# Towards $p$-adic boundary value problems

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# Outline

1. $p$-adic elliptic divergence operators
2. Ultrametric Manifolds

# 0. Desideratum

Would like to imitate partial derivatives from real analysis,

$\rightarrow$ but this time on $p$-adic domains

$\rightarrow$ and even ultrametric manifolds!

**Project Goal.** Boundary Value Problems on Ultrametric Analytic Manifolds. [*Ongoing habilitation project*]

# 1. *p*-Adic Elliptic Divergence Operators

Let $\mathbb{Q}_p$ be the field of $p$-adic numbers.

Let $F \subset \mathbb{Q}_p^d$ be a compact clopen subset.

$$\pi_i \colon \mathbb{Q}_p^d \to \mathbb{Q}_p \quad \text{projection on } i\text{-th coordinate}$$

Fix a finite disjoint covering $\mathcal{U}_i$ of $\pi_i(F)$:

$$\pi_i(F) = \bigsqcup_{k_i=1}^{N_i} B_{i,k_i}\,,$$

where $B_{i,k_i} \subset \mathbb{Q}_p$ are $p$-adic disks.

Obtain a covering

$$F = \bigsqcup_{\underline{k} \in \underline{N}} B_{\underline{k}}$$

with polydisks

$$B_{\underline{k}} = \prod_{i=1}^{d} B_{i,k_i} \in \mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_d$$

# 1. $p$-adic elliptic operators

Haar measure $dx$ on $\mathbb{Q}_p^d$
[*corresponds to the Lebesgue measure on* $\mathbb{R}^d$]:

$$dx = dx_1 \wedge \cdots \wedge dx_d$$

with

$$\int_{\mathbb{Z}_p} dx_i = 1, \quad i = 1, \ldots, d$$

Write

$$\mu(A) = \int_A dx$$

Have

$$\mu(B_{\underline{k}}) = \prod_{i=1}^{d} \mu_i(B_{i,k}) = \prod_{i=1}^{d} \int_{B_{i,k}} dx_i = p^{-(k_1 + \cdots k_d)}$$

for some $(k_1, \ldots, k_d) \in \mathbb{Z}^d$.

## 1.1 Component Parisi-Zúñiga Operators

Let $i \in \{1, \ldots, d\}$.

$$\mathcal{L}_{i,\alpha_i} u(x) = \int_{\pi_i(F)} L_i(\xi_i, \eta_i)(u(\xi_1, \ldots, \xi_i, \ldots, \xi_d) - u(\xi_1, \ldots, \eta_i, \ldots, \xi_d))\, d\mu_i(\eta_i)$$

with $\alpha_i > 0$, $x = (\xi_1, \ldots, \xi_d) \in F$, and

$$L_i(\xi_i, \eta_i) = \begin{cases} |\xi_i - \eta_i|_p^{-\alpha_i}, & U_i(\xi_i) = U_i(\eta_i),\ \xi_i \neq \eta_i \\ w_i(U_i(\xi_i), U_i(\eta_i)), & U_i(\xi_i) \neq U_i(\eta_i), \end{cases}$$

where $U_i(\zeta_i) \in \mathcal{U}_i$ is unique with $\zeta_i \in U_i(\zeta_i)$ for $\zeta_i \in \pi_i(F)$, and

$$w_i(U_i(\xi_i), U_i(\eta_i)) \geq 0$$

is symmetric in $\mathcal{U}_i \times \mathcal{U}_i$ outside the diagonal.

# 1.1 Component Parisi-Zúñiga Operator

**Lemma.** It holds true that

$$\mathcal{L}_{i,\alpha_i} \circ \mathcal{L}_{j,\alpha_j} = \mathcal{L}_{j,\alpha_j} \circ \mathcal{L}_{i,\alpha_i}$$

for $i, j = 1, \ldots, d$.

This is a $p$-adic version of Schwarz's Theorem!

# 1.1 Component Parisi-Zúñiga Operator

Push-forward via $i$-th coordinate projection:

$$\pi_{i,*}\mathcal{L}_{i,\alpha_i}f(\xi_i) = \int_{\pi_i(F)} L_i(\xi_i, \eta_i)(f(\xi_i) - f(\eta_i))\, d\mu_i(\eta_i)$$

for $i = 1, \ldots, d$.

