mechanics: $(L_2(A), W(z), U(t))$
  - adelic Hilbert space, $L_2(A)$
  - Weyl quantization of complex-valued functions on adelic classical phase space, $W(z)$
  - unitary representation of an adelic evolution operator, $U(t)$
- The form of adelic wave function

$$\psi = \psi_\infty(x_\infty) \cdot \prod_{p\in M} \psi_p(x_p) \cdot \prod_{p\notin M} \Omega(|x|_p)$$

# Adelic Quantum Theory

- **Exactly soluble p-adic and adelic quantum mechanical models:**
- a free particle and harmonic oscillator [VVZ, Dragovich]
- a particle in a constant field, [G. Dj, Dragovich]
- a free relativistic particle[G. Dj, Dragovich, Nesic]
- a harmonic oscillator with time-dependent frequency [G. Dj, Dragovich]
- **Resume of AQM:** AQM takes in account ordinary as well as p-adic effects and may me regarded as a starting point for construction of more complete quantum cosmology (and string theory … ) . In the low energy limit AQM effectively becomes the ordinary one.
- Other approaches: A. Khrennikov, W. A. Zuniga-Galindo and others

# (ADELIC) QUANTUM COSMOLOGY

- **In the very beginning the Universe was in a quantum state, which should be described by a wave function (complex valued and depends on some real parameters).**
- **…**
- There is no Schroedinger and Wheeler-De Witt equation for cosmological models.
- Feynman's path integral method was exploited and minisuperspace cosmological models are investigated as a model of adelic quantum mechanics [Dragovich (1995), G Dj, Dragovich, Nesic and Volovich (2002), G.Dj and Nesic (2005, 2008, 2016) …
- Aspects of an "Adelic DBI Quantum Cosmology" will be discussed elsewhere

# Quantum Theory – path integrals

- The main problem in our approach is computation of *p*-adic transition amplitude in Feynman's PI method

$$K_p(x'',t'';x',t') = \int_{(x',t')}^{(x'',t'')} \chi_p\left(-\frac{1}{h}\int_{t'}^{t''} L(\dot{q},q,t)\right)Dq$$

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$$\times\left|\det\begin{pmatrix} -\frac{\partial^2\bar{S}}{\partial x'\partial x''} & -\frac{\partial^2\bar{S}}{\partial x'\partial y''} & -\frac{\partial^2\bar{S}}{\partial x'\partial z''} \\ -\frac{\partial^2\bar{S}}{\partial y'\partial x''} & -\frac{\partial^2\bar{S}}{\partial y'\partial y''} & -\frac{\partial^2\bar{S}}{\partial y'\partial z''} \\ -\frac{\partial^2\bar{S}}{\partial z'\partial x''} & -\frac{\partial^2\bar{S}}{\partial z'\partial y''} & -\frac{\partial^2\bar{S}}{\partial z'\partial z''} \end{pmatrix}\right|_p^{1/2} \chi_p\left(-\bar{S}(x'',y'',z'',t'';x',y',z't')\right)$$

# DBI Lagrangians

- String Theory
- A. Sen – proposed (effective) tachyon field action (for the *Dp*-brane in string theory):

$$S = -\int d^{n+1}x V(T)\sqrt{1+\eta^{ij}\partial_i T \partial_j T}$$

$$\eta_{00} = -1$$

$$\eta_{\mu\nu} = \delta_{\mu\nu} \quad \mu,\nu = 1,\ldots,n$$

- $T(x)$ - tachyon field
- $V(T)$ - tachyon potential
- Non-standard Lagrangian and DBI Action!

