Nonlocal de Sitter $\sqrt{dS}$ gravity model and its local description

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Nonlocal Modification of GR

Nonlocal square root gravity model

$$S = \frac{1}{16\pi G} \int_M \sqrt{R - 2\Lambda}\, F(\Box)\, \sqrt{R - 2\Lambda}\, \sqrt{-g}\, d^4x,$$

where $F(\Box) = 1 + \mathcal{F}(\Box) = 1 + \sum_{n=1}^{\infty} f_n \Box^n + \sum_{n=1}^{\infty} f_{-n} \Box^{-n}$.

- Construction

$$\begin{aligned} R - 2\Lambda = \sqrt{R - 2\Lambda}\,\sqrt{R - 2\Lambda} &\to \sqrt{R - 2\Lambda}\, F(\Box)\, \sqrt{R - 2\Lambda} \\ &= R - 2\Lambda + \sqrt{R - 2\Lambda}\, \mathcal{F}(\Box)\, \sqrt{R - 2\Lambda} \end{aligned}$$

Equations of motion

Equation of motion are

$$-\frac{1}{2}g_{\mu\nu}\sqrt{R-2\Lambda}\mathcal{F}(\Box)\sqrt{R-2\Lambda}+R_{\mu\nu}W-K_{\mu\nu}W+\frac{1}{2}\Omega_{\mu\nu}=-(G_{\mu\nu}+\Lambda g_{\mu\nu})$$

$$\Omega_{\mu\nu}=\sum_{n=1}^{\infty}f_n\sum_{l=0}^{n-1}S_{\mu\nu}\left(\Box^l\sqrt{R-2\Lambda},\Box^{n-1-l}\sqrt{R-2\Lambda}\right)$$

$$-\sum_{n=1}^{\infty}f_{-n}\sum_{l=0}^{n-1}S_{\mu\nu}\left(\Box^{-(l+1)}\sqrt{R-2\Lambda},\Box^{-(n-l)}\sqrt{R-2\Lambda}\right),$$

$$K_{\mu\nu}=\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box,$$

$$S_{\mu\nu}(A,B)=g_{\mu\nu}\nabla^\alpha A\nabla_\alpha B-2\nabla_\mu A\nabla_\nu B+g_{\mu\nu}A\Box B,$$

$$W=\frac{1}{\sqrt{R-2\Lambda}}\mathcal{F}(\Box)\sqrt{R-2\Lambda}.$$

Eigenvalue problem

If we assume $\Box\sqrt{R-2\Lambda} = q\sqrt{R-2\Lambda}$, EOM are simplified to

$$W = \mathcal{F}(p)$$
$$\Omega_{\mu\nu} = \mathcal{F}'(q) S_{\mu\nu}(\sqrt{R-2\Lambda}, \sqrt{R-2\Lambda}),$$

$$(G_{\mu\nu} + \Lambda g_{\mu\nu})(1+\mathcal{F}(q)) + \frac{1}{2}\mathcal{F}'(q) S_{\mu\nu}(\sqrt{R-2\Lambda}, \sqrt{R-2\Lambda}) = 0.$$

It is evident that EOM are satisfied if $\mathcal{F}(q) = -1$ and $\mathcal{F}'(q) = 0$.

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As an example we can take the following simple form of a function $\mathcal{F}(\Box)$:

$$\mathcal{F}(\Box) = a\frac{\Box}{q}e^{1-\frac{\Box}{q}} + (1-a)\frac{q}{\Box}e^{1-\frac{q}{\Box}}, \qquad a \in \mathbb{R}.$$

