Scalar-tensor cosmology: unifying metric, Palatini and hybrid approaches

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ST gravity and solution-equivalent frames

Most general action for the scalar-tensor (ST) theories of gravity can be written as (four functions defining the frame: $\mathcal{A}(\Phi), \mathcal{B}(\Phi), \mathcal{V}(\Phi), \alpha(\Phi)$) [1]:

$$S[g_{\mu\nu}, \Phi, \chi] = \frac{1}{2\kappa^2} \int_\Omega d^n x \sqrt{-g} \Big[\mathcal{A}(\Phi)\mathcal{R} - \mathcal{B}(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \mathcal{V}(\Phi) \Big] + S_{\text{matter}} \left[e^{2\alpha(\Phi)} g_{\mu\nu}, \chi \right], \quad T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \tag{1}$$

Then the corresponding field equations can be recast into the form:

$$\mathcal{A}(\Phi)\mathcal{G}_{\mu\nu}(g) - (\nabla^g_\mu \nabla^g_\nu - g_{\mu\nu}\Box^g)\mathcal{A}(\Phi) = \Xi^\Phi_{\mu\nu} + \kappa^2 T_{\mu\nu} \tag{2}$$

$$\mathcal{A}'(\Phi)\mathcal{R}(g) + \mathcal{B}'(\Phi)(\partial\Phi)^2 + 2\mathcal{B}(\Phi)\Box^g\Phi - \mathcal{V}'(\Phi) = -2\kappa^2\alpha'(\Phi) T \tag{3}$$

$\frac{n(n+1)}{2} + 1$ equations, where $n > 2$ is a dimension of spacetime.

[1] L. Järv, P. Kuusk, M. Saal, and O. Vilson, Phys. Rev. D **91** (2015) 024041.

where $(\partial\Phi)^2 = g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi$ and

$$\Xi^{\Phi}_{\mu\nu} = \mathcal{B}(\Phi)\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}(\mathcal{B}(\Phi)(\partial\Phi)^2 + \mathcal{V}(\Phi))g_{\mu\nu} \tag{4}$$

$$= -(\partial\Phi)^2\,\mathcal{B}(\Phi)u_\mu\,u_\nu - \frac{1}{2}(\mathcal{B}(\Phi)(\partial\Phi)^2 + \mathcal{V}(\Phi))g_{\mu\nu}\,.$$

mimics a perfect fluid with velocity $u_\mu = \partial_\mu\Phi/\sqrt{-(\partial\Phi)^2}$. The matter stress-energy tensor is **not conserved** (chameleon mechanism) in general:

$$\nabla_\mu T^{\mu\nu} = \alpha'(\Phi)T\partial^\nu\Phi, \tag{5}$$

where $T = g^{\mu\nu}T_{\mu\nu}$. For the case $\alpha = const, \mathcal{A} = const$

$$\nabla_\mu T^{\mu\nu} = 0 = \nabla_\mu \Xi^{\Phi\,\mu\nu}\,. \tag{6}$$

ST gravity and solution-equivalent frames

By applying conformal transformation and field re-definition [2]

$$\bar{g}_{\mu\nu} = e^{2\gamma(\Phi)} g_{\mu\nu}, \tag{7a}$$

$$\bar{\Phi} = f(\Phi) \quad \text{i.e.} \quad \Phi = \check{f}(\bar{\Phi}). \tag{7b}$$

the solutions in some initial frame transform to solutions in a new one

$$\bar{\mathcal{A}}(\bar{\Phi}) = e^{(n-2)\check{\gamma}(\bar{\Phi})}\mathcal{A}(\check{f}(\bar{\Phi})), \tag{8a}$$

$$\bar{\mathcal{B}}(\bar{\Phi}) = e^{(n-2)\check{\gamma}(\bar{\Phi})}\Bigg(\left(\frac{d\Phi}{d\bar{\Phi}}\right)^2 \mathcal{B}(\check{f}(\bar{\Phi})) - (n-1)(n-2)\left(\frac{d\check{\gamma}}{d\bar{\Phi}}\right)^2 \mathcal{A}(\check{f}(\bar{\Phi})) - 2(n-1)\frac{d\check{\gamma}}{d\bar{\Phi}}\frac{d\mathcal{A}}{d\Phi}\frac{d\Phi}{d\bar{\Phi}}\Bigg), \tag{8b}$$

$$\bar{\mathcal{V}}(\bar{\Phi}) = e^{n\check{\gamma}(\bar{\Phi})}\mathcal{V}(\check{f}(\bar{\Phi})), \tag{8c}$$

$$\bar{\alpha}(\bar{\Phi}) = \alpha(\check{f}(\bar{\Phi})) + \check{\gamma}(\bar{\Phi}) = \alpha(\check{f}(\bar{\Phi})) - \gamma(\bar{\Phi})\,. \tag{8d}$$

We call such frames solution equivalent frames, where $\check{\gamma}_i = -\gamma_i \circ f$, $\check{f} = f^{-1}$.

