On Relations between the Real Kähler and the Complex Hilbert Spaces Quantum Mechanics

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Nonlinearity, Nonlocality and Ultrametricity

Dragovich 80

Outlook:

Continuation of the previous talk

- **Formulation of real QM in real Kähler space**
- **Example: real QM in real Kähler space $\mathcal{K}^2$**

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- **Formulation of real QM in real Kähler space**
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- **Main subtleties:**
  - **Tensor product**
  - **Degeneration of the spectrum**

Quantum Mechanics in Real Kähler Space

Mathematical notions are borrowed from the previous talk by I.Volovich

- i) To a physical system one assigns a real Kähler space $\mathcal{K}$
  - its state is represented by vectors $\eta \in \mathcal{K}$, $g(\eta,\eta) = 1$, $g(\cdot,\cdot)$ - an inner product in $\mathcal{K}$,
  - or by a density matrix ρ

Quantum Mechanics in Real Kähler Space

- ii) To the observable L corresponds the $\mathcal{K}$-Hermitian operator $\mathcal{L}$, which spectrum is observable

The spectral decomposition for $\mathcal{L}$:

$$\sum_{i=1}^{n} \lambda_i \mathcal{E}_i = \mathcal{L}, \quad \sum_{i=1}^{n} \mathcal{E}_i = \mathcal{I}, \quad \text{rank}(\mathcal{E}_i) \geq 2$$

Quantum Mechanics in Real Kähler Space

- **iii) Born rule:**

  if we measure $\boldsymbol{L}$ in the normalized state $\boldsymbol{\eta}$,

  the probability of obtaining result $\boldsymbol{\lambda_i}$ is

$$\frac{g(\eta, \mathcal{E}_i \eta)}{\mathrm{rank}(\mathcal{E}_i)}$$

Quantum Mechanics in Real Kähler Space

- iv) the Kähler space $\mathcal{K}$ corresponding to the composition of two systems $\mathfrak{N}$ and $\mathfrak{M}$ is

$$\mathcal{K}_{\mathfrak{N}} \bigotimes_{\mathcal{K}} \mathcal{K}_{\mathfrak{M}}$$

Some subtleties with $\otimes$

Quantum Mechanics in Real Kähler Space

- v*) A compact Lie group $\mathfrak{G}$ of internal symmetries is realized in the Kähler space $\mathcal{K}$ by symplectic orthogonal representation $\mathcal{U}(\mathfrak{g}), \mathfrak{g} \in \mathfrak{G}$.

- vi*) time and space -symplectic orthogonal representation of the Galilean group

Examples of $\mathbb{C}^n$ and $\mathcal{K}^{2n}$ correspondence

$\mathbb{C}^1$ and $\mathcal{K}^2$ correspondence

- Real 2×2 symmetric matrices $\ell = \begin{pmatrix} s_1 & a \\ a & s_2 \end{pmatrix}$
- Complex structure $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- $\mathcal{K}$-Hermiticity $\ell j - j\ell = 0$ implies: $a = 0$, $s_1 = s_2 \equiv s$ we left with $\ell_0 = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$ with equal e.v. $\lambda_{1,2} = s$
- Two projectors on corresponding e.v.'s $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad jP_1 - P_2 j = 0$
- The resolution of the identity

$$P_1 + P_2 = 1, \quad s(P_1 + P_2) = \ell_0.$$

$\mathbb{C}^2$ (qubit) and $\mathcal{K}^4$ correspondence

- General Hermitian matrix acting in $\mathbb{C}^2$

$$L = S + iA, \quad \text{where} \quad S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$$

s_{ik}, $i = 1, 2$ and a are real

$$L = \begin{pmatrix} s_{11} & s_{12} + ia \\ s_{12} - ia & s_{22} \end{pmatrix}$$

- L has the following orthonormal eigenvectors V_α, $\alpha = 1, 3$

$$V_1 = n_- \begin{pmatrix} \frac{w_-(w_0+i)}{1+w_0^2} \\ 1 \end{pmatrix}, \quad V_2 = n_+ \begin{pmatrix} \frac{w_+(w_0+i)}{1+w_0^2} \\ 1 \end{pmatrix}$$