## 1.1 Component Parisi-Zúñiga Operators

*Kozyrev wavelets.* Let $a \in \mathbb{Q}_p$.

$$B_n(a) = \left\{ x \in \mathbb{Q}_p \mid |x-a|_p \le p^{-n} \right\}$$

$$\chi\colon \mathbb{Q}_p \to S^1 \quad \text{a unitary character}$$

like

$$\chi(x) := e^{2\pi i \{x\}_p},$$

where

$$\{x\}_p = \sum_{k=-n}^{-1} \alpha_k p^k \quad \text{for} \quad x = \sum_{k=-n}^{\infty} \alpha_k p^k \in \mathbb{Q}_p$$

Then with $j \in \{1, \dots, p-1\}$

$$\psi_{B_n(a),j}\colon \mathbb{Q}_p \to \mathbb{C},\ \xi \mapsto p^{\frac{n}{2}}\,\chi(p^{-(n+1)} j\xi)\,1_{B_n(a)}(\xi)$$

is a *Kozyrev wavelet.* [*corresponds to Haar wavelet over* $\mathbb{R}$]

# 1.1 Component Parisi-Zúñiga Operators

**Theorem** (PEB, ÁML, Kozyrev). The Hilbert space $L^2(\pi_i(F), \mu_i)$ has an orthonormal eigenbasis for $\pi_{i,*}\mathcal{L}_{i,\alpha_i}$ consisting of Kozyrev wavelets $\psi_{B_n(a),j}$, $j = 1, \ldots, p-1$, supported in $B_n(a) \subset \pi_i(F)$, plus associated graph eigenfunctions. The eigenvalue corresponding to $\psi = \psi_{B_n(a),j}$ is

$$\lambda_\psi = p^{n(1+\alpha_i)}(p^{-m(1+\alpha_i)} + 1) + \sum_{U_i(b) \neq U_i(a)} w_i(U_i(a), U_i(b))\mu_i(U_i(b)) - 1\,,$$

assuming that $U_i(a) = B_m(a) \supseteq B_n(a)$. The operator is self-adjoint, positive definite, the multiplicity of each eigenvalue is finite.

*The eigenvalue should remind us of Kozyrev's eigenvalue calculation for his integral operators!*

## 1.1 Component Parisi-Zúñiga Operators

For $d = 1$, the Zúñiga-Parisi Operators have been studied in recent work by Á.M. Ledezma and P.E.B.:

- In the context of local ultrametric approximations of graph Laplacian diffusion
- And their finite approximations
- Graph Laplacians appear also in the context of multi-topologies and diffusion on such
- Also non-autonomous $p$-adic diffusion on time-dependent graphs. . .

## 1.1 Component Parisi-Zúñiga Operator

**Theorem** (PEB, ÁML). There exists a probability measure $p_t(x,\cdot)$ with $t \geq 0$, $\xi \in \pi_i(F)$, on the Borel $\sigma$-algebra of $\pi_i(F)$ such that the Cauchy problem

$$\frac{\partial}{\partial t} u(\xi,t) + \pi_{i,*}\mathcal{L}_{i,\alpha_i} u(\xi,t) = 0$$
$$u(\xi,0) = u_0(\xi) \in C(\pi_i(F))$$

for $\alpha_i > 0$ has a unique solution in $C^1((0,\infty) \times C(F))$ of the form

$$u(\xi,t) = \int_{\pi_i(F)} L_i(\xi,\eta) p_t(x, d\eta), \quad \xi \in \pi_i(F)\,.$$

In addition, $p_t(x,\cdot)$ is the transition function of a strong Markov process whose paths are càdlàg [*i.e. a jump process*].

## 1.2 Boundary Conditions

Let $U \subseteq F$ open. The *i-th component outer boundary* of $U$ is

$$\delta_i^+ U = \{\eta_i \in \pi_i(F \setminus U) \mid \exists \xi_i \in \pi_i(U) \colon L_i(\xi_i, \eta_i) \neq 0\},$$

the *outer boundary* of $U$ w.r.t. $\mathcal{L} = (\mathcal{L}_{1,\alpha_1}, \ldots, \mathcal{L}_{d,\alpha_d})$ is

$$\delta^+ U = \bigsqcup_{i=1}^{d} (U \sqcup \delta_1^+ U) \times \cdots \times \delta_i^+ \times \cdots \times (U \sqcup \delta_d^+ U),$$

and

$$\mathrm{cl}_{\mathcal{L}} U = U \sqcup \delta^+ U$$

is the $\mathcal{L}$-closure of $U$ in $F$.