[2] Solution for the metric **does not depend** on redefinition of Φ.

Remark We have a group action

$$(f,\gamma) \triangleright (g_{\mu\nu}, \Phi) = (\exp{(2\gamma(\Phi))} g_{\mu\nu}, f \circ \Phi) \equiv (\bar{g}_{\mu\nu}, \bar{\Phi})\,,$$

which extends to the action $(\mathcal{A}, \mathcal{B}, \mathcal{V}, \alpha) \to (\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{\mathcal{V}}, \bar{\alpha})$ (where $f \in \mathtt{Diff}(\mathbb{R})$ and $\gamma \in C^1(\mathbb{R})$) and

$$(\bar{f}, \bar{\gamma}) \circ (f,\gamma) = (\bar{f} \circ f, \bar{\gamma} + \gamma \circ \bar{f}^{-1}).$$

One can define the following (frame independent) invariants:

1. $\mathcal{I}_{\mathcal{A}}(\Phi) = \frac{\mathcal{A}(\Phi)}{e^{(n-2)\alpha(\Phi)}}$,
2. $\mathcal{I}_{\mathcal{V}}(\Phi) = \frac{\mathcal{V}(\Phi)}{(\mathcal{A}(\Phi))^{\frac{n}{n-2}}}$,
3. $d\Psi = \sqrt{\pm\frac{(n-2)\mathcal{A}(\Phi)\mathcal{B}(\Phi)+(n-1)(\mathcal{A}'(\Phi))^2}{n\mathcal{A}^2(\Phi)}}d\Phi$.

We may introduce invariant metrics, remaining unchanged under a conformal transformation:

$$\text{EF}: \ \hat{g}_{\mu\nu} := (\mathcal{A}(\Phi))^{\frac{2}{n-2}} g_{\mu\nu}\,, \tag{9a}$$

$$\text{JF}: \ \tilde{g}_{\mu\nu} := e^{2\alpha(\Phi)} g_{\mu\nu}\,. \tag{9b}$$

$f(R)$ versus Scalar-Tensor gravity

$$S_F[g_{\mu\nu}] = \frac{1}{2\kappa^2}\int_\Omega \mathrm{d}^n x\sqrt{-g}F(R) + S_{\mathrm{matter}}(g_{\mu\nu},\chi), \tag{10}$$

Dynamically equivalent constrain system

$$S[g_{\mu\nu},.,\Xi] = \frac{1}{2\kappa^2}\int_\Omega \mathrm{d}^n x\sqrt{-g}\left(F'(\Xi)(R-\Xi)+F(\Xi)\right) + S_{\mathrm{matter}}(g_{\mu\nu},\chi). \tag{11}$$

by solving the constraint equation $\Xi = R$. Introducing a scalar field $\Phi = F'(\Xi)$ one obtains equivalent STT **Jordan Frame** action with a non-dynamical scalar field

$$S[g_{\mu\nu},.,\Phi] = \frac{1}{2\kappa^2}\int_\Omega \mathrm{d}^n x\sqrt{-g}\left(\Phi R - U_F(\Phi)\right) + S_{\mathrm{matter}}(g_{\mu\nu},\chi) \tag{12}$$

with the potential describing self-interaction of Φ:

$$U_F(\Phi) \equiv R(\Phi)\Phi - F(R(\Phi)), \tag{13}$$

is the result of Legendre' transformation. This is a special case of Brans-Dickey action

$$S_{BD}[g_{\mu\nu},\Phi] = \frac{1}{2\kappa^2}\int_\Omega \mathrm{d}^n x\sqrt{-g}\left(\Phi R - \frac{\omega_{BD}}{\Phi}\partial_\mu\Phi\partial^\mu\Phi - U(\Phi)\right). \tag{14}$$