Cosmological solutions

There are eleven known solutions of the EOM

$a(t)$	k	q
$At^{2/3}e^{\frac{\Lambda t^2}{14}}$	0	$-\frac{3}{7}\Lambda$
$Ae^{\frac{\Lambda t^2}{6}}$	0	$-\Lambda$
$A\cosh^{\frac{2}{3}}\left(\sqrt{\frac{3}{8}\Lambda}\, t\right)$	0	$\frac{3}{8}\Lambda$
$A\sinh^{\frac{2}{3}}\left(\sqrt{\frac{3}{8}\Lambda}\, t\right)$	0	$\frac{3}{8}\Lambda$
$A\sqrt[3]{1+\sin\left(\sqrt{-\frac{3}{2}\Lambda}\, t\right)}$	0	$\frac{3}{8}\Lambda$
$A\sqrt[3]{1-\sin\left(\sqrt{-\frac{3}{2}\Lambda}\, t\right)}$	0	$\frac{3}{8}\Lambda$
$A\sin^{\frac{2}{3}}\left(\sqrt{-\frac{3}{8}\Lambda}\, t\right)$	0	$\frac{3}{8}\Lambda$
$A\cos^{\frac{2}{3}}\left(\sqrt{-\frac{3}{8}\Lambda}\, t\right)$	0	$\frac{3}{8}\Lambda$
$Ae^{\sqrt{\frac{\Lambda}{6}}\, t}$	± 1	$\frac{1}{3}\Lambda$
$A\cosh^{\frac{1}{2}}\left(\sqrt{\frac{2}{3}\Lambda t}\right)$	± 1	$\frac{1}{3}\Lambda$
$A\sinh^{\frac{1}{2}}\left(\sqrt{\frac{2}{3}\Lambda t}\right)$	± 1	$\frac{1}{3}\Lambda$

Effective energy density and pressure

The EOM can be rewritten in the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu}.$$

Corresponding effective density and pressure are

- $\rho = \frac{3a'(t)^2}{8\pi G a(t)^2} - \frac{\Lambda}{8\pi G}$,
- $p = -\frac{a''(t)}{4\pi G a(t)} - \frac{a'(t)^2}{8\pi G a(t)^2} + \frac{\Lambda}{8\pi G}$.

Effective pressure and densities $k = 0$

$a(t)$	ρ	p
$At^{2/3}e^{\frac{\Lambda t^2}{14}}$	$\frac{9\Lambda t^2\left(\Lambda t^2-7\right)+196}{1176\pi G t^2}$	$\frac{\Lambda\left(7-3\Lambda t^2\right)}{392\pi G}$
$Ae^{\frac{\Lambda t^2}{6}}$	$\frac{\Lambda\left(\Lambda t^2-3\right)}{24\pi G}$	$\frac{\Lambda-\Lambda^2 t^2}{24\pi G}$
$A\cosh^{\frac{2}{3}}\left(\sqrt{\frac{3}{8}\Lambda}\ t\right)$	$\frac{\Lambda\left(\tanh^2\left(\sqrt{\frac{3}{8}\Lambda}t\right)-2\right)}{16\pi G}$	$\frac{\Lambda}{16\pi G}$
$A\sinh^{\frac{2}{3}}\left(\sqrt{\frac{3}{8}\Lambda}\ t\right)$	$\frac{\Lambda\left(\coth^2\left(\sqrt{\frac{3}{8}\Lambda}t\right)-2\right)}{16\pi G}$	$\frac{\Lambda}{16\pi G}$
$A\sqrt[3]{1+\sin\left(\sqrt{-\frac{3}{2}\Lambda}\ t\right)}$	$\frac{\Lambda\left(-\frac{\cos^2\left(\sqrt{-\frac{3}{2}\Lambda}t\right)}{\left(\sin\left(\sqrt{-\frac{3}{2}\Lambda}t\right)+1\right)^2}-2\right)}{16\pi G}$	$\frac{\Lambda}{16\pi G}$
$A\sqrt[3]{1-\sin\left(\sqrt{-\frac{3}{2}\Lambda}\ t\right)}$	$\frac{\Lambda\left(-\frac{\cos^2\left(\sqrt{-\frac{3}{2}\Lambda}t\right)}{\left(\sin\left(\sqrt{-\frac{3}{2}\Lambda}t\right)-1\right)^2}-2\right)}{16\pi G}$	$\frac{\Lambda}{16\pi G}$
$A\sin^{\frac{2}{3}}\left(\sqrt{-\frac{3}{8}\Lambda}\ t\right)$	$\frac{\Lambda\left(-\cot^2\left(\sqrt{-\frac{3}{8}\Lambda}t\right)-2\right)}{16\pi G}$	$\frac{\Lambda}{16\pi G}$
$A\cos^{\frac{2}{3}}\left(\sqrt{-\frac{3}{8}\Lambda}\ t\right)$	$\frac{\Lambda\left(-\tan^2\left(\sqrt{-\frac{3}{8}\Lambda}t\right)-2\right)}{16\pi G}$	$\frac{\Lambda}{16\pi G}$