**Lemma.** The set $\mathrm{cl}_{\mathcal{L}}$ is clopen in $F$.

# 1.2 Boundary Conditions

The *i-th component inner boundary of $U$* is

$$\delta_i^- u = \{\xi_i \in \pi_i(U) \mid \exists \eta_i \in \pi_i(F \setminus U) \colon L_i(\xi_i, \eta_i) \neq 0\}$$

The inner boundary of $U$ w.r.t. $\mathcal{L} = (\mathcal{L}_{1,\alpha_1}, \dots, \mathcal{L}_{d,\alpha_d})$ is

$$\delta^- U = \bigsqcup_{i=1}^{d} U \times \cdots \times \delta_i^- \times \cdots \times U$$

**Lemma.** It holds true that

$$u|_{\delta_i^- U}(x) = 0 \quad \Leftrightarrow \quad u(x) \int_{\delta_i^+ U} L_i(\xi_i, \eta_i)\, d\eta_i = 0$$

for $x \in U$, $i = 1, \dots, d$.

# 1.3 Sobolev Spaces

Let $q > 0$, $k \in \mathbb{N}$. Define

$$W^{k,q}(U) = \left\{ f \in L^q(U) \mid \forall \underline{\ell} \in \mathbb{N}^d : \ |\underline{\ell}| \leq k \Rightarrow \left\| \mathcal{L}^{\underline{\ell}} f \right\|_{L^q(U)} < \infty \right\}$$

$$W_0^{k,q}(U) = \left\{ f \in W^{k,q}(U) \mid f|_{\delta^- U} = 0 \right\},$$

where

$$\mathcal{L}^{\underline{\ell}} = \mathcal{L}_{1,\alpha_1}^{\ell_1} \cdots \mathcal{L}_{d,\alpha_d}^{\ell_d}$$

with

$$\underline{\ell} = (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d$$

# 1.3 Sobolev Spaces

Norm on Sobolev space: $f \in W^{q,k}(U)$, then

$$\|f\|_{W^{q,k}(U)} = \left( \sum_{|\underline{\ell}|<k} \left\| \mathcal{L}^{\underline{\ell}} f \right\|_{L^q(U)} \right)^{\frac{1}{d}}$$

**Proposition.** The spaces $W^{q,k}(U)$ for $1 \le q < \infty$, $k \in \mathbb{N}$, are Banach spaces, and $W_0^{q,k}(U)$ is a closed subspace of $W^{q,k}(U)$. Furthermore, $W^{2,k}(U)$ is a Hilbert space for $k \in \mathbb{N}$.

Proof.

Imitate the classical case. □

## 1.3 Sobolev Spaces

**Proposition** (Poincaré Inequality). Let $u \in W^{1,2}(U)$. Then there exists some $C > 0$ such that

$$\|u\|_{L^2} \leq C \, \|\mathcal{L}_{i,\alpha_i} u\|_{L^2}$$

for $i = 1, \ldots, d$.

Proof.

Use eigendecomposition w.r.t. $\mathcal{L}_{i,\alpha_i}$: $u = \sum_{\psi} \alpha_\psi \psi$, where $\psi = \psi_1 \cdots \psi_d$ with $\psi_i$ eigenfunction of $\pi_{i,*}\mathcal{L}_{i,\alpha_i}$, and thus

$$\|u\|_{L^2}^2 = \sum_{\psi} |\alpha_\psi|^2 \leq C \sum_{\psi} \lambda_\psi^2 \, |\alpha_\psi|^2 = C \, \|\mathcal{L}_{i,\alpha_i} u\|_{L^2} \ ,$$

because $\lambda_\psi \to \infty$ for $\operatorname{supp} \psi \to \{pt\}$. □

# 1.4 Elliptic Divergence Operators

Let $A \subseteq \mathbb{Q}_p^d$.

$$\mathcal{D}(A) = \{f : A \to \mathbb{R} \mid f \text{ is locally constant with compact support}\}$$

Then define

$$\mathcal{D}_0(U) = \{f \in \mathcal{D}(U) \mid f|_{\delta^- U} = 0\}$$

Homogeneous second-order divergence operator on $\mathcal{D}(U)$:

$$P_2(\mathcal{L})u = \sum_{i,j=1}^{d} \mathcal{L}_{j,\alpha_j}\left(a^{ij}\mathcal{L}_{i,\alpha_i}u\right)$$

with $a^{ij} : F \to \mathbb{R}$ such that

$$a^{ij} = a^{ji}$$

for $i, j = 1, \ldots, d$.