Hybrid[3] metric-Palatini generalization

The action functional is given by

$$S[g_{\mu\nu},\Gamma^{\alpha}_{\mu\nu}] = \frac{1}{2\kappa^2}\int_{\Omega} d^n x\sqrt{-g}[\Omega_A \mathcal{R}(g) + F(\hat{R}(g,\Gamma))] + S_{\text{matter}}[g_{\mu\nu},\chi] \tag{15}$$

where Ω_A is a coupling constant and $\hat{R}(g,\Gamma) = g^{\mu\nu}R_{\mu\nu}(\Gamma)$ is a Palatini Ricci scalar. It can be shown that we end up in purely metric STT (see below)

$$S[g_{\mu\nu},\Phi] = \frac{1}{2\kappa^2}\int_{\Omega} \mathrm{d}^n x\sqrt{-g}\left((\Omega_A+\Phi)\mathcal{R}(g) + \frac{n-1}{(n-2)\Phi}\partial_\mu\Phi\partial^\mu\Phi - U_F(\Phi)\right) + S_{\text{matter}}[g_{\mu\nu},\chi]. \tag{16}$$

where $\omega_{BD} = -\frac{n-1}{(n-2)}$ is again 'patological'. $\Omega_A = 0$ give rise to Palatini $f(R)$ gravity.

[3] T. Harko, T. S. Koivisto, F. S. N. Lobo, G. J. Olmo, Phys. Rev. D 85 (2012) 084016; S. Capozziello, T. Harko, T.S. Koivisto, F.S.N. Lobo, G.J. Olmo, , Universe 1 (2015) 2, 199.

Metric -Palatini-hybrid ST gravity[5]

$$S[g_{\mu\nu},\Gamma^{\alpha}_{\mu\nu},\Phi] = \frac{1}{2\kappa^2}\int_\Omega d^n x\sqrt{-g}\Big[\mathcal{A}_1(\Phi)\mathcal{R}(g) + \mathcal{A}_2(\Phi)\hat{R}(g,\Gamma) - \mathcal{B}(\Phi)g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - Q^{\mu}(g,\Gamma)\mathcal{C}_1(\Phi)\partial_\mu\Phi - \bar{Q}^{\mu}(g,\Gamma)\mathcal{C}_2(\Phi)\partial_\mu\Phi - \mathcal{V}(\Phi)\Big] + S_{\text{matter}}[e^{2\alpha(\Phi)}g_{\mu\nu},\chi]. \quad (17)$$

The action depend on the non-metricities of a torsionless connection Γ: $Q_\mu = g^{\alpha\beta}\hat{\nabla}_\mu g_{\alpha\beta}$ and $\bar{Q}_\mu = -g^{\alpha\beta}\hat{\nabla}_\alpha g_{\beta\mu}$, where $\hat{R}(g,\Gamma) = g^{\mu\nu}R_{\mu\nu}(\Gamma)$.

The case $\mathcal{A}_2(\Phi) = 0$ corresponds to a purely metric while $\mathcal{A}_1(\Phi) = 0$ purely Palatini STT (see A. Kozak, AB., Eur.Phys.J.C 79 (2019)).
It is form invariant with respect to the following transformation[4]

$$\bar{g}_{\mu\nu} = e^{2\gamma_1(\Phi)}g_{\mu\nu}, \quad \bar{\Phi} = f(\Phi), \quad (18a)$$

$$\bar{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + 2\delta^{\alpha}_{(\mu}\partial_{\nu)}\gamma_2(\Phi) - g_{\mu\nu}g^{\alpha\beta}\partial_\beta\gamma_3(\Phi) \quad (18b)$$

$\mathcal{A}_1 + \mathcal{A}_2 \neq 0$. In particular, $\mathcal{A}_1 = 0$ gives purely Palatini case. In contrast $\mathcal{A}_2 = \mathcal{C}_1 = \mathcal{C}_2 = 0$ provides pure metric formalism.

[4] $\gamma_2 = \gamma_3$ implies Weyl transformation of Γ.
[5] AB, A. Kozak, JCAP 07 (2020) 003

The solution equivalent frame functions transform accordingly as (for $\gamma_1 = \gamma_2 = \gamma_3, \mathcal{C}_1 = \mathcal{C}_2 = 0$ reconstruct the metric case):

$$\bar{\mathcal{A}}_1(\bar{\Phi}) = e^{(n-2)\check{\gamma}_1(\bar{\Phi})}\mathcal{A}_1(\check{f}(\bar{\Phi})), \qquad \bar{\mathcal{A}}_2(\bar{\Phi}) = e^{(n-2)\check{\gamma}_1(\bar{\Phi})}\mathcal{A}_2(\check{f}(\bar{\Phi}))$$