Effective pressure and densities $k = \pm 1$

$a(t)$	ρ	p
$Ae^{\sqrt{\frac{\Lambda}{6}}\,t}$	$-\frac{\Lambda A^2 - 6ke^{-\sqrt{\frac{2}{3}\Lambda}t}}{16\pi GA^2}$	$\frac{\Lambda A^2 - 2ke^{-\sqrt{\frac{2}{3}\Lambda}t}}{16\pi GA^2}$
$A\cosh^{\frac{1}{2}}\left(\sqrt{\frac{2}{3}\Lambda}t\right)$	$\rho = \frac{3}{8\pi G}\left(\frac{k}{A^2\cosh\sqrt{\frac{2}{3}\Lambda}t} + \frac{\Lambda\left(\tanh^2\left(\sqrt{\frac{2}{3}\Lambda}t\right)-2\right)}{6}\right)$ $p = \frac{\frac{4}{3}A^2\Lambda\left(\tanh^2\left(\sqrt{\frac{2}{3}\Lambda}t\right)+2\right) - \frac{8k}{\cosh\sqrt{\frac{2}{3}\Lambda}t}}{64\pi A^2 G}$	
$A\sinh^{\frac{1}{2}}\left(\sqrt{\frac{2}{3}\Lambda}t\right)$	$\rho = \frac{3}{8\pi G}\left(\frac{k}{A^2\sinh\sqrt{\frac{2}{3}\Lambda}t} + \frac{\Lambda\left(\coth^2\left(\sqrt{\frac{2}{3}\Lambda}t\right)-2\right)}{6}\right)$ $p = \frac{\frac{4}{3}A^2\Lambda\left(\coth^2\left(\sqrt{\frac{2}{3}\Lambda}t\right)+2\right) - \frac{8k}{\sinh\sqrt{\frac{2}{3}\Lambda}t}}{64\pi A^2 G}$	

The standard action for the local scalar field φ in minimal coupling with gravity is

$$S = \frac{1}{16\pi G}\Big(\int \sqrt{-g}(R - 2\Lambda)\ d^4x + \frac{1}{8\pi G}\int \sqrt{-g}(-\frac{1}{2}\ \nabla_\mu\varphi\nabla^\mu\varphi - V(\varphi))d^4x\Big),$$

where φ is dimensionless scalar field.
Variation of action S with respect to metric $g^{\mu\nu}$ gives

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\nabla^\rho\varphi\nabla_\rho\varphi + g_{\mu\nu}V(\varphi) - \nabla_\mu\varphi\nabla_\nu\varphi = 0.$$

The corresponding equations of motion in the usual form are:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\ T_{\mu\nu}, \qquad \Box\varphi = V'(\varphi),$$

where $'$ denotes derivative with respect to φ.

Hence, we obtained the relations between effective energy density and effective pressure and scalar field φ and potential V.

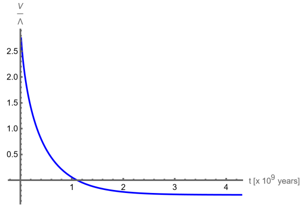
$$\begin{aligned} 8\pi G\rho &= \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \\ 8\pi G p &= \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \\ 8\pi G(\rho + p) &= \dot{\varphi}^2, \\ 4\pi G(\rho - p) &= V(\varphi). \end{aligned}$$

Taking into account $\rho(t)$ and $p(t)$ from before, we can calculate scalar field $\varphi(t)$ and potential $V(t)$ as functions of time t. In some cases, potential V can be explicitly expressed as a function of φ.