# Tachyons – History of an Idea

- A. Somerfeld - first discussed about possibility of particles to be faster than light (100 years ago).
- G. Feinberg - called them tachyons: Greek word, means fast, swift (almost 50 years ago).
- According to Special Relativity: $m^2 < 0, \qquad v = \frac{p}{\sqrt{p^2 + m^2}}.$
- From a more modern perspective the idea of faster-than-light propagation is abandoned and the term "tachyon" is recycled to refer to a quantum field with $m^2 = V'' < 0.$

# Tachyon field

- Field Theory
- Standard Lagrangian (real scalar field):

$$L(\varphi, \partial_\mu \varphi) = T - V = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - V(\varphi_0) - V'(\varphi_0)\varphi - \frac{1}{2}V''(\varphi_0)\varphi^2 - \ldots$$

- Extremum (min or max of the potential): $V'(\varphi_0) = 0$
- Mass term: $V''(\varphi_0) = m^2$
- Clearly $V''$ can be negative (about a maximum of the potential). Fluctuations about such a point will be unstable: tachyons are associated with the presence of instability.

$$L(\phi, \partial_\mu \phi) = L_{kin} - V = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 + const$$

# Tachyons-From Field Theory to Classical Analogue – DBI and Sen approach

- Equation of motion (EoM):

$$\ddot{T}(t) - \frac{1}{V(T)}\frac{dV}{dT}\dot{T}^2(t) = -\frac{1}{V(T)}\frac{dV}{dT}$$

- Can we transform EoM of a class of non-standard Lagrangians in the form which corresponds to Lagrangian of a canonical form, even quadratic one? Some classical canonical transformation (CCT)?

# Classical Canonical Transformation and Quantization

- CCT: $T, P \quad \mapsto \quad \tilde{T}, \tilde{P}$
- Generating function: $G(\tilde{T}, P) = -PF(\tilde{T})$

$$T = -\frac{\partial G}{\partial P} = F(\tilde{T}) \qquad\qquad \tilde{P} = -\frac{\partial G}{\partial \tilde{T}} \Rightarrow P = \left(\frac{dF(\tilde{T})}{d\tilde{T}}\right)^{-1} \tilde{P}$$

- EoM transforms to

$$\ddot{\tilde{T}} + \left( \frac{\frac{d^2F(\tilde{T})}{d\tilde{T}^2}}{\frac{dF(\tilde{T})}{d\tilde{T}}} - \frac{dF(\tilde{T})}{d\tilde{T}} \frac{d\ln V(F)}{dF} \right) \dot{\tilde{T}}^2 + \frac{1}{\frac{F(\tilde{T})}{d\tilde{T}}} \frac{d\ln V(F)}{dF} = 0$$

# Classical Canonical Transformation and Quantization

- Choice: $F^{-1}(T) = \int_{T_0}^{T} \frac{dX}{V(X)}$
- EoM reduces to:

$$\ddot{\tilde{T}} + \frac{1}{F'}\frac{d\ln V(F)}{dF} = 0!!!$$

- This EoM can be obtained from the standard type Lagrangians $\mathcal{L} = L_{kin} - V$

# Classical Canonical Transformation and Quantization

- Example: $V(T) = \dfrac{1}{\cosh(\beta T)}$

$$F^{-1}(T) = \int^{T} \frac{dx}{V(x)} = \frac{1}{\beta}\sinh(\beta T)$$

- Generating function: $G(\tilde{T}, P) = -PF(\tilde{T}) = -\dfrac{P}{\beta}\operatorname{arcsinh}(\beta T)$
- EoM: $\ddot{\tilde{T}}(t) - \beta^2 \tilde{T}(t) = 0$
- This EoM can be obtained from the standard-type (quadratic) Lagrangian

$$\mathcal{L}_{quad}(\tilde{T}, \dot{\tilde{T}}) = \frac{1}{2}\dot{\tilde{T}}^2 + \frac{1}{2}\beta^2 \tilde{T}^2$$

# Classical Canonical Transformation and Quantization

- Action (quadratic): $S_{cl} = \int_0^{\tau} \mathcal{L}_{quad} dt = \frac{\beta}{2}\left( \left( \tilde{T}_1^2 + \tilde{T}_2^2 \right) \coth(\beta\tau) - \frac{2\tilde{T}_1\tilde{T}_2}{\sinh(\beta\tau)} \right)$
- Quantization: Transition (adelic!?) amplitude, $v = \infty, 2, 3, \ldots p, \ldots$

$$\mathcal{K}_v(\tilde{T}_2, \tau; \tilde{T}_1, 0) = \lambda_v\left(\frac{1}{2\tau}\right)\left| -\frac{1}{\tau} \right|_v^{1/2} \chi_v\left( -S_{cl}(\tilde{T}_2, \tau; \tilde{T}_1, 0) \right)$$