$$\begin{aligned}\bar{\mathcal{B}}(\bar{\Phi}) = e^{(n-2)\check{\gamma}_1(\bar{\Phi})}\Big[&\mathcal{B}(\check{f}(\bar{\Phi}))(\check{f}'(\bar{\Phi}))^2 + (n-1)\Big(n\mathcal{A}_2(\check{f}(\bar{\Phi}))\check{\gamma}_2'(\bar{\Phi})\check{\gamma}_3'(\bar{\Phi}) - \mathcal{A}_2(\check{f}(\bar{\Phi}))\left(\check{\gamma}_2'(\bar{\Phi})\right)^2 \\ &- \mathcal{A}_2(\check{f}(\bar{\Phi}))\left(\check{\gamma}_3'(\bar{\Phi})\right)^2 - \frac{d\mathcal{A}_2(\check{f}(\bar{\Phi}))}{d\bar{\Phi}}(\check{\gamma}_2'(\bar{\Phi}) + \check{\gamma}_3'(\bar{\Phi})) - 2\frac{d\mathcal{A}_1(\check{f}(\bar{\Phi}))}{d\bar{\Phi}}\check{\gamma}_1'(\bar{\Phi}) \\ &- (n-2)\mathcal{A}_2(\check{f}(\bar{\Phi}))\check{\gamma}_1'(\bar{\Phi})(\check{\gamma}_2'(\bar{\Phi}) + \check{\gamma}_3'(\bar{\Phi})) - (n-2)\mathcal{A}_1(\check{f}(\bar{\Phi}))(\check{\gamma}_1'(\bar{\Phi}))^2\Big) \\ &+ \check{f}'(\bar{\Phi})\Big(\mathcal{C}_1(\check{f}(\bar{\Phi}))(2n\check{\gamma}_1'(\bar{\Phi}) - 2(n+1)\check{\gamma}_2'(\bar{\Phi}) + 2\check{\gamma}_3'(\bar{\Phi})) \\ &- \mathcal{C}_2(\check{f}(\bar{\Phi}))(2\check{\gamma}_1'(\bar{\Phi}) - (n+3)\check{\gamma}_2'(\bar{\Phi}) + (n+1)\check{\gamma}_3'(\bar{\Phi}))\Big)\Big],\end{aligned}$$

$$\bar{\mathcal{C}}_1(\bar{\Phi}) = e^{(n-2)\check{\gamma}_1(\bar{\Phi})}\left[\check{f}'(\bar{\Phi})\mathcal{C}_1(\check{f}(\bar{\Phi})) - \mathcal{A}_2(\check{f}(\bar{\Phi}))\left(\frac{n-1}{2}\check{\gamma}_2'(\bar{\Phi}) + \frac{n-3}{2}\check{\gamma}_3'(\bar{\Phi})\right)\right],$$

$$\bar{\mathcal{C}}_2(\bar{\Phi}) = e^{(n-2)\check{\gamma}_1(\bar{\Phi})}\left[\check{f}'(\bar{\Phi})\mathcal{C}_2(\check{f}(\bar{\Phi})) - \mathcal{A}_2(\check{f}(\bar{\Phi}))\left((n-1)\check{\gamma}_2'(\bar{\Phi}) - \check{\gamma}_3'(\bar{\Phi})\right)\right],$$

$$\bar{\mathcal{V}}(\bar{\Phi}) = e^{n\check{\gamma}_1(\bar{\Phi})}\mathcal{V}(\check{f}(\bar{\Phi})),$$

$$\bar{\alpha}(\bar{\Phi}) = \alpha(\check{f}(\bar{\Phi})) + \check{\gamma}_1(\bar{\Phi}),$$

Invariant quantities can be generalized as well:

$$\mathcal{I}_{\mathcal{A}}(\Phi) = \frac{\mathcal{A}_1(\Phi)}{\mathcal{A}_2(\Phi)}\,,$$

$$\mathcal{I}_V^{(1)}(\Phi) = \frac{\mathcal{V}(\Phi)}{(\mathcal{A}_1(\Phi))^{\frac{n}{n-2}}}, \qquad \mathcal{I}_V^{(2)}(\Phi) = \frac{\mathcal{V}(\Phi)}{(\mathcal{A}_2(\Phi))^{\frac{n}{n-2}}}\,,$$

$$\mathcal{I}_\alpha^{(1)}(\Phi) = \frac{\mathcal{A}_1(\Phi)}{e^{(n-2)\alpha}(\Phi)}, \qquad \mathcal{I}_\alpha^{(2)}(\Phi) = \frac{\mathcal{A}_2(\Phi)}{e^{(n-2)\alpha}(\Phi)}\,,$$

$$d\Psi(\Phi) = \frac{d\Phi}{(\mathcal{A}_1(\Phi)+\mathcal{A}_2(\Phi))}\Big|(n-1)(n-2)\mathcal{B}(\Phi)(\mathcal{A}_1(\Phi)+\mathcal{A}_2(\Phi))$$
$$+ (1+\mathcal{I}_{\mathcal{A}}^{-1}(\Phi))[(n-1)\mathcal{A}_1'(\Phi)]^2 + (\mathcal{I}_{\mathcal{A}}(\Phi)+1)[-4\mathcal{C}_1^2(\Phi) + (n^2-5)\mathcal{C}_2^2(\Phi)$$
$$- 2(n^2-n-4)\mathcal{C}_1(\Phi)\mathcal{C}_2(\Phi) + 2(n-1)\mathcal{A}_2'(\Phi)(\mathcal{C}_2(\Phi) - n\mathcal{C}_1(\Phi))]\Big|^{\frac{1}{2}}\,.$$