## 1.4 Elliptic Divergence Operators

General second-order divergence operator:

$$P(\mathcal{L}) = P_2(\mathcal{L}) + P_1(\mathcal{L}) + P_0(\mathcal{L})$$

with

$$P_1(\mathcal{L})u = \sum_{i=1}^{d} b^i \mathcal{L}_{i,\alpha_i} u$$

$$P_0(\mathcal{L})u = cu$$

with $b^i, c \colon F \to \mathbb{R}$.

**Assumption.** It is assumed that

$$a^{i,j}, b^i, c \in L^\infty(U)$$

for $i, j = 1, \ldots, d$.

## 1.4 Elliptic Divergence Operators

**Definition.** The operator $P(\mathcal{L})$ is called *elliptic*, if the matrix

$$A = (a^{ij}(x)) \in \mathbb{R}^{d \times d}$$

is positive definite for almost all $x \in F$, and the smallest eigenvalue of $A$ is in this case always at least $\theta > 0$.

# 1.5 Poisson Equation

$u \in W_0^{1,2}(U)$ is a weak solution of the Poisson equation, if

$$\int_U (P(\mathcal{L}) + \mu)\, u(x)\phi(x)\, dx = \int_U f(x)\phi(x)\, dx$$

for all $\phi \in W_0^{1,2}(U)$.

**Theorem.** There is a number $\gamma \geq 0$ such for all $\mu \geq \gamma$ and every $f \in L^2(U)$, there exists a weak solution $u \in W_0^{1,2}(U)$ of the boundary value problem

$$\begin{cases} P(\mathcal{L})u(x) + \mu u(x) = f(x), \quad x \in U \\ u|_{\delta U} = 0 \end{cases}$$

for $U \subseteq F$ open.

Proof.

Prove energy estimates just like in the classical case. □

## 1.6 Spectrum

**Theorem.** Let $P(\mathcal{L})$ with $a^{i,j}, b^i, c \in \mathcal{D}(U)$ for $i,j = 1,\dots,d$ acting on $L^2(U)$ with $U \subseteq F$ open. Assume that the eigenspaces of $\pi_{1,*}\mathcal{L}_{1,\alpha_1} \otimes \cdots \otimes \pi_{d,*}\mathcal{L}_{d,\alpha_d}$ are invariant under the multiplication with $b^i$, $i = 1,\dots,d$, or that $P_1(\mathcal{L})$ is normal. Moreover, assume that

$$P_k(\mathcal{L})P_\ell(\mathcal{L}) = P_\ell(\mathcal{L})P_k(\mathcal{L})$$

for $k,\ell = 0,1,2$. Then $P(\mathcal{L})$ is unitarily diagonalisable, its spectrum is a point spectrum, and all eigenvalues have only finite multiplicity.

# 1.6 Spectrum

Sketch of proof.

Let $\phi' \in \mathcal{E}$, where

$$\mathcal{E} = \text{the product eigenbasis for the } \pi_{1,*}\mathcal{L}_{i,\alpha_1}, \ldots, \pi_{d,*}\mathcal{L}_{d,\alpha_s}$$

Then

$$P(\mathcal{L})\phi = \sum_{\phi' \in \mathcal{E}} \left\langle \phi \left[ \sum_{i=1}^{d} \left( \sum_{j=1}^{d} \lambda_{\phi,i} a^{ij} \lambda_{\phi',j} \right) + \lambda_{\phi,i} b^i + c \right], \phi' \right\rangle \phi'$$

Since the $a^{ij}, b^i, c$ are locally constant with compact support, there is a finite-dimensional subspace $V_\phi \subset L^2(U)$ invariant under $P(\mathcal{L})$ for each $\phi \in \mathcal{E}$. [*The sums here are just finite!*]

*[Not yet done]* □

# 1.6 Spectrum

Continuation of proof.