- The necessary condition for the existence of a *p*-adic (adelic) quantum model is the existence of a *p*-adic quantum-mechanical ground (vacuum) state in the form of a characteristic $\Omega$ -function; we get expression which defines constraints on parameters of the theory

$$\int_{|\tilde{T}_1|_p \leq 1} \mathcal{K}_p(\tilde{T}_2, \tau; \tilde{T}_1, 0) d\tilde{T}_1 = \Omega(| \tilde{T}_2 |_p)$$

# Constraints on existence of *p*-adic vacuum state

- Using *p*-Adic Gauss integral

$$\int_{|y|_p \le 1} \chi_p(ay^2 + by)dy = \begin{cases} \Omega(|b|_p), & |a|_p \le 1 \\ \dfrac{\lambda_p(a)}{|a|_p^{1/2}} \chi_p(-\dfrac{b^2}{4a})\Omega(|\dfrac{b}{a}|_p), & |a|_p > 1 \end{cases}$$

- we get (in the case of an inverse power-law potential) $V \sim \tilde{T}^{-n}, n = 1$

$$\lambda_p\left(\frac{1}{2\tau}\right)|\tau|_p^{-1/2}\ \chi_p\left(-\frac{1}{2\tau}\tilde{T}_2^2 - \frac{1}{2}k\tau\tilde{T}_2 + \frac{1}{24}k^2\tau^3\right) \times I_{Gauss} = \Omega(|\tilde{T}_2|_p)$$

# Constraints on existence of *p*-adic vacuum state

- Case 1 $|\tau|_p > 1$ impossible to fulfill
- Case 2 $|\tau|_p = 1$

$$\chi_p\Big(-\frac{1}{2\tau}\tilde{T}_2^2 - \frac{1}{2}k\tau\tilde{T}_2 + \frac{1}{24}k^2\tau^3\Big)\Omega(|\frac{\tilde{T}_2}{\tau} - \frac{1}{2}k\tau|_p) = \Omega(|\tilde{T}_2|_p)$$

- Case 3 $|\tau|_p < 1$

$$\chi_p\Big(-k\tau\tilde{T}_2 + \frac{1}{6}k^2\tau^3\Big)\Omega(|-2\tilde{T}_2 + k\tau^2|_p) = \Omega(|\tilde{T}_2|_p)$$

# Quadratic and almost quadratic systems (*p*-adic case) – new results

- We consider DBI-type Lagrangian in 4d
- $L_T = -V(T)\sqrt{1 + (\partial T)^2}$
- $ds_p^2 = -c^2 + a(t)^2(dx^2 + dy^2 + dz^2)$
- $a(t)$ – scale factor, $T(t)$ – tachyon field,
- $p = 1(mod\ 4)$
- Equation of motion, homogeneous and isotropic space
- $\ddot{T} + \boxed{3H(t)\dot{T}(1 - \dot{T}^2)} + \frac{1}{V(T)}\frac{dV(T)}{dT}(1 - \boxed{\dot{T}^2}) = 0$
- Where, Hubble parameter is $H = \frac{\dot{a}}{a}$

# The method of Darboux

- After performing necessary and straightforward integration

$$L_T \equiv a^3(t)\mathcal{L}_T = a^3(t)(-V(t)\sqrt{1-\dot{T}^2})$$

- **The method of Darboux**
- The problem of reconstructiong an adequate Lagrangian, starting from EoM – the inverse problem.
- The procedure of constructiong a Lagrangian is generaly simplied when:

$$\ddot{q} + A(q, \dot{q}, t) = 0$$

# Equivalent, quadratic Lagrangian?

- Jacobi multiplayer $\Lambda(q,\dot{q},t) = \chi_{\nu}\left(\int \frac{\partial A(q,\dot{q},t)}{\partial \dot{q}}\,dt\right)$