In particular for $a(t) = At^{2/3}e^{\frac{\Lambda t^2}{14}}$ we erarlier obtained

$$\rho = \frac{9\Lambda t^2\left(\Lambda t^2 - 7\right) + 196}{1176\pi G t^2}, \quad p = \frac{\Lambda\left(7 - 3\Lambda t^2\right)}{392\pi G}.$$

This gives the following values of φ and V.



$$\varphi(t) = \pm\frac{2}{\sqrt{3}}\left(\sqrt{1 - \frac{3}{14}\Lambda t^2} - \operatorname{arctanh}\sqrt{1 - \frac{3}{14}\Lambda t^2}\right) + C,$$

$$V(t) = -\frac{2\Lambda}{7} + \frac{3\Lambda^2 t^2}{49} + \frac{2}{3t^2}.$$

We cannot write the explicit dependence of the potential V on φ. If $\Lambda > 0$, time is restricted by $\Lambda t^2 \leq \frac{14}{3}$.

This table provides an overview of eight cases when the potential V can be explicitly expressed as function of the scalar field φ.

$a(t) = Ae^{\frac{\Lambda t^2}{6}}$	$k = 0$	$V(\varphi) = -\frac{1}{2}\Lambda(\varphi - C)^2 - \frac{2\Lambda}{3}$ $\varphi = \pm\sqrt{-\frac{2}{3}\Lambda}\, t + C$
$a(t) = A\cosh^{\frac{2}{3}}\left(\sqrt{\frac{3}{8}\Lambda}\, t\right)$	$k = 0$	$V(\varphi) = -\frac{1}{8}\Lambda\left(\cosh\left(\sqrt{3}(\varphi - C)\right) + 5\right)$ $\varphi = \pm\frac{4i}{\sqrt{3}}\arctan\left(\tanh\sqrt{\frac{3}{32}\Lambda}\, t\right) + C$
$a(t) = A\sinh^{\frac{2}{3}}\left(\sqrt{\frac{3}{8}\Lambda}\, t\right)$	$k = 0$	$V(\varphi) = -\frac{1}{8}\Lambda\left(-\cosh\left(\sqrt{3}(\varphi - C)\right) + 5\right)$ $\varphi = \pm\frac{2}{\sqrt{3}}\ln\left(\tanh\sqrt{\frac{3}{32}\Lambda}\, t\right) + C$
$a(t) = A\sqrt[3]{1 + \sin\left(\sqrt{-\frac{3}{2}\Lambda}\, t\right)}$	$k = 0$	$V(\varphi) = -\frac{1}{8}\Lambda\left(\cosh(\sqrt{3}(\varphi - C \mp \frac{\ln\left(3-2\sqrt{2}\right)}{\sqrt{3}})) + 5\right)$ $\varphi = \pm\frac{4}{\sqrt{3}}\arctan\frac{\tanh\sqrt{\frac{3}{32}\Lambda}\, t - 1}{\sqrt{2}} + C$
$a(t) = A\sqrt[3]{1 - \sin\left(\sqrt{-\frac{3}{2}\Lambda}\, t\right)}$	$k = 0$	$V(\varphi) = -\frac{1}{8}\Lambda\left(\cosh(\sqrt{3}(\varphi - C \mp \frac{\ln\left(3+2\sqrt{2}\right)}{\sqrt{3}})) + 5\right)$ $\varphi = \pm\frac{4}{\sqrt{3}}\arctan\frac{\tanh\sqrt{\frac{3}{32}\Lambda}\, t + 1}{\sqrt{2}} + C$
$a(t) = A\sin^{\frac{2}{3}}\left(\sqrt{-\frac{3}{8}\Lambda}\, t\right)$	$k = 0$	$V(\varphi) = -\frac{1}{8}\Lambda\left(\cosh\left(\sqrt{3}(\varphi - C)\right) + 5\right)$ $\varphi = \pm\frac{2}{\sqrt{3}}\ln\left(\tanh\sqrt{\frac{3}{32}\Lambda}\, t\right) + C$
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$a(t) = Ae^{\pm\frac{\sqrt{\Lambda}\, t}{\sqrt{6}}}$	$k = \pm 1$	$V(\varphi) = \frac{1}{6}\Lambda\left((\varphi - C)^2 - 3\right)$ $\varphi = \pm\sqrt{\frac{12k}{\Lambda A^2}e^{\mp\sqrt{\frac{2}{3}\Lambda}\, t}} + C$