Their e.v.'s are $\lambda_1 = \frac{1}{2}(-\kappa + s_{11} + s_{22})$, $\quad \lambda_2 = \frac{1}{2}(\kappa + s_{11} + s_{22})$

$$w_\pm = \frac{\pm\kappa + s_{11} - s_{22}}{2a}, \quad w_0 = \frac{s_{12}}{a}, \quad n_\pm = \frac{\sqrt{1 + w_0^2}}{\sqrt{1 + w_\pm^2 + w_0^2}},$$

$$\kappa = \sqrt{4a^2 + s_{11}^2 - 2s_{11}s_{22} + 4s_{12}^2 + s_{22}^2},$$

From L to $\mathcal{L}$

- L on a 2-component complex vector Z

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + i \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = X + iP$$

can be written as

$Z' = LZ = X' + iP' = (S + iA)(X + iP) = SX - AP + i(AX + SP)$

and we have $X' = SX - AP, \quad P' = AX + SP$.
Or in 2-component notations $\begin{pmatrix} X' \\ P' \end{pmatrix} = \begin{pmatrix} S & -A \\ A & S \end{pmatrix} \begin{pmatrix} X \\ P \end{pmatrix}$

- $\mathcal{L}$ on a 4-component real vectors $\begin{pmatrix} X \\ P \end{pmatrix}$

$$\mathcal{L} = \begin{pmatrix} s_{11} & s_{12} & 0 & -a \\ s_{12} & s_{22} & a & 0 \\ 0 & a & s_{11} & s_{12} \\ -a & 0 & s_{12} & s_{22} \end{pmatrix}.$$

From L to $\mathcal{L}$

- **According our scheme to the vector Z in $\mathbb{C}^2$ corresponds the vector in $\mathbb{R}^4$**

$$Z = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} \Leftrightarrow \mathcal{Z} = \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix}$$

Since $LZ = Z' = \begin{pmatrix} x_1' + ip_1' \\ x_2' + ip_2' \end{pmatrix}$, **and** $x_i' = s_{ik}x_k - a_{ik}p_k$,

$$p_i' = a_{ik}x_k + s_{ik}p_k$$

we get $$\mathcal{Z}' = \begin{pmatrix} x_1' \\ x_2' \\ p_1' \\ p_2' \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & 0 & -a \\ s_{12} & s_{22} & a & 0 \\ 0 & a & s_{11} & s_{12} \\ -a & 0 & s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix}$$

where $a_{12} = -a$ **and** $a_{21} = a$.

Eigenvectors of $\mathcal{L}$

- Eigenvalues of the matrix $\mathcal{L}_4$ are $\lambda_1 = \lambda_3, \quad \lambda_2 = \lambda_4$ where

$$\lambda_1 = \frac{1}{2}(-\kappa + s_{11} + s_{22}), \quad \lambda_2 = \frac{1}{2}(\kappa + s_{11} + s_{22})$$

- The two first orthogonal eigenvectors corresponding to eigenvalues λ_1 and λ_2 are

$$\mathcal{V}_1 = \gamma(V_1) = n_- \gamma \begin{pmatrix} \frac{w_-(w_0+i)}{1+w_0^2} \\ 1 \end{pmatrix} = n_- \begin{pmatrix} w_0 \rho_- \\ 1 \\ \rho_- \\ 0 \end{pmatrix},$$

$$\mathcal{V}_2 = \gamma(V_2) = n_+ \gamma \begin{pmatrix} \frac{w_+(w_0+i)}{1+w_0^2} \\ 1 \end{pmatrix} = n_+ \begin{pmatrix} w_0 \rho_+ \\ 1 \\ \rho_+ \\ 0 \end{pmatrix},$$

where

$$\rho_\pm = \frac{w_\pm}{(1 + w_0^2)}$$

Eigenvectors of $\mathcal{L}$

- $\mathcal{V}_3$ and $\mathcal{V}_4$ can be obtained by acting of the complex operator $\mathcal{J}$ on $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively,

$$\mathcal{V}_3 = \mathcal{J}\,\mathcal{V}_1, \quad \mathcal{V}_4 = \mathcal{J}\,\mathcal{V}_2,$$