Varying with respect to all independent variables entering the action one gets

Metric:

$$\begin{aligned}&\mathcal{A}_1(\Phi)\mathcal{G}_{\mu\nu}(g) + \mathcal{A}_2(\Phi)\hat{G}_{\mu\nu}(g,\Gamma) + (\mathcal{A}_1''(\Phi) + \frac{1}{2}\mathcal{B}(\Phi) - \mathcal{C}_1'(\Phi))(\partial\Phi)^2 g_{\mu\nu} + \frac{1}{2}\mathcal{V}(\Phi)g_{\mu\nu}\\ &-(\mathcal{A}_1''(\Phi) + \mathcal{B}(\Phi) - \mathcal{C}_2'(\Phi))\partial_\mu\Phi\partial_\nu\Phi + (\mathcal{C}_2(\Phi)\hat{\nabla}_\mu\partial_\nu - \mathcal{C}_1(\Phi)g_{\mu\nu}\hat{\Box})\Phi - \mathcal{A}_1'(\Phi)(\nabla^g_\mu\partial_\nu - g_{\mu\nu}\Box^g)\Phi\\ &+ Q_{\beta\lambda\zeta}\Big[\frac{1}{2}\mathcal{C}_2(\Phi)\delta^\sigma_{(\nu}\delta^\beta_{\mu)}g^{\lambda\zeta} - \mathcal{C}_1(\Phi)\left(\frac{1}{2}g_{\mu\nu}g^{\sigma\beta}g^{\lambda\zeta} - g_{\mu\nu}g^{\sigma\lambda}g^{\beta\zeta} + \delta^\sigma_{(\mu}\delta^\beta_{\nu)}g^{\lambda\zeta}\right)\Big]\partial_\sigma\Phi = \kappa^2 T_{\mu\nu}\,,\end{aligned} \tag{21}$$

Connection:

$$\begin{aligned}&\hat{\nabla}_\alpha\left[\sqrt{-g}\left(g^{\alpha(\zeta}\delta^{\lambda)}_\beta - g^{\lambda\zeta}\delta^\alpha_\beta\right)\right] =\\ &= \sqrt{-g}\partial_\alpha\Phi\left[g^{\alpha(\zeta}\delta^{\lambda)}_\beta\left(\frac{\mathcal{C}_2(\Phi) - 2\mathcal{C}_1(\Phi) - \mathcal{A}_2'(\Phi)}{\mathcal{A}_2(\Phi)}\right) - g^{\lambda\zeta}\delta^\alpha_\beta\left(\frac{-\mathcal{C}_2(\Phi) - \mathcal{A}_2'(\Phi)}{\mathcal{A}_2(\Phi)}\right)\right]\,,\end{aligned} \tag{22}$$

Scalar field:

$$\begin{aligned}&\mathcal{A}_1'(\Phi)\mathcal{R}(g) + \mathcal{A}_2'(\Phi)\hat{R}(g,\Gamma) + \mathcal{B}'(\Phi)g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + 2\mathcal{B}(\Phi)\Box^g\Phi + 2\mathcal{B}(\Phi)\partial_\mu\Phi Q_{\nu\alpha\beta}\\ &\times\left(\frac{1}{2}g^{\mu\nu}g^{\alpha\beta} - g^{\alpha\mu}g^{\beta\nu}\right) + \mathcal{C}_1(\Phi)\nabla^g_\mu Q^\mu + \mathcal{C}_2(\Phi)\nabla^g_\mu\bar{Q}^\mu - \mathcal{V}'(\Phi) = -2\kappa^2\alpha'(\Phi)T\,.\end{aligned} \tag{23}$$