- Our initial equation now takes the form

$$\ddot{T} + 3H(t)\left(1 + \frac{2}{3}\frac{\dot{H}(t)}{H^2(t)}\right)\dot{T} + \frac{V'(T)}{V(T)}\left(1 - \dot{T}^2\right) = 0$$

# Equivalent, quadratic Lagrangian?

$$A(T,\dot{T},t) = 3H(t)(1+\frac{2}{3}\frac{\dot{H}(t)}{H^2(t)})\dot{T} + \frac{V'(T)}{V(T)}(1-\dot{T}^2)$$

$$\Lambda(T,\dot{T},t) = \frac{a^3(t)H^2(t)}{V^2(T)}$$

- We get the initial DBI Lagrangian in a new form!

$$L = a^3(t)H^2(t)\left[\frac{1}{2}\left(\frac{\dot{T}}{V(T)}\right)^2 + \frac{1}{2}\frac{1}{V^2(T)}\right]$$

$$\dot{\phi} = \frac{\dot{T}}{V(T)} \qquad \phi = \int^T \frac{dT}{V(T)}$$

$$L = a(t)\dot{a}^2(t)[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2V^2(T(\phi))}]$$

# And some examples

- If we choose, for example, *exp* potential
  $V(T) = V_0 e^{-\omega T}, \quad V_0 = const,\ \omega = const$

- Inverted harmonic oscilator with time dependent mass

$$L = \frac{1}{2} m(t)\dot{x}^2 + \frac{1}{2} m(t)\omega^2 x^2$$

# The new, extended, model

- Inverted harmonic oscillator with time-dependent mass and frequency

$$L = \frac{1}{2}m(t)\dot{x}^2 + \frac{1}{2}m(t)\omega^2(t)x^2$$

- EOM

$$\ddot{x} + 2\frac{\dot{\eta}(t)}{\eta(t)}\dot{x} - \dot{\omega}^2(\mathrm{t}) = 0, \quad \eta(t) = \sqrt{m(t)}$$

- Let consider a solution of the form $x(\mathrm{t}) = \frac{\alpha(t)}{\eta(t)}\left(Ae^{\gamma(t)} + Be^{-\gamma(t)}\right)$

$$\left(\frac{\ddot{\alpha}}{\eta} - \frac{\alpha\ddot{\eta}}{\eta^2} + \frac{\alpha\dot{\gamma}^2}{\eta} - \frac{\alpha\omega^2}{\eta}\right)(\mathrm{Ae}^{\gamma} + \mathrm{Be}^{-\gamma}) + \left(\frac{\alpha\ddot{\gamma}}{\eta} + 2\frac{\dot{\alpha}\dot{\gamma}}{\eta}\right)(\mathrm{Ae}^{\gamma} - \mathrm{Be}^{-\gamma}) = 0$$

- We obtain

$$\ddot{\alpha} - \left(\omega^2(\mathrm{t}) + \frac{\ddot{\eta}}{\eta} - \dot{\gamma}^2\right)\alpha = 0$$

$$\ddot{\gamma} + 2\frac{\dot{\alpha}\dot{\gamma}}{\alpha} = 0 \qquad \alpha^2\dot{\gamma} = \mathrm{const}$$

# We get the final form

$$x(t) = \frac{\alpha}{\eta \sinh(\gamma'' - \gamma')} \left( \frac{\eta' x'}{\alpha'} \sinh(\gamma'' - \gamma) - \frac{\eta'' x''}{\alpha''} \sinh(\gamma' - \gamma) \right)$$

$$K(x'', t''; x', t') = F(t'', t') e^{iS_{cl}(x'', t''; x', t')/\hbar}$$

$$F(t'', t') = \frac{i}{2\pi\hbar} \frac{\partial^2}{\partial x' \partial x''} S_{cl}(x'', t''; x', t')$$