- Therefore, the spectrum is double degenerated

$$\begin{aligned} (\mathcal{L} - \lambda_1)\mathcal{V}_1 &= 0 \quad (\mathcal{L} - \lambda_2)\mathcal{V}_2 = 0 \\ (\mathcal{L} - \lambda_1)\mathcal{V}_3 &= 0, \quad (\mathcal{L} - \lambda_2)\mathcal{V}_4 = 0. \end{aligned}$$

These properties follow immediately from $\mathcal{J} = \mathcal{J}\mathcal{L}$.

Vectors $\mathcal{V}_i$, $i = 1, ...4$ are orthonormal

Spectral decomposition for $\mathcal{L}$

Defining
$$\mathcal{P}_1 = \mathcal{V}_1 \otimes \mathcal{V}_1; \qquad \mathcal{P}_3 = -\mathcal{J}\mathcal{P}_1\mathcal{J}$$
$$\mathcal{P}_2 = \mathcal{V}_2 \otimes \mathcal{V}_2; \qquad \mathcal{P}_4 = -\mathcal{J}\mathcal{P}_2\mathcal{J}$$

we get the following spectral decomposition

$$\sum_{i=1}^{4} \lambda_i \mathcal{P}_i = \mathcal{L}_4, \qquad \sum_{i=1}^{4} \mathcal{P}_i = \mathcal{I}_4$$

here $\mathcal{I}_4$ is the unit 4×4 matrix.

Relations between $U(2)$ rotations in $\mathbb{C}^2$ and $O(4)\cap Sp(4)$ in $\mathcal{K}^4$

$$G_1=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\quad G_2=\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix},$$

$$G_3=\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix},\quad G_4=\begin{pmatrix}0&-1&0&0\\1&0&0&1\\0&0&0&1\\0&0&-1&0\end{pmatrix}.$$

These generators span the Lie algebra of $O(4)\cap Sp(4)$.

Relations between $U(2)$ rotations in $\mathbb{C}^2$ and $O(4) \cap Sp(4)$ in $\mathcal{K}^4$

These generators possess the following properties:

- **Skew-Symmetry: All generators satisfy $G_i^T = -G_i$, ensuring their membership in $\mathfrak{o}(4)$.**
- **Preservation of the Symplectic Form: For each G_i, the condition $G_i J = J G_i$ holds, where $\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, guaranteeing membership in $\mathfrak{sp}(4)$.**
- **Isomorphism with $U(2)$:**
  - **Generators G_1, G_2, G_3 correspond to $SU(2)$**
  - **Generator G_4 corresponds to $U(1)$, note that $G_2 = -\mathcal{J}$**

Tensor Products on Kahler spaces

To define the tensor product of two linear vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ it suffices to define the tensor product on elements of bases

$$\ell_A \otimes \ell_B = \ell_{AB}$$

In $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^N$ there is a basis

$$\{\ell_A, A = 1, ...2n\} = \{e_a|+\rangle,\, h_a|-\rangle, a = 1, ...N\},$$

as noted above we take $h_a = e_a$. Therefore the basis ℓ_{AB} in term of $e_a|\pm\rangle$ can be parametrize as

$$\begin{aligned} \ell_{AB} &= e_a|\pm\rangle \otimes e_b|\pm\rangle = e_a \otimes e_b \otimes |i\rangle \otimes |j\rangle, \qquad (7) \\ A, B &= 1,,, 2n,\; a, b = 1, ...n,\; i, j = \pm \end{aligned}$$