Embedding into metric STT

There is an on-shell dynamics preserving "projection" from more general Palatini-hybrid frame $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{V}, \alpha)$ to the corresponding metric one $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \mathcal{V}, \alpha)$, where[6]

$$\tilde{\mathcal{A}}(\Phi) = \mathcal{A}_1(\Phi) + \mathcal{A}_2(\Phi)\,.$$

$$\tilde{\mathcal{B}}(\Phi) = \frac{(n-2)\mathcal{A}_2(\Phi)\mathcal{B}(\Phi) - (n-1)(\mathcal{A}_2'(\Phi))^2 + 2\mathcal{A}_2'(\Phi)[\mathcal{C}_2(\Phi) - n\mathcal{C}_1(\Phi)]}{(n-2)\mathcal{A}_2(\Phi)} + \frac{(n^2-5)\mathcal{C}_2(\Phi)^2 - 4\mathcal{C}_1(\Phi)^2 + 2(4+n-n^2)\mathcal{C}_1(\Phi)\mathcal{C}_2(\Phi))}{(n-2)(n-1)\mathcal{A}_2(\Phi)}\,,$$

leaving the metric and the scalar field solutions unchanged.

The other two frame functions $\mathcal{V}(\Phi), \alpha(\Phi)$ remain the same.

[6] AB, A. Kozak, JCAP 07 (2020) 003; e-Print: 2003.02741 [gr-qc]

Because the equations of motion admit the following generic solutions for the connection:

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ \begin{array}{c} \alpha \\ \mu\nu \end{array} \right\}_g + 2\,\mathcal{F}_1(\Phi)\delta^{\alpha}_{(\mu}\partial_{\nu)}\Phi - \mathcal{F}_2(\Phi)g_{\mu\nu}g^{\alpha\beta}\partial_\beta\Phi, \qquad (24)$$

and the non-metricity :

$$Q^{\mu\nu}_{\alpha} = \nabla_\alpha g^{\mu\nu} = 2\left(\mathcal{F}_1(\Phi) - \mathcal{F}_2(\Phi)\right)\delta^{(\mu}_{\alpha}g^{\nu)\rho}\partial_\rho\Phi + 2\,\mathcal{F}_1(\Phi)g^{\mu\nu}\partial_\alpha\Phi,$$

where $\mathcal{F}_1(\Phi) = \frac{2\mathcal{C}_1(\Phi)+(n-3)\mathcal{C}_2(\Phi)+(n-1)\mathcal{A}_2'(\Phi)}{\mathcal{A}_2(\Phi)(n-1)(n-2)}$, and $\mathcal{F}_2(\Phi) = \frac{2\mathcal{C}_1(\Phi)-\mathcal{C}_2(\Phi)+\mathcal{A}_2'(\Phi)}{\mathcal{A}_2(\Phi)(n-2)}$. Then by a suitable choice of functions γ_2, γ_3 one can obtained transformed $\tilde{\mathcal{C}}_1 = \tilde{\mathcal{C}}_2 = -\mathcal{A}_2'$. It means that $\tilde{\mathcal{F}}_1 = \tilde{\mathcal{F}}_2 = 0$ and the connection becomes Levi-Civita, i.e. non-metricity vanishes on shall.

Adopting Ehlers-Pirani-Schild (EPS) (1972 volume in honor J.L. Synge)[7] idea one can use geodesics of the original connection to explain galactic curves (**intergalactic DM**) (see e.g. A. Wojnar, C. Sporea, AB, Galaxies 6 (2018) 3, 70, e-Print: 1804.09620 [gr-qc], Eur.Phys.J. C78 (2018) 4, 308; e-Print: 1705.04131 [gr-qc]).

[7] Reprinted in GRG Golden Oldies 2012 (A. Tratman).

$f(R)$ examples

	$\tilde{\mathcal{A}}$	$\tilde{\mathcal{B}}$	$\mathcal{V}$	α
metric	Φ	0	$U_F(\Phi - \Omega_A)$	0
Palatini	Φ	$-\frac{n-1}{n-2}\frac{1}{\Phi}$	$U_F(\Phi)$	0
hybrid	$\Omega_A + \Phi$	$-\frac{n-1}{n-2}\frac{1}{\Phi}$	$U_F(\Phi)$	0

Table: JF metric SST frames for three cases of $\Omega_A R + F(R)$ gravity.

where

$$U_F(\Phi) \equiv R(\Phi)\Phi - F(R(\Phi)), \tag{25}$$

ST FLRW cosmology; $n = 4$

Substitution of FLRW metric ($n = 4$):

$$g_{\mu\nu} = \text{diag}\left(-N(t)^2, \frac{a(t)^2}{1-Kr^2}, a(t)^2 r^2, a(t)^2 r^2 \sin^2\theta\right) \tag{26}$$

yields:

$$R = \frac{6K}{a^2} + \frac{6}{N^2}\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} - \frac{\dot{a}}{a}\frac{\dot{N}}{N}\right) . \tag{27}$$

Useful choices: $N(\tau) = a(\tau)$, $N(\tau) = H(\tau)$.
Stress-energy tensor = barotropic perfect fluid (non-interacting components) :