- For $p \equiv 1 (\bmod 4)$

$$K_p(x'', t''; x', t') = \lambda_p(-\tfrac{1}{2h} F(t'', t')) \left| F(t'', t') \right|_p^{1/2} \times \chi_p\left( -\tfrac{1}{h} S_{\mathrm{cl}}(x'', t''; x', t') \right)$$

# Inverted Caldirola-Kanai oscillator

$$m(t) = me^{rt}, \quad r = \text{const}$$

- We find

$$\alpha = \text{const}$$

$$\gamma(t) = \Omega t, \quad \Omega^2 = \sqrt{\omega^2 + \frac{r^2}{4}}$$

$$K(x'',t'';x',t') = \left(\frac{m\Omega}{2\pi i\hbar \sinh \Omega t}\right)^{\frac{1}{2}} \times e^{\frac{1}{4}rt} \times \exp\left(\frac{im}{4\hbar}(x'^2 - e^{rt}x''^2)\right)$$

$$\times \exp\left(\frac{im\Omega}{2\hbar \sinh \Omega t}\left[(e^{rt}x''^2 + x'^2)\cosh \Omega t - 2x'x''e^{\frac{1}{2}rt}\right]\right).$$

# DBI inflation – an effective approach

- Consider the tachyonic field *T* minimally coupled to Einstein's gravity

$$S = -\frac{1}{16\pi G}\int \sqrt{-g} R d^4x + S_T$$

- Where *R* is Ricci scalar, *g* – determinant of the metric tensor and tachyon action

$$S_T = \int \sqrt{-g}\mathcal{L}(T, \partial_\mu T) d^4x$$
$$\mathcal{L} = -V(T)\sqrt{1 + g^{\mu\nu}\partial_\mu T \partial_\nu T}$$

- Friedman equation:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_{Pl}^2}\frac{V}{(1-\dot{T}^2)^{1/2}}$$

- Energy-momentum conservation equation:

$$\dot{\rho} = -3H(P + \rho)$$
$$\frac{\ddot{T}}{1-\dot{T}^2} + 3H\dot{T} + \frac{V'}{V} = 0$$

**Energy density and pressure:**

$$\rho = \frac{V(T)}{\sqrt{1-\dot{T}^2}}$$
$$P = -V(T)\sqrt{1-\dot{T}^2}$$

# Tachyon inflation

- The slow-roll parameters

$$\epsilon_{i+1} \equiv \frac{d \ln | \epsilon_i |}{dN}, \quad i \geq 0, \quad \epsilon_0 \equiv \frac{H_*}{H}$$

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = \frac{1}{H}\frac{\ddot{H}}{\dot{H}} + 2\epsilon_1$$

$$\epsilon_1 = \frac{3}{2}\dot{T}^2, \quad \epsilon_2 = 2\frac{\ddot{T}}{H\dot{T}}$$