Schwarzschild-de Sitter-type metric

We want to investigate our model outside the spherically symmetric massive body. Since this model is a nonlocal generalization of general relativity with the cosmological constant Λ, it is natural to consider a generalization of the Schwarzschild-de Sitter metric starting from the standard Schwarzschild expression

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2.$$

The corresponding scalar curvature R of above metric is

$$R = \frac{2}{r^2} - \frac{1}{B(r)}\Big(\frac{A''(r)}{A(r)} - \frac{A'(r)B'(r)}{2A(r)B(r)} - \frac{A'(r)^2}{2A(r)^2} + \frac{2A'(r)}{rA(r)} - \frac{2B'(r)}{rB(r)} + \frac{2}{r^2}\Big).$$

As we concluded earlier, to find a solution of EoM it is sufficient to solve an eigenvalue problem $\Box\sqrt{R-2\Lambda} = q\sqrt{R-2\Lambda}$. Note that here d'Alembertian $\Box$ acts in the following way:

$$\Box u(r) = \frac{1}{B(r)}\left(\triangle u(r) + \frac{1}{2}\frac{d}{dr}\left(\ln\frac{A(r)}{B(r)}\right)u'(r)\right),$$

where $u(r)$ is any differentiable scalar function and $\triangle$ is a Laplacian in spherical coordinates.

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where $u(r)$ is any differentiable scalar function and $\triangle$ is a Laplacian in spherical coordinates. Thus we ought to solve the equation

$$\Box u(r) = \frac{1}{B(r)}\left(\triangle u(r) + \frac{1}{2}\frac{d}{dr}\left(\ln\frac{A(r)}{B(r)}\right)u'(r)\right) = qu(r).$$

Local case

Recall, that in the local case, around the spherically symmetric body of mass M, we have

$$A(r) = A_0(r) = 1 - \frac{\mu}{r} - \frac{1}{3}\Lambda r^2,$$
$$B(r) = B_0(r) = \frac{1}{A_0(r)},$$

where $\mu = \frac{2GM}{c^2}$ is Schwarzschild radius. Hence it is useful to take $A(r) = A_0(r) - \alpha(r)$ and $B(r) = \frac{1}{A_0(r) - \alpha(r)}$.

$$R = 4\Lambda + \frac{2\alpha}{r^2} + \frac{4\alpha'}{r} + \alpha'',$$
$$\Box u = (A_0 - \alpha)\triangle u + (A_0' - \alpha')u' = qu, \qquad u = \sqrt{R - 2\Lambda}.$$

Approximation

In the remainder of this presentation we limit ourselves to weak gravity field approximation, which means $A(r) \approx 1$, or in particular

$$\frac{\mu}{r} \ll 1, \quad \frac{\Lambda r^2}{3} \ll 1, \quad |\alpha(r)| \ll 1.$$

Moreover, we can replace $\Box$ with $\triangle$ and our eigenvalue problem is simplified to

$$\frac{d^2}{dr^2}\sqrt{R - 2\Lambda} + \frac{2}{r}\frac{d}{dr}\sqrt{R - 2\Lambda} = q\sqrt{R - 2\Lambda}.$$