$P(\mathcal{L})$ acts on $V_\phi$ as

$$W_\phi = \sum_{i,j=1}^{d} C_{\phi,ij} + C_{\phi,i} + C_\phi\,,$$

where

$$C_{\phi,ij} = D_{\phi,i} A_{\phi,ij} D_{\phi,j}\,, \quad C_{\phi,i} = D_{\phi,i} B_{\phi,i}$$

with $D_{\phi,i}, D_{\phi,j}$ diagonal matrices, and $A_{\phi,ij}, B_{\phi,i}, C_\phi$ symmetric matrices representing multiplication with $a^{ij}, b^i, c$ on $V_\phi$.

- Diagonalisablility of $P(\mathcal{L})$ and orthogonality property of eigenbasis follows from assumptions.

- Finiteness of eigenvalues of $P(\mathcal{L})$ and point spectrum property follow from that of the eigenvalues of $\mathcal{L}_{i,\alpha_i}$. □

## 1.6 Spectrum

$P_1(\mathcal{L})$. The property

$$C_{\phi,i} = D_{\phi,i} B_{\phi,i}$$

with $B_{\phi,i}$ symmetric and $D_{\phi,i}$ diagonal matrix is the *detailed balance property*, and $D_{\phi,i}$ corresponds to a stationary distribution for $P_1(\mathcal{L})$.

$P_2(\mathcal{L})$. The property

$$C_{\phi,ij} = D_{\phi,i} A_{\phi,ij} D_{\phi,j}$$

is also a kind of detailed balance property for $P_2(\mathcal{L})$.

$P(\mathcal{L})$. Together, the operator $P(\mathcal{L})$ can be viewed as a balanced process.

# 1.6 Spectrum

**Corollary.** Under the hypothesis of the Theorem,

$$L_0^2(U) = \{u \in L^2(U) \mid u|_{\delta^- U} = 0\}$$

is invariant under $P(\mathcal{L})$, and this operator is also unitarily diagonalisable with point spectrum, and with eigenfunctions in $\mathcal{D}_0(U)$.

## 1.7 Heat Kernels and Green function

**Assumption.** It is assumed that $P(\mathcal{L})$ is elliptic, satisfies

$$a^{ij}, b^i, c \in \mathcal{D}(U), \quad i, j = 1, \ldots d\,,$$

the eigenspaces of $\pi_{i,*}\mathcal{L}_{i,\alpha_i}$ are invariant under the multiplication with $b^j$, or that $P_1(\mathcal{L})$ is normal, and that the eigenvalues of $P(\mathcal{L})$ are non-negative.

**Lemma.** The semigroup $e^{-tP(\mathcal{L})}$ acts compactly on $W_0^{k,2}(U)$ for $t > 0$, $k \in \mathbb{N}$.

Proof.

The operators $e^{-tP(\mathcal{L})}$ for $t > 0$ are trace-class operators acting on the Hilbert spaces $W_0^{k,2}(U)$ by Assumption. □

## 1.7 Kernels and Green functions

Let $x_0 \in U$. The Green function for the diffusion equation

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) + P(\mathcal{L})u(x,t) &= 0 \\ u|_{\delta^-(U)} &= 0 \end{aligned}$$

is given by the Poisson equation

$$\begin{cases} P(\mathcal{L})G(x,x_0) = \delta(x-x_0), & x \in U \\ \qquad G(x,x_0) = 0, & x \in \delta^-(U) \end{cases}$$

i.e. we can take $\gamma = \mu = 0$, here by Assumption.

## 1.7 Heat Kernels and Green functions

Relation between Green function and heat kernel:

$$G(x,y) = \int_0^\infty h(x,y,t)\,dt$$

with

$$h(x,y,t) = \sum_{\substack{\psi \\ \lambda_\psi > 0}} e^{-\lambda_\psi t} \psi(x)\overline{\psi(y)}$$

as part of the heat kernel

$$H(x,y,t) = h(x,y,t) + \sum_{\substack{\psi \\ \lambda_\psi = 0}} \psi(x)\overline{\psi(y)}$$

with $\psi$ running through an eigenbasis of $W_0^{k,2}(U)$ for $P(\mathcal{L})$.

