Ensemble averaging, singularities and LLM geometries

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Black holes & chaos

- Fast scrambling (Susskind) $\Rightarrow$ chaos bound (Maldacena, Shenker & Stanford)
- Black hole horizons imply strong chaos in dual holographic quantum field theory (SYK model, Yang-Mills plasmas etc)
- Near-horizon metric (AdS_2 throat) and its $\mathrm{SL}(2,\mathbb{R})$ isometry imply fast scrambling and maximum Lyapunov exponent $2\pi T$
- The same symmetry arguments lead to *integrable* geodesics in the bulk (black hole geometry)
- Qualitatively: integrable geodesics (AdS) $\leftrightarrow$ maximum chaos (CFT)

Black holes & microstates

- Black hole must be a quantum many-body system in the Hilbert space of quantum gravity
- In string theory: black hole solutions – horizonful vs microstate solutions – horizonless but look like a black hole from far away
- Microstate program: Bena, Warner et al – impressive wealth of solutions but problems persist (no holography etc)

Black holes & averaging

- Big question from the replica wormhole solution of the black hole information paradox: are black holes ensemble-averaged solutions?
- JT gravity (Saad, Shenker, Stanford, Iliesiu) and AdS_3 gravity (Belin, Perlmutter): ensemble average over *theories*
- In higher dimension: unlikely, but perhaps ensemble average over *solutions* or *states*
- Exciting!

Black holes & averaging

- Big question from the replica wormhole solution of the black hole information paradox: are black holes ensemble-averaged solutions?
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- Does it explain black hole thermodynamics?
- Does it relate smooth microstate solutions to black holes?

Idea: check this picture on an explicit example

- LLM solutions provide a controlled top-down stringy system where one can perform averaging and check if the result is black-hole-like
- Study the dynamics and statistics of geodesics
- Do we obtain black-holish behavior after averaging?

1. LLM solutions

2. Geodesic chaos in black & white geometries

3. Weak geodesic chaos and averaging in grayscale geometries

1 LLM solutions

LLM solution

- Lin, Lunin & Maldacena 2004: Giant gravitons wrapped around D-branes
- Very simple dual matrix model: free fermion in 2D with a constraint, solution specified by the Fermi surface
- Metric:

$$
\begin{aligned}
ds^2 &= \frac{1}{h^2}\Big[-(dt + V_a dx^a)^2 + h^4 \left(d\xi^2 + dx_a dx^a\right) + \\
&\quad + \left(\frac{1}{2} - z\right) d\tilde{\Omega}_3^2 + \left(\frac{1}{2} + z\right) d\Omega_3^2 \Big]
\end{aligned}
$$

- Two 3-spheres (Ω_3 and $\tilde{\Omega}_3$) times the static (t, x_1, x_2, ξ) manifold: $\mathrm{SO}(4) \times \mathrm{SO}(4) \times \mathbb{R}$

$$
h^2 = \frac{1}{\xi}\sqrt{\frac{1}{4} - z^2}, \quad \partial_a^2 z + \xi \partial_\xi \left(\frac{\partial_\xi z}{\xi}\right) = 0, \quad \partial_\xi V_a = \frac{\epsilon_{ab}\partial_b z}{\xi}
$$

Black & white patterns and bubbling AdS

- Finite curvature requires $z = +1/2$ ("black") or $z = -1/2$ ("white") in the LLM plane $\xi = 0$: dual to particles/holes in 2D Fermi liquid (Berenstein 2004)
- Geometry of black & white patterns:
  - Black disk – AdS
  - Multi-disk patterns – bubbling AdS
  - Black half-plane – pp-wave limit
  - Small deformations (rings, droplets etc) – small fluctuations
- A thing ring is roughly a giant graviton excitation

Grayscale solutions

- $-1/2 < z < 1/2 \Rightarrow$ naked singularity – but a "good" singularity a la Gubser (can be enclosed by a horizon)
- Matrix-wise: coarse-grained Young tableaux – smoothen the edges
- Natural arena for averaging: we expect to get grayscale physics by averaging over small deformations of black & white solutions