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$$\frac{d^2}{dr^2}\sqrt{R - 2\Lambda} + \frac{2}{r}\frac{d}{dr}\sqrt{R - 2\Lambda} = q\sqrt{R - 2\Lambda}.$$

The next step is to expand $\sqrt{R - 2\Lambda}$ into power series for $|R| \ll 2\Lambda$, which gives

$$\sqrt{R - 2\Lambda} \approx \frac{-\mathrm{i}}{\sqrt{8\Lambda}}(R - 4\Lambda).$$

Finally, expressing everything in terms of $\alpha(r)$ one gets linear equation in the form

$$\alpha'''' + \frac{6}{r}\alpha''' + \frac{2}{r^2}\alpha'' - \frac{4}{r^3}\alpha' + \frac{4}{r^4}\alpha = q(\alpha'' + \frac{4}{r}\alpha' + \frac{2}{r^2}\alpha).$$

The general solution is

$$\alpha(r) = \frac{C_1}{r} + \frac{C_2}{r^2} + C_3 e^{-\sqrt{q}r}\left(\frac{1}{qr} + \frac{2}{q^{\frac{3}{2}}r^2}\right) + C_4 e^{\sqrt{q}r}\left(\frac{1}{qr} - \frac{2}{q^{\frac{3}{2}}r^2}\right),$$

where $q = \zeta\Lambda$.

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where $q = \zeta\Lambda$. The term $e^{\sqrt{q}r}$ diverges to infinity for large distances r and hence we take $C_4 = 0$.
The other requirement is that for $\zeta \to 0$ we get the local case, i.e. $A(r) \to A_0(r)$. The appropriate choice of integration constants is

$$C_1 = -\frac{\delta}{\sqrt{q}}, \quad C_2 = \frac{2\delta}{q}, \quad C_3 = -\delta\sqrt{q}, \quad C_4 = 0.$$

The final expression for $\alpha(r)$ and $A(r)$ are

$$\alpha(r) = -\frac{\delta}{\sqrt{q}r}\left(1 + e^{-\sqrt{q}r}\right) + \frac{2\delta}{qr^2}\left(1 - e^{-\sqrt{q}r}\right),$$

$$A(r) = 1 - \frac{\mu}{r} - \frac{\Lambda r^2}{3} + \frac{\delta}{\sqrt{q}r}\left(1 + e^{-\sqrt{q}\ r}\right) - \frac{2\delta}{qr^2}\left(1 - e^{-\sqrt{q}\ r}\right).$$

We have two dimensionless parameters δ and ζ. To determine values of δ and ζ we will use galaxy rotation curves of Milky way and spiral galaxy M33.

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We have two dimensionless parameters δ and ζ. To determine values of δ and ζ we will use galaxy rotation curves of Milky way and spiral galaxy M33. Gravitational potential and acceleration are given by

$$\Phi = \frac{1}{2}c^2(1 - A(r)), \quad a = -\Phi'(r).$$

The velocity of a circular orbit is then obtained by

$$v^2(r) = \frac{GM}{r} - \frac{\Lambda c^2 r^2}{3} + \frac{\delta c^2}{\sqrt{q}r}\left(\frac{2}{\sqrt{q}r} - \frac{1}{2}\right) + \delta c^2\left(\frac{1}{2} + \frac{1}{2\sqrt{q}r} - \frac{2}{qr^2}\right)e^{-\sqrt{q}r}$$

To obtain the values of the parameters δ and ζ we use the least squares fit for the galaxy rotation curves of Milky way and M33 spiral galaxy.