**Strategy.** Prove convergence of $H(x,y,t)$, and solve the Poisson equation for the Green function.

# 1.7 Heat Kernels and Green functions

**Markov property.**

**Theorem.** The operator $-P(\mathcal{L})$ generates a contraction semigroup $e^{-tP(\mathcal{L})}$ with $t \geq 0$ on $W_0^{k,2}(\mathcal{L})$ for $k \in \mathbb{N}$, and the action satisfies the Markov property if $k \geq 2$.

Sketch of proof.

*Contraction semigroup property.* Show that

$$\left\| \int_0^t e^{-\tau P(\mathcal{L})} u \, d\tau \right\|_{W_0^{k,2}(U)} \leq t \, \|u\|_{W_0^{k,2}(U)}$$

and use that

$$R(\lambda)u = \lambda \int_0^\infty e^{-\lambda t} \int_0^t e^{-\tau P(\mathcal{L})} u \, d\tau \, dt$$

expresses the resolvent $R(\lambda) = (\lambda + P(\mathcal{L}))^{-1}$.

not finished

## 1.7 Heat Kernels and Green functions

Continued proof.

It follows that

$$\|R(\lambda)u\|_{W_0^{k,2}(U)} \leq \lambda \|u\|_{W_0^{k,2}(U)} ,$$

i.e.

$$\|\lambda + P(\mathcal{L})\|^{-1} \leq \frac{1}{\lambda} .$$

Then use Hille-Yosida shows that $e^{-tP(\mathcal{L})}$ is a contraction semigroup on $W_0^{k,2}(U)$ for $t \geq 0$, $k \in \mathbb{N}$.

*Markov property.* First, show for $k \geq 2$ that

$$0 \leq f \leq 1 \text{ a.e.} \quad \Rightarrow \quad 0 \leq e^{-tP(\mathcal{L})f} \leq 1 \qquad (1)$$

$$e^{-tP(\mathcal{L})}1_U = 1_U \qquad (2)$$

not finished □

# 1.7 Heat Kernels and Green functions

Continued proof.

$f \geq 0$ means $f$ is a positive linear combination of eigenfunctions invariant under $\left(\mathbb{F}_p^\times\right)^d$ via

$$x \mapsto (j_1\xi_1, \dots, j_d\xi_d)$$

for $x = (\xi_1, \dots, \xi_d) \in U$, and $(j_1, \dots, j_d) \in \left(\mathbb{F}_p^\times\right)^d$. The eigenspaces of $P(\mathcal{L})$ are invariant under this action. Hence, $e^{-tP(\mathcal{L})}f$ is invariant. Non-Positivity of eigenvalues thus show (1).

(2): $1_U$ is an eigenfunction with eigenvalue 0.

Next, find an invariant measure for the semigroup $e^{-tP(\mathcal{L})}$. Use the invariant measures $\pi_\phi$ for the finite-dimensional invariant spaces $V_\phi$, and show that $\pi = \sum_{V_\phi} \pi_\phi$ is an invariant measure for $t > 0$, and $k \geq 2$. [*$k \geq 2$ is needed due to infinite quadratic sums with eigenvalues in the proof.*] □

# 1.7 Heat Kernels and Green functions

**Corollary.** The semigroup $e^{-tP(\mathcal{L})}$ with $t \geq 0$ has a kernel representation $p_t(x, \cdot)$ for $t \geq 0$, $x \in U$, i.e. the map $A \mapsto p_t(x, A)$ is a Borel measure, and it holds true that

$$\int_U p_t(x, dy) f(y) = e^{-tP(\mathcal{L})} f(x)$$

for $f \in W_0^{k,2}(U)$ with $k \geq 2$.

Proof.

The theory of Markov diffusion operators shows this. □

# 1.7 Heat Kernels and Green functions

**Theorem.** The Markov semigroup $e^{-tP(\mathcal{L})}$ on $W_0^{k,2}(U)$ has a heat kernel function given by

$$H(x, y, t) \in L^\infty(U \times U)$$

for $t > 0$, $k \geq 2$.

Proof.

Need only show that $H(x, y, t) \in L^\infty(U \times U)$.