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu + p g_{\mu\nu} , \qquad p = w\rho \tag{28}$$

with $u^\mu = \left(\frac{1}{N}, 0, 0, 0\right)$.
Not conserved, in general, the stress-energy tensor is :

$$\nabla_\mu T^{\mu\nu} = \alpha'(\Phi) T \partial^\nu \Phi, \tag{29}$$

is solved by (barotropic fluids $\rho_i = w_i p_i$):

$$\rho(a, \Phi) = \sum_i \rho_{0,i} a^{-3(1+w_i)} e^{(1-3w_i)\alpha(\Phi)} . \tag{30}$$

Field equations

One obtains a closed (over-determined) system of the second-order ODE for two functions $a(t), \Phi(t)$

$$3H^2 = \frac{\kappa^2\,\rho(a,\Phi)}{\mathcal{A}(\Phi)} + \frac{\mathcal{B}(\Phi)}{2\,\mathcal{A}(\Phi)}\dot{\Phi}^2 - 3\frac{\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)}H\dot{\Phi} + \frac{\mathcal{V}(\Phi)}{2\mathcal{A}(\Phi)}\,, \tag{31a}$$

$$\begin{aligned} 2\dot{H} + 3H^2 = &-\frac{\kappa^2\,p(a,\Phi)}{\mathcal{A}(\Phi)} - \frac{\mathcal{B}(\Phi) + 2\mathcal{A}''(\Phi)}{2\,\mathcal{A}(\Phi)}\dot{\Phi}^2 \\ &+ \frac{\mathcal{V}(\Phi)}{2\,\mathcal{A}(\Phi)} - \frac{\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)}\left(2H\dot{\Phi} + \ddot{\Phi}\right)\,, \end{aligned} \tag{31b}$$

$$\begin{aligned} &\left(3(\mathcal{A}'(\Phi))^2 + 2\mathcal{A}(\Phi)\mathcal{B}(\Phi)\right)\ddot{\Phi} = -3(3(\mathcal{A}'(\Phi))^2 + 2\mathcal{A}(\Phi)\mathcal{B}(\Phi))H\dot{\Phi} \\ &- \left((\mathcal{A}(\Phi)\mathcal{B}(\Phi))' + 3\mathcal{A}'(\Phi)\mathcal{A}''(\Phi)\right)\dot{\Phi}^2 + \left(2\mathcal{V}(\Phi)\mathcal{A}'(\Phi) - \mathcal{V}'(\Phi)\mathcal{A}(\Phi)\right) \\ &\quad + \kappa^2(\rho - 3p)(a,\Phi)\left[\mathcal{A}'(\Phi) - 2\alpha'(\Phi)\mathcal{A}(\Phi)\right]\,. \end{aligned} \tag{31c}$$

which can solved and compared with LCDM model at least numerically, after choosing required frame functions $\{\mathcal{A},\mathcal{B},\mathcal{V},\alpha\}$ and imposing initial conditions that respect the zero Hamiltonian energy constraints (31a).

Minisuperspace reformulation with chameleon mechanism [8]

$$S_{\rm MSS}[a,\Phi] = \frac{1}{2\kappa^2}\int dt\left[\frac{1}{N}\left(-6a\mathcal{A}\dot{a}^2 - 6a^2\mathcal{A}'\dot{a}\dot{\Phi} + a^3\mathcal{B}\dot{\Phi}^2\right) - NV_{\rm MSS}\right], \tag{32}$$

with

$$V_{\rm MSS}(a,\Phi) = -6Ka\mathcal{A}(\Phi) + a^3\mathcal{V}(\Phi) + V_{\rm matter}(a,\Phi)\,, \tag{33}$$

where

$$V_{\rm matter}(a,\Phi) = 2\kappa^2\, a^3\rho \equiv 2\kappa^2 \sum_i \rho_{i,0}\, a^{-3w_i} e^{(1-3w_i)\alpha(\Phi)}\,.$$

One gets MSS Lagrangian:

$$L_{\rm MSS}(N,x,\dot{x}) = \frac{1}{2N} m_{jk}(x)\dot{x}^j\dot{x}^k - NV_{\rm MSS}(x) \tag{34}$$

with

$$m_{ij} \equiv m_{ij}(a,\Phi) = \begin{pmatrix} -12a\mathcal{A}(\Phi) & -6a^2\mathcal{A}'(\Phi) \\ -6a^2\mathcal{A}'(\Phi) & 2a^3\mathcal{B}(\Phi)\end{pmatrix}\,. \tag{35}$$

[8] AB, A. Kozak, Phys.Rev.D 105 (2022) 4, 044011; e-Print: 2108.13324 [gr-qc]