Milky way

r [kpc]	v [km/s]	Δv [km/s]
9.5	221.75	3.17
10.5	223.32	3.02
11.5	220.72	3.47
12.5	222.92	3.19
13.5	224.16	3.48
⋮	⋮	⋮
22.5	197.00	3.81
23.5	191.62	12.95
24.5	187.12	8.06
25.5	181.44	19.58
26.5	175.68	24.68

Milky way

r [kpc]	v [km/s]	Δv [km/s]	$\bar{v}$ [km/s]	**relative error [%]**
9.5	221.75	3.17	217.36	1.98
10.5	223.32	3.02	220.19	1.40
11.5	220.72	3.47	221.93	0.55
12.5	222.92	3.19	222.72	0.09
13.5	224.16	3.48	222.66	0.67
⋮	⋮	⋮	⋮	⋮
22.5	197.00	3.81	195.42	0.80
23.5	191.62	12.95	190.17	0.75
24.5	187.12	8.06	184.57	1.36
25.5	181.44	19.58	178.62	1.55
26.5	175.68	24.68	172.32	1.91

Milky way

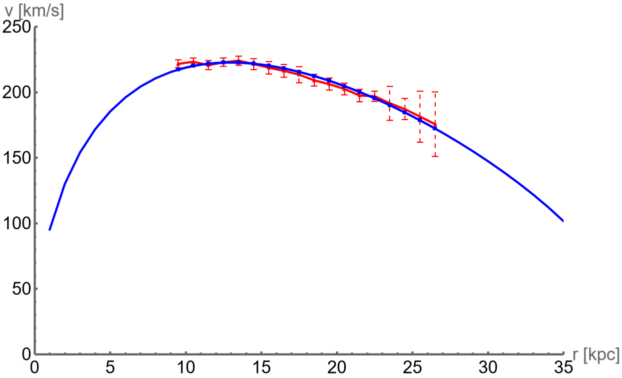

Figure: Rotation curve for the Milky Way galaxy. Red points are observational values and blue line is computed $v(r)$ by our model, where $\delta = 1.9 \cdot 10^{-5}$, $\zeta = 4.40 \cdot 10^{10}$.

M33

r [kpc]	v [km/s]	Δv [km/s]
0.5	42.0	2.4
1.0	58.8	1.5
1.5	69.4	0.4
⋮	⋮	⋮
9.8	117.2	2.5
10.3	116.5	6.5
10.8	115.7	8.1
11.2	117.4	8.2
⋮	⋮	⋮
22.5	101.2	27.4
23.0	123.5	39.1
23.5	115.3	26.7

M33

r $[kpc]$	v $[km/s]$	Δv $[km/s]$	$\bar{v}$ $[km/s]$	**relative error [%]**
0.5	42.0	2.4	35.62	15.18
1.0	58.8	1.5	49.61	15.63
1.5	69.4	0.4	59.83	13.79
⋮	⋮	⋮	⋮	⋮
9.8	117.2	2.5	117.27	0.06
10.3	116.5	6.5	118.24	1.49
10.8	115.7	8.1	119.07	2.91
11.2	117.4	8.2	119.63	1.90
⋮	⋮	⋮	⋮	⋮
22.5	101.2	27.4	111.88	10.56
23.0	123.5	39.1	110.81	10.27
23.5	115.3	26.7	109.69	4.86

M33

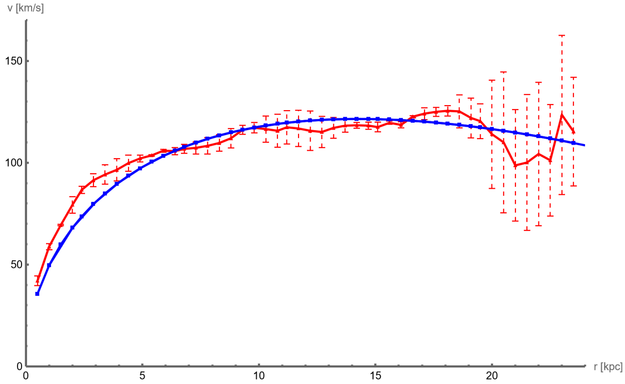

Figure: Rotation curve for spiral galaxy M33. Red points are observational values and blue line is computed our model, where $\delta = 5.7 \cdot 10^{-6}$, $\zeta = 3.62 \cdot 10^{10}$

Some relevant references

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THANK YOU FOR YOUR ATTENTION!