$x = y$. $H(x, x, t)$ is the trace of $e^{-tP(\mathcal{L})}$, and is finite.

$x \neq y$. Since $\left|\psi(x)\overline{\psi(y)}\right| \leq \mu(U)$, it follows that

$$|H(x, y, t)| \leq \sum_{\psi} e^{-t\lambda_\psi} < \infty \,,$$

and the assertion follows. □

# 1.7 Heat Kernels and Green functions

**Corollary.** The Green function $G(x,y)$ for $-P(\mathcal{L})$ exists and is given by

$$G(x,y) = \sum_{\substack{\psi \\ \lambda_\psi > 0}} \lambda_\psi^{-1} \psi(x) \overline{\psi(y)}$$

for $x, y \in U$.

Proof.
The expression for $G(x,y)$ is given by integration from its relation with the heat kernel. Convergence follows from the unbounded growth of the eigenvalues $\lambda_\psi \in O\left(p^{2dn(1+\alpha)}\right)$ with $p^{-dn}$ the volume of the support of $\phi \in \mathcal{E}$ making up $\psi$ for $n >> 0$, and

$$\alpha = \max\{\alpha_1, \ldots, \alpha_d\},$$

as well as $\left|\psi(x)\overline{\psi(y)}\right| \leq \mu(U)$ for $x, y \in U$. □

# 2 Ultrametric Manifolds

Now we sketch how to possibly generalise the previous results to ultrametric manifolds.

# 2 Ultrametric Manifolds

**Definition.** A Cantor set is a totally disconnected compact metrisable space without isolated points.

Notice that up to homeomorphism, there is precisely one Cantor set.

**Definition.** A locally compact local Cantor set is a second countable Hausdorff space in which each point has an open neighbourhood which is a Cantor set.

**Definition.** An ultrametric $d$ on a Cantor set $C$ is regular, if $d$ generates the topology of $C$. The pair $(C, d)$ is then called a regular ultrametric Cantor set.

## 2 Ultrametric Manifolds

**Definition.** A chart of a locally compact local Cantor set $X$ is a map $\phi\colon U \to V$ such that $U$ is open in $X$, $V$ is a Cantor set, $\phi(U)$ is an open of $V$, and $\phi$ is a homeomorphism onto its image. An ultrametric $n$-chart of $X$ is a tuple

$$c = (U, \phi; d_1, \dots, d_n)$$

where $\phi\colon U \to V$ is a chart, such that $V$ is given the structure of

$$V = (C, d_1) \times \cdots \times (C, d_n),$$

where $C$ is a Cantor set, and each $d_i$ is a regular ultrametric on $C$. The number $n \in \mathbb{N}$ is the dimension of the ultrametric chart.

# 2 Ultrametric Manifolds

Fix a Cantor set $V$.

Given two charts $\phi_\alpha\colon U_\alpha \to V$, $\phi_\beta\colon U_\beta \to V$ of a locally compact local Cantor set $X$, there is a transition map

$$\tau_{\alpha\beta}\colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

given by

$$\tau_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$$

which is a homeomorphism.

*We would like to define analyticity for maps between ultrametric sets.*

## 2 Ultrametric Manifolds

Let $B_{\underline{\rho}}(z)$ be the polydisk centred in

$$z \in (C_1, d_1) \times \cdots \times (C_n, d_n),$$

where each $(C_i, d_i)$ is a regular ultrametric Cantor set.

Fix a Radon measure $\nu_i$ on $(C_i, d_i)$, $i = 1, \ldots, n$, and let

$$\nu = \nu_1 \wedge \cdots \wedge \nu_n$$

be the product measure.

## 2 Ultrametric Manifolds

**Definition.** A homeomorphism

$$\tau\colon (C_1, d_1) \times \cdots \times (C_n, d_n) \to (C_1, d_1) \times \cdots \times (C_m, d_m)$$

is *analytic*, if it takes polydisks to polydisks, and for each $z \in (C_1, d_1) \times \cdots \times (C_n, d_n)$, the value

$$\alpha_{ij}(z) = -\log\left(\frac{\nu_i(B_{\rho_i}(z))}{\nu_j\left(\pi_j\left(\tau\left(B_{\underline{\rho}}(z)\right)\right)\right)}\right)$$

is constant for $\left\|\underline{\rho}\right\|_1 << \infty$ with $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Here,

$$\pi_k\colon (C_1, d_1) \times \cdots \times (C_n, d_n) \to (C_k, d_k)$$

is projection onto the $k$-th factor.