- Number of e-folds

$$N(t) = \int_{t_i}^{t_e} H(t)dt$$

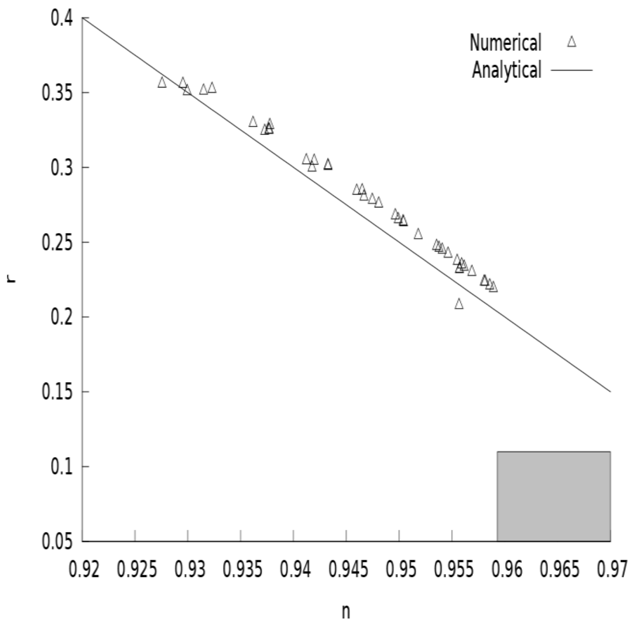
- In the slow-roll aproximation

$$N(x) = X_0^2 \int_{x_i}^{x_e} \frac{U(x)^2}{| U'(x) |}dx, \qquad \text{where } \varepsilon_1(x_e) = 1$$

- Observational parameters
  - The scalar spectral index $n = 1 - 2\epsilon_1(x_i) - \epsilon_2(x_i)$
  - The tensor-to-scalar ratio $r = 16\epsilon_1(x_i)$

# Tachyon inflation

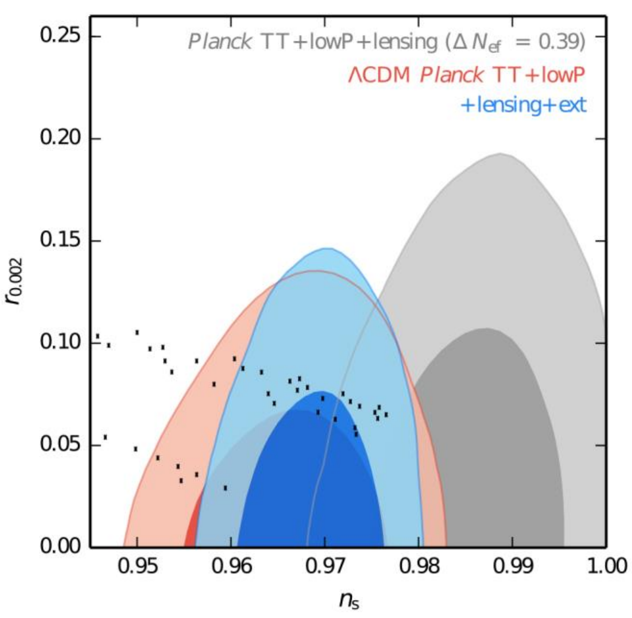
- Numerical results




$$45 \leq N \leq 75,\ 5 \leq X_0 \leq 25$$

$$U(x) = \frac{1}{\tilde{x}^4} \qquad \text{(left)}$$

$$U(x) = \frac{1}{\cosh(\tilde{x})} \qquad \text{(right)}$$



# Primordial Black Hole formation – DBI Tachyon Inflation

- The Lagrangian and the Hamiltonian

$$\mathcal{L} = -U(\varphi)\sqrt{1-X}, \quad X \equiv g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} \qquad \mathcal{H} = \frac{U}{\sqrt{1-X}} = \sqrt{U^2+\eta^2}$$

- The Hamilton equations

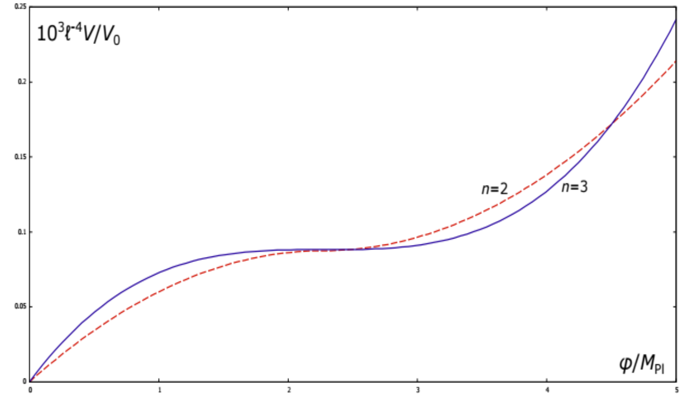
$$\frac{d\varphi}{dN} = \frac{\eta}{H\sqrt{U^2+\eta^2}} \qquad \frac{d\eta}{dN} = -3\eta - \frac{U}{H\sqrt{U^2+\eta^2}}\frac{\partial U}{\partial\varphi}$$