2 Geodesic chaos in black & white geometries

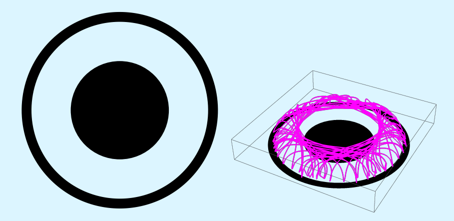
Equations of motion

- Geodesic Hamiltonian:

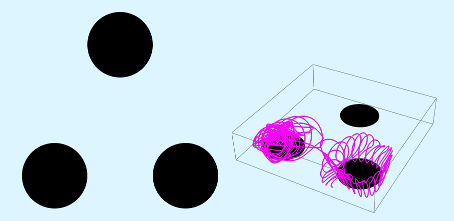
$$\mathcal{H} = \frac{1}{2h^2}\left[P_\xi^2 + (P_x + EV_x)^2 + (P_y + EV_y)^2 - h^4\left(E^2 - \frac{2L^2}{1-2z} - \frac{2\tilde{L}^2}{1+2z}\right)\right]$$

- Integrals of motion: E on (ξ, x_1, x_2) and the full set of angular momenta on 3-spheres
- Two representative configurations: disk+ring and 3-disk
- Both are nonintegrable but disk+ring has P_ϕ as an extra integral of motion $\Rightarrow$ 2 degrees of freedom instead of 2 and a half

Backgrounds and geodesics



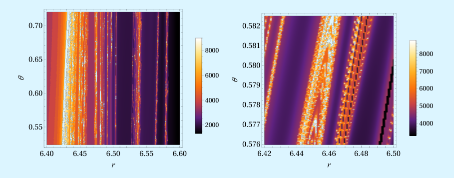
Disk+ring background and geodesic



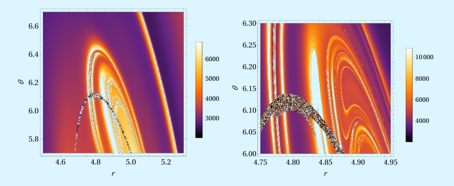
3 disks background and geodesic

- Two representative configurations: disk+ring and 3-disk
- Both are nonintegrable but disk+ring has P_ϕ as an extra integral of motion $\Rightarrow$ 2 degrees of freedom instead of 2 and a half

Chaos in disk+ring case



Disk+ring: typical mixed phase-space with remnants of KAM tori and the chaotic sea

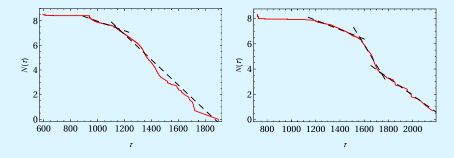


3 disks: KAM tori still present but do not present a barrier (3 degrees of freedom)

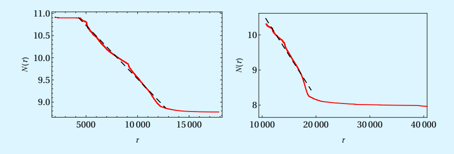
- Sticky trajectories provide trapping and mimic the black hole behavior
- Measure of sticky trajectories μ defines the surface gravity and effective temperature as

$$2\pi T_{\rm eff} = \mu$$

Escape rates and the fractal structures



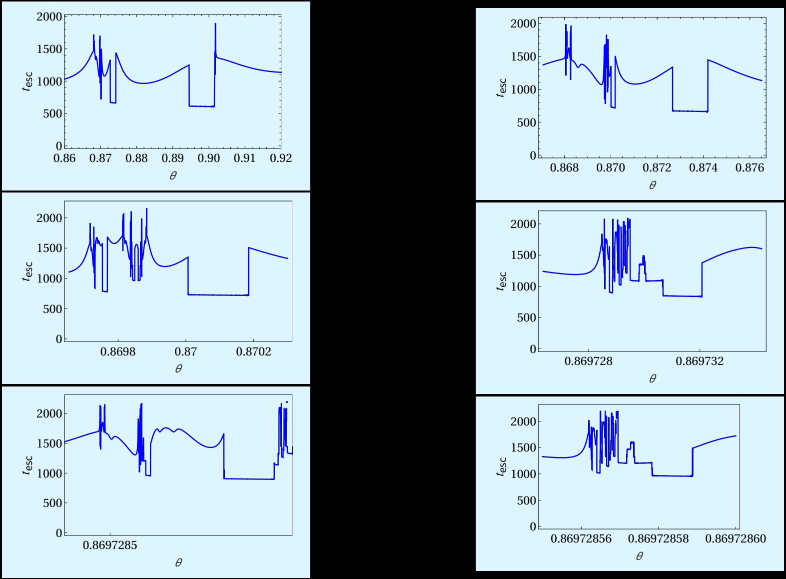
Disk+ring: several populations with different escape rates $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, plus sticky trajectories with very slow (subexponential) escape.