Tensor Product $\underset{\mathbb{R}}{\otimes}$ on Kähler Spaces

Denoting $\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \eta_1, \quad \begin{pmatrix} q_2 \\ p_2 \end{pmatrix} = \eta_2$, **we consider the tensor product in the sense of** (7) **that is indicated as** $\underset{\mathbb{R}}{\otimes}$

$$\eta_1 \underset{\mathbb{R}}{\otimes} \eta_2 = \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \underset{\mathbb{R}}{\otimes} \begin{pmatrix} q_2 \\ p_2 \end{pmatrix}$$

$$= \sum_a (q_{1a} e_a |+\rangle + p_{1a} e_a |-\rangle) \underset{\mathbb{R}}{\otimes} \sum_b (q_{2b} e_b |+\rangle + p_{2b} e_b |-\rangle)$$

$$= \sum_{a,b} \Big(q_{1a} q_{2b} \, |+\rangle \otimes |+\rangle + p_{1a} q_{2b} |-\rangle \otimes |+\rangle + q_{1a} p_{2b} \, |+\rangle \otimes |-\rangle$$

$$+ p_{1a} p_{2b} |-\rangle \otimes |-\rangle \Big) \otimes e_a \otimes e_b$$

Tensor Product $\underset{\mathbb{R}}{\otimes}$ on Kähler Spaces

The scalar product and the symplectic form of two vectors in the form $\eta_1 \underset{\mathbb{R}}{\otimes} \eta_2$ are given by the formula

$$g\left(\eta_1 \underset{\mathbb{R}}{\otimes} \eta_2, \chi_1 \underset{\mathbb{R}}{\otimes} \chi_2\right) = g(\eta_1, \chi_1) \cdot g(\eta_2, \chi_2)$$

$$\omega\left(\eta_1 \underset{\mathbb{R}}{\otimes} \eta_2, \chi_1 \underset{\mathbb{R}}{\otimes} \chi_2\right) = \omega(\eta_1, \chi_1) \cdot g(\eta_2, \chi_2) + g(\eta_1, \chi_1) \cdot \omega(\eta_2, \chi_2)$$

Tensor Product $\underset{\mathcal{K}}{\otimes}$ on Kahler Spaces

$$\eta_1 \underset{\mathcal{K}}{\otimes} \eta_2 = \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \underset{\mathcal{K}}{\otimes} \begin{pmatrix} q_2 \\ p_2 \end{pmatrix} = \sum_{a,b} \begin{pmatrix} q_{1a}q_{2b} - p_{1a}p_{2b} \\ q_{1a}p_{2b} + p_{1a}q_{2b} \end{pmatrix} e_a \otimes e_b$$

Comparing $\underset{\mathcal{K}}{\otimes}$ and $\underset{\mathbb{R}}{\otimes}$ we get

$$\eta_1 \underset{\mathcal{K}}{\otimes} \eta_2 = \mathbb{P}\, \eta_1 \underset{\mathbb{R}}{\otimes} \eta_2$$

$$\mathbb{P} = \Big[|+\rangle\Big(\langle+| \otimes \langle+| - \langle-| \otimes \langle+|\Big) + |-\rangle\Big(\langle+| \otimes \langle-| + \langle-| \otimes \langle+|\Big)\Big]$$

Comparison of tensor product $\underset{\mathbb{C}}{\otimes}$ on Hilbert space $\mathbb{C}^n$ and $\underset{\mathcal{K}}{\otimes}$ on $\mathcal{K}^{2n}$

In $\mathbb{C}^N$ there is a basis $\{e_a, a = 1, ...N\}$ and the tensor product of two element $\psi = \sum_a (q_{1a} + ip_{1a})e_a$ and $\phi = \sum_b (q_{2b} + ip_{2b})\, e_b$ is

$$\psi \underset{\mathbb{C}}{\otimes} \phi = \sum_{a,b} \Big(q_{1a}q_{2b} - p_{1a}p_{2b} + i(q_{1a}p_{2b} + p_{1a}q_{2b})\Big) e_a \otimes e_b$$

Full agreement!

Conclusion

- **Formulation of real QM in real Kähler space and proof of the equivalence of real Kähler QM to QM in Hilbert space are given**