Varying w.r.t. lapse function provides zero-Hamiltonian energy constraints (MSS Hamiltonian is conserved)

$$\frac{\delta L_{\mathrm{MSS}}}{\delta N} \equiv \frac{1}{2N} m_{ij} \dot{x}^i \dot{x}^j + N\, V_{\mathrm{MSS}} = 0, \tag{36}$$

Which after choosing the gauge $N = \mathrm{const} = 1$ reduces to the Friedman type eq. :

$$3\mathcal{H}^2 = -3\frac{\mathcal{A}'}{\mathcal{A}}\mathcal{H}\dot{\Phi} + \frac{\mathcal{B}}{2\mathcal{A}}\dot{\Phi}^2 - \frac{3K}{a^2} + \frac{\mathcal{V}}{2\mathcal{A}} + \frac{V_{\mathrm{matter}}}{2a^3\mathcal{A}}\,. \tag{37}$$

Remaining equations (assuming $\det m \neq 0$):

$$\ddot{x}^i + G^i_{jk}\dot{x}^j\dot{x}^k - \frac{\dot{N}}{N}\dot{x}^i = -N^2 m^{ij}\partial_j V_{\mathrm{MSS}}, \tag{38}$$

well suited for numerical calculations with MSS Levi-Civita connection

$$G^i_{\ kl} = \frac{1}{2} m^{ip}\left(\partial_k m_{lp} + \partial_l m_{kp} - \partial_p m_{kl}\right)\,. \tag{39}$$

Imposing Cauchy data of LCDM type

In order to solve above system of the second order ODFs one needs to impose the present day data $(a_0, \Phi_0, \dot{a}_0, \dot{\Phi}_0)$, where 0 refers to the age of the Universe, and $\dot{a}_0 = \mathcal{H}_0$ being a Hubble constant after normalization $a_0 = 1$. Remaining data $(\Phi_0, \dot{\Phi}_0)$ are constrained by the Hubble relation. Assuming further that the scalar field **has no dynamics at the present epoch**, i.e. $\dot{\Phi}_0 = 0$, one gets ΛCDM type relation

$$1 = \Omega_{\Lambda 0} + \Omega_K + \frac{1}{2\mathcal{A}(\Phi_0)} \sum_i \Omega_{0,i}\, e^{(1-3w_i)\alpha(\Phi_0)} = \Omega_\Lambda + \Omega_K + \sum_i \tilde{\Omega}_{0,i} \ ,$$

where $\Omega_{i,0} = \frac{\kappa^2 \rho_{0,i}}{3\mathcal{H}_0^2}$, $\Omega_K = -\frac{K}{3\mathcal{H}_0^2}$ and $\Omega_\Lambda = \frac{\mathcal{V}(\Phi_0)}{6\mathcal{H}_0^2 \mathcal{A}(\Phi_0)}$ could play a role of cosmological constant and Φ_0 is a free parameter.
In such scenario the **observed matter** $\widetilde{\Omega}_{0,i}$ **differs from 'true' baryonic matter** represented by a cosmic dust $\Omega_{0,\mathrm{dust}}$ by a factor $\frac{\exp \alpha(\Phi_0))}{2\mathcal{A}(\Phi_0)}$ imitating **cosmic DM**. The remaining terms
$\Omega_\Lambda = \frac{\mathcal{V}(\Phi)}{6\mathcal{H}^2 \mathcal{A}(\Phi)} - \frac{\mathcal{A}'}{\mathcal{A}} \frac{\dot{\Phi}}{\mathcal{H}} + \frac{\mathcal{B}}{6\mathcal{A}} (\frac{\dot{\Phi}}{\mathcal{H}})^2$ can be considered as a **dynamical DE**.

Summary

- A new class of scalar-tensor theories (STT) including a non-metricity that unifies metric, Palatini and hybrid metric-Palatini gravitational actions with non-minimal interaction is proposed and investigated from the point of view of its consistency with generalized conformal transformations. Generalized invariants are indicated.
- It is also shown that every such theory can be represented on-shell by a purely metric STT possessing the same solutions for metric and scalar field. Therefore, a connection provides additional degrees of freedom which can be used, according the EPS formalism, to imitate **intergalactic DM**.
- We introduced minisuperspace formulation of ST FLRW cosmology with perfect fluid non-minimally coupled to matter (chameleon mechanism).
- An appropriate choice of Cauchy data allows to approximate LCD era and can be used to explain DM-DE effects: **cosmological DM**, and running "cosmological constant"

HAPPY BIRTHDAY DEAR BRANCO !

Figure: Good health and many more to come ...