3 disks: uniform escape rate γ. Sticky trajectories are still present but do not divide the phase space into disjoint populations.

- Expect multifractal scaling for disk+ring
- The difference will be important later for averaging

Escape rates and the fractal structures



Multifractal spectrum for disk+ring with 4 exponents

Photon ring?

- For a black hole: the only unstable periodic orbit at $r = r_*$, positive Lyapunov exponent but no chaos (no skeleton of unstable periodic orbits)
- Cardoso et al: Lyapunov exponent on the photon ring determines the real part of the quasi-normal mode spectrum for $n \gg 1$
- Here: chaotic dynamics, infinite skeleton of unstable periodic orbits $\Rightarrow$ photon ring has no special significance
- For simplicity consider only in-plane dynamics: $\xi = P_\xi = 0$
- WKB approximation:

$$\frac{Q(E_*, r_*)}{\sqrt{2Q''(r_*, E_*)}} = -\imath\left(n + \frac{1}{2}\right), \quad n \in \mathbb{N}$$

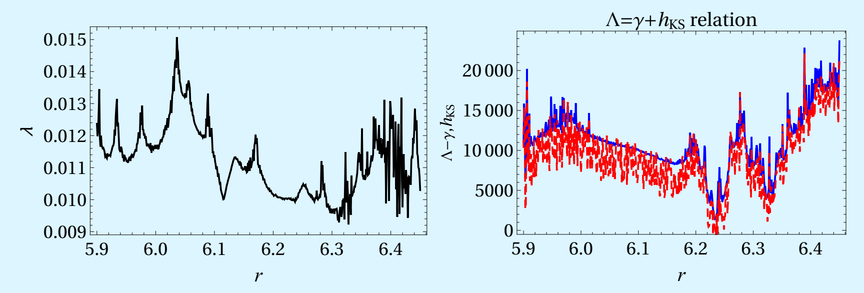
$$E_n = -(2n+1)\frac{\sqrt{Q''(r_*, E_*)}}{\partial_E Q(r_*, E_*)\sqrt{2}}$$

- When the dust settles: $E_n = E_* - \imath(2n+1)\lambda$ – the Cardoso relation

Photon ring insignificant for the Lyapunov spectrum

Lyapunov exponent (left) and the Pesin relation for the sum of positive Lyapunov exponents Λ, Kolmogorov-Sinai entropy h_{KS} and escape rate γ (right):

$$\Lambda \equiv \sum_{\lambda_i > 0} \lambda_i = h_{\mathrm{KS}} + \gamma$$



- At the photon ring we have $\lambda_* \approx 0.001$ – much less than the typical exponent
- Cardoso relation remains but it does not influence dynamics and presumably observable quantities

Horizonless microstates do not mimic black holes

- Chaos in geodesic motion leads to (observable) differences with respect to black holes
- Trapping is reproduced but but photon rings no
- The bottom line: microstates do not look like black holes in terms of geodesics

3 Weak geodesic chaos in black & white geometries

Grayscale geodesics

- Same Hamiltonian $\Rightarrow$ still nonintegrable. But different z function leads to a potential well which is never present in black & white:

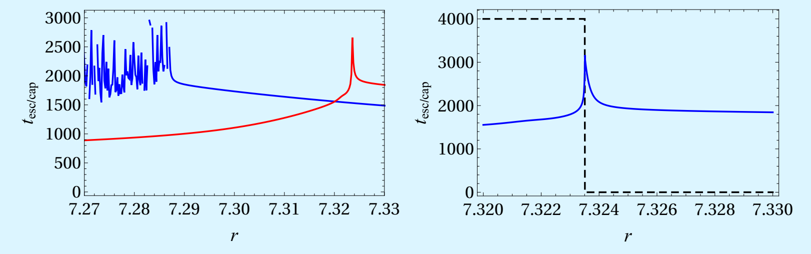
$$V_{\mathrm{eff;BW}}(\xi) = \frac{J_-^2\Theta(\rho - R_i) + J_+^2\Theta(R_i - \rho)}{\xi^2} \geq 0$$

$$V_{\mathrm{eff;gray}}(\xi) = \frac{-(\frac{E}{2})^2\left(1 - g^2\right) + \frac{J_-^2 + J_+^2}{2}\frac{g}{2}\left(J_-^2 - J_+^2\right)\mathrm{sgn}(\rho - R_i)}{\xi^2}$$

- Now both escapes and captures by the singularity are possible
- Despite nonintegrability the dynamics are now much simpler $\Rightarrow \sum_{\lambda_i > 0} \lambda_i \approx \gamma + \gamma_s$, h_{KS} almost zero

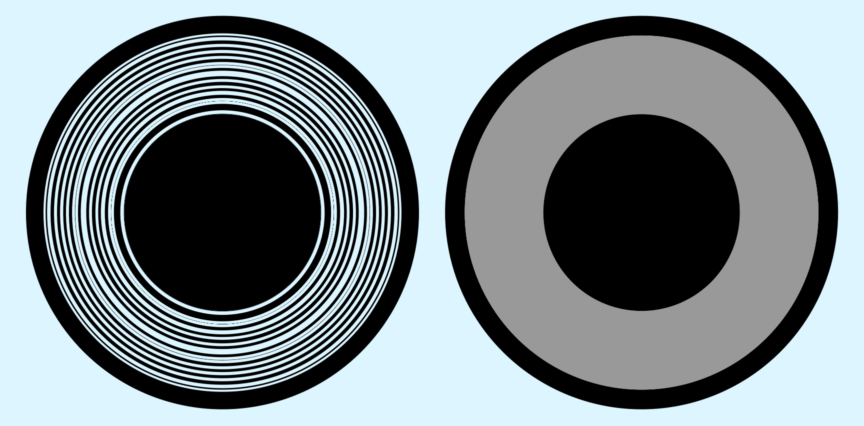
Grayscale escape rates

- Now both escapes and captures by the singularity are possible
- Left: escape/capture rate for black & white (blue) vs gray (red)
- Right: the photon ring separates captures ($r < r_*$) and escapes ($r > r_*$)



- Despite nonintegrability the grayscale dynamics are much simpler $\Rightarrow \sum_{\lambda_i>0} \lambda_i \approx \gamma + \gamma_s$, h_{KS} almost zero
- Smooth escape rate dependence, no fractal structure
- The photon ring is again observable and crucial: separates captures from escapes
- More black-holish than black & white

Averaged black & white geodesics vs. gray geodesics

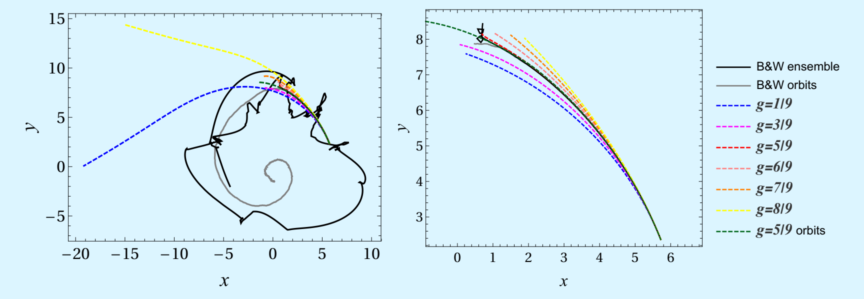


Disk + multiring + ring vs. disk + gray area + ring

- Gray background = average of black & white backgrounds with appropriate averaged (gray) flux g
- Does the same hold for geodesics?
- Idea: generate an ensemble of disk + multiring + ring backgrounds, compute geodesics, average them over the ensemble