- The Hubble parameter

$$H^2 = \frac{\sqrt{U^2+\eta^2}}{3\ell^4 M_{\rm Pl}^2}$$

- The potential

$$U = \ell^4 V_0\left[\exp\left(-\lambda\frac{|\varphi_0-\varphi|^n\,\mathrm{sgn}(\varphi_0-\varphi)}{\ell^{2n}M_{\rm Pl}^n}\right) - \exp\left(-\lambda\frac{\varphi_0^n}{\ell^{2n}M_{\rm Pl}^n}\right)\right]$$

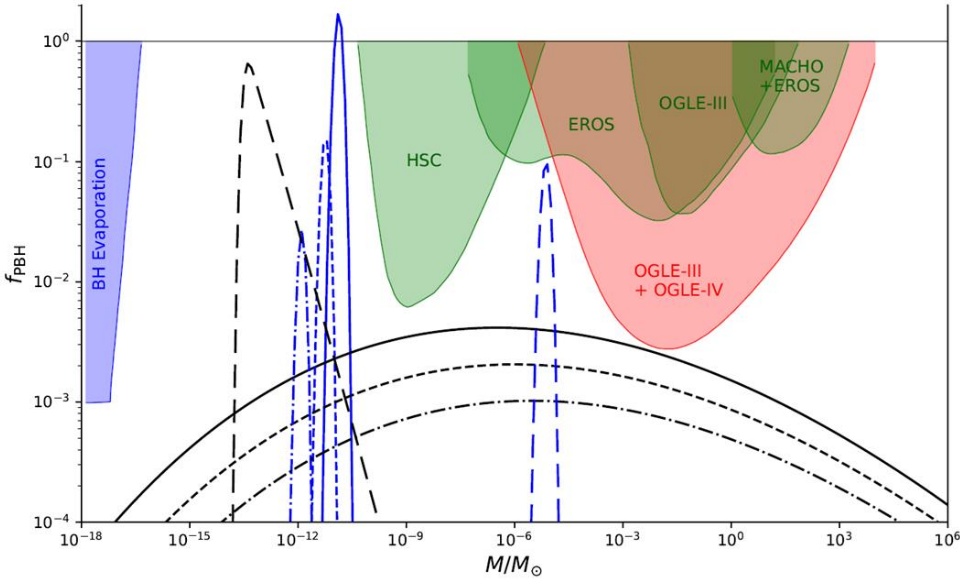


# Primordial Black Hole formation – DBI Tachyon Inflation

- The curvature power spectrum is obtained by numerically solving equation

$$\frac{d^2\zeta_q}{dN^2}+\left(3+\varepsilon_2-\varepsilon_1-2\frac{c_s'}{c_s}\right)\frac{d\zeta_q}{dN}+\frac{c_s^2q^2}{a^2H^2}\zeta_q=0 \qquad \mathcal{P}_S=c_0\frac{q^3}{2\pi^2}|\zeta_q(N_q)|^2$$

- The fraction of PBH dark matter versus the PBH mass




Black lines – the Tachyon model

Blue lines – the PLLS model

# Beyond

- Our understanding of quantum aspects of DBI dynamics is still pure.
- Perturbative solutions for classical particles analogous to the tachyons offer many possibilities in quantum mechanics, quantum and string field theory and cosmology on archimedean and nonarchimedean spaces.
- Reverse Engineering Method-REM remains a valuable auxiliary tool for investigation on tachyonic–universe evolution for nontrivial models.
- It was shown that the theory of *p*-adic inflation can be compatible with CMB observations. Quantization of tachyons would allow us to consider even more realistic inflationary models including quantum fluctuations and to test their adelic and “real-effective” spectra
- Despite quantization of *p*-adic DBI systems for $p = 1(mod\ 4)$ has been done, their “full” adelization remains as a real challenge!

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*Thank you!*

*Хвала Бранко!*