# 2 Ultrametric Manifolds

In the case of overlap between an ultrametric $n$-chart $\phi_\alpha$ and an $m$-chart $\phi_\beta$, the sets

$$\phi_\alpha(U_\alpha \cap U_\beta), \phi_\beta(U_\alpha \cap U_\beta)$$

are disjoint unions of products of ultrametric Cantor sets.

**Definition.** We say that

$$\tau_{\alpha\beta} \colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is *locally analytic*, if every $z \in \phi_\alpha(U_\alpha \cap U_\beta)$ has a clopen neighbourhood $C = (C_1, d_1) \times \cdots \times (C_n, d_n)$ such that $\tau_{\alpha\beta}$ restricted to $C$ is an analytic homeomorphism onto its image.

Obtain in the case $m = n$ a family

$$d\tau_{\alpha\beta}(x) := e^{A_{\alpha\beta}(x)} \in \mathrm{GL}_n(\mathbb{R}),$$

where for $x \in U_\alpha \cap U_\beta$: $A_{\alpha\beta}(x) = (\alpha_{ij}(\tau_{\alpha\beta}(x))_{i,j=1,\dots,n} \in \mathbb{R}^{n\times n}$.

## 2 Ultrametric Manifolds

**Definition.** Two ultrametric charts

$$c_\alpha = (U_\alpha, \phi_\alpha, d_{\alpha,1}, \ldots, d_{\alpha,n_\alpha}),\ c_\beta = (U_\beta, \phi_\beta, d_{\beta,1}, \ldots, d_{\beta,n_\beta})$$

of a locally compact local Cantor set $X$ are *compatible*, if the maps

$$\phi_\alpha(U_\alpha \cap U_\beta) \underset{\tau_{\beta\alpha}}{\overset{\tau_{\alpha\beta}}{\rightleftarrows}} \phi_\beta(U_\alpha \cap U_\beta)$$

are locally analytic homeomorphisms. Two compatible charts are $C^0$-*compatible*, if $n_\alpha = n_\beta$ and

$$d\tau_{\alpha\beta} \colon X \to \mathrm{GL}_n(\mathbb{R}),\ x \mapsto d\tau_{\alpha\beta}(x)$$

is continuous, in the case that $U_\alpha \cap U_\beta \neq \emptyset$.

# 2 Ultrametric Manifolds

**Definition.** An ultrametric $C^0$-atlas $\mathcal{A}$ of a locally compact local Cantor set $X$ is a family of ultrametric charts $c_\alpha$, $\alpha \in I$, which are mutually $C^0$-compatible, and such that $\{U_\alpha \mid \alpha \in I\}$ is a cover of $X$. Two atlantes $\mathcal{A}, \mathcal{A}'$ are $C^0$-compatible, if $\mathcal{A} \cup \mathcal{A}'$ is a $C^0$-atlas of $X$.

- An ultrametric $C^0$-atlas of $X$ is *full*, if any ultrametric chart $C^0$-compatible with any chart in $\mathcal{A}$ already belongs to $\mathcal{A}$.

- $C^0$-compatibiblity of atlantes is an equivalence relation.

- Each equivalence class of $C^0$-compatible atlantes of X is readily seen to contain a unique full atlas.

**Definition.** An *ultrametric analytic $C^0$-manifold* is a pair $(X, \mathcal{A})$ with $X$ a locally compact local Cantor set, and $\mathcal{A}$ a full ultrametric $C^0$-atlas of $X$.

# 2 Ultrametric Manifolds

**How to continue from here:**

- Tangent bundle as a "mixed" kind of "manifold"!
- Vector fields on ultrametric manifolds via tangent spaces as real vector spaces!
- Imitate the Laplace-Beltrami operator using *local* coordinate Laplacians [*Vladimirov-Pearson operators*]!
- Imitate differential forms!
- See if all this works also for ultrametric manifolds whose local dimension is not constant!
- Study the heat equation!
- Define general elliptic operators like pdo's, but on ultrametric manifolds, and do the same!

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## Dear Branko,

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