Averaged black & white geodesics vs. gray geodesics

- For small timescales ($t < t_s$): perfect agreement $x^\mu_{\rm gray}(t) = \langle x^\mu_{\rm BW}(t)\rangle$
- Averaging over orbits is roughly equivalent: $x^\mu_{\rm gray}(t) = \bar{x}^\mu_{\rm BW}(t)$, also for $t < t_s$
- For longer timescales: no notion of averaging
- Same story for averages over Euclidean wormholes: self-averaging for $t < 1/2\pi T$



Averaged black & white potential vs. gray potential

- Gaussian ensemble of disk + multiring + ring solutions
- Partition function for $2N+1$ disks:

$$Z = \frac{\pi^{N-1}}{(M_+ M_-)^{\frac{N-1}{2}}} \prod_{j=2}^{2N-1} e^{\frac{R_{j;0}^2}{4\sigma^2 M_{(-1)^j}}} \left[1 - \mathrm{Erf}\left(\frac{R_{j;0}}{2\sigma\sqrt{M_{(-1)^j}}}\right)\right]$$

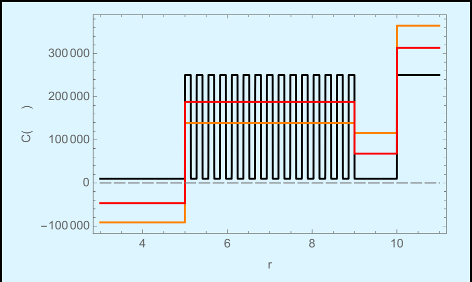
- Averaged effective potential:

$$\langle V_{\mathrm{eff}}(\xi)\rangle = \frac{1}{2\xi^2}\int d\lambda e^{-\imath\lambda\Sigma_-} \sum_{j=2}^{2N-1} \frac{A_j}{Z_{1;(j)}}$$

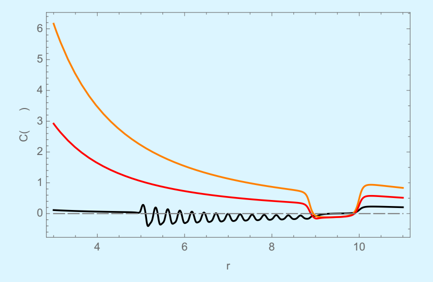
$$\frac{A_j}{Z_{1;(j)}} = \frac{\left(J_-^2 + J_+^2\right)\mathrm{Erf}\left(x_j + \rho\sqrt{M_{(-1)^j}}\right) + J_+^2 - J_-^2\mathrm{Erf}(x_j)}{1 - \mathrm{Erf}(x_j)}$$

$$x_j \equiv \frac{R_{j;0}}{2\sigma\sqrt{M_{(-1)^j}}}$$

Averaged black & white potential vs. gray potential



Effective potential (disk+rings): black – microscopic; red – average over ensembles of disk patterns; orange – grayscale



Same as left for the 3-disk configuration
No AdS asymptotics, bubbles with flat asymptotics

- Averaged potential has a potential well of depth $\langle \min V_{\text{eff;BW}} \rangle = -E_s$; always $E_s < E_g = \min V_{\text{eff;gray}}$
- Self-averaging epoch: $t_a \sim \hbar/E_s \sim N^2/E_s$

Conclusions

- Black & white LLM microstates vs gray LLM states vs black holes: no horizon and geodesic chaos vs naked singularity and weak chaos vs horizon and integrable geodesics
- To Do: Check that dual CFT (matrix model) exhibits no fast scrambling

- Geodesic dynamics and trapping are self-averaging quantities at short enough time scales (shown analytically for V_{eff}!)
- To Do: Lessons from JT suggest that averaging is a consequence of UV incompleteness (inserting UV branes eliminates the need for averaging). Is